The combinatorial encoding of disjoint convex sets in the plane

JACOB E. GOODMAN*
Department of Mathematics
City College, CUNY
New York, NY 10031
USA
e-mail: jgoodman@ccny.cuny.edu

RICHARD POLLACK†
Courant Institute, NYU
251 Mercer St.
New York, NY 10012
USA
e-mail: pollack@courant.nyu.edu

November 26, 2006

Abstract

We introduce a new combinatorial object, the double-permutation sequence, and use it to encode a family of mutually disjoint compact convex sets in the plane in a way that captures many of its combinatorial properties. We use this encoding to give a new proof of the Edelsbrunner-Sharir theorem that a collection of n compact convex sets in the plane cannot be met by straight lines in more than $2n-2$ combinatorially distinct ways. The encoding generalizes the authors' encoding of point configurations by "allowable sequences" of permutations. Since it applies as well to a collection of compact connected sets with a specified pseudoline arrangement $\mathcal{A}$ of separators and double tangents, the result extends the Edelsbrunner-Sharir theorem to the case of geometric permutations induced by pseudoline transversals compatible with $\mathcal{A}$.

1 Introduction

Approximately twenty-five years ago **[7]**, the authors introduced a combinatorial encoding of planar point configurations designed to open problems on configurations to purely combinatorial investigation. This encoding, which assigned to each planar configuration of n points a circular sequence of permutations of $1, \ldots, n$, has been used in a number of papers since then, in dual form (as an encoding of line arrangements) as well as in primal form; because the same object encodes pseudoline arrangements as well, it has also been used to derive results on pseudoline

*Supported in part by NSA grant MDA904-03-I-0087 and PSC-CUNY grant 65440-0034.
†Supported in part by NSF grant CCR-9732101.

arrangements. For a survey of results obtained by this technique, see, e.g., **[6]**. Recent applications include **[15]** and **[16]**.

In the present paper, we extend the encoding of point configurations by circular sequences of permutations to an encoding of planar families of disjoint compact convex sets by circular sequences of what we call "double permutations." It turns out that this new encoding applies as well to a more general class of objects: families of compact *connected* sets in the plane with a specified arrangement of pairwise tangent and pairwise separating *pseudolines*, and thereby permits us to extend combinatorial results from convex sets to these more general objects. In particular, we use the double-permutation sequence encoding to prove the theorem of Edelsbrunner and Sharir that a collection of n mutually disjoint compact convex sets in the plane has no more than $2n-2$ "geometric permutations," and thereby establish this result in greater generality as well.

In 1990, Edelsbrunner and Sharir solved a problem in geometric transversal theory that had resisted several earlier efforts, and proved that a collection of n pairwise disjoint compact convex sets in the plane had no more than $2n-2$ "geometric permutations," i.e., could be met by straight lines in no more than $2n-2$ distinct orders (and their reverses) **[4]**. Earlier, Katchalski *et al.* **[13]** had shown that this many geometric permutations were indeed possible for all $n \geq 4$ (their well-known example is shown in Figure 1), so that the Edelsbrunner-Sharir bound was tight.

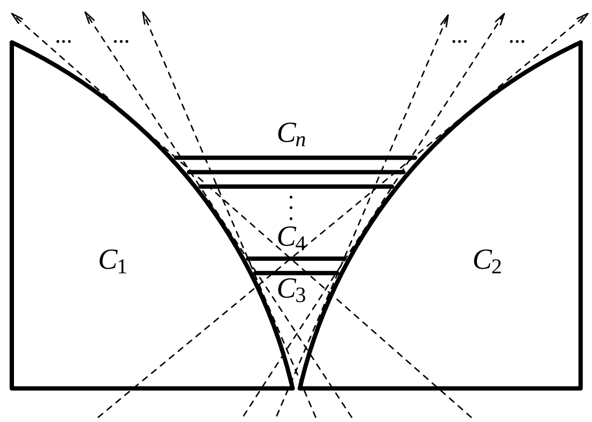

Figure 1

In **[11]**, Branko Grünbaum asked how essential "straightness" was for combinatorial results on line arrangements, and suggested in particular that many of the same combinatorial properties that had been established for an arrangement of straight lines should hold as well for an arrangement of *pseudolines*, i.e., finite sets of curves homeomorphic to straight lines, any two of which meet just once and cross there, as straight lines do.

To fix our ideas, we adopt Grünbaum's model from **[12]**: the affine plane is represented by the interior of a closed disk Δ, and a pseudoline by an arc joining a pair of antipodal points on $\partial\Delta$. Unless noted to the contrary, we insist that every two of our pseudolines meet in the *interior* of the disk, i.e., we eschew "parallel" pseudolines. If C is a point set in the plane, we say that a pseudoline T is *tangent* to C if T meets C, and if C is contained in one of the two closed (pseudo)halfplanes determined by T. If C_1 and C_2 are two disjoint sets, we call T a *double tangent* to the pair C_1, C_2 if T is tangent to both; it is *externally tangent* if C_1 and C_2 lie in the same closed halfplane determined by T; *internally tangent* otherwise.

We will often work with directed pseudolines, i.e, we will specify one of the endpoints of a pseudoline as its terminal point.

The new encoding works as follows. Given a planar family $\mathcal{C} = \{C_1, \ldots, C_n\}$ of mutually disjoint compact convex sets, we project the sets onto a directed line L and denote the endpoints of each projected set C_i by i, i', according to their order on L. This gives, in general, a permutation of the $2n$ indices $1, \ldots, n, 1', \ldots, n'$. We then rotate L counterclockwise, and record the circular sequence consisting of all the "double permutations" of $1, \ldots, n$ that arise in this way. (Notice that if the convex sets are points, this encoding reduces to the encoding by circular sequences of permutations in the sense of **[7]**.)

It turns out that this simple-minded encoding is strong enough to capture all of the features of the family $\mathcal{C}$ that are essential in determining the (partial and complete) transversals that $\mathcal{C}$ possesses, and that it extends in a natural way to the case where the sets C_i are merely connected.

Our main application of this new encoding is

Theorem 1. *Suppose $\mathcal{C} = \{C_1, ..., C_n\}$ is a family of compact connected sets in the affine plane, each pair C_i, C_j separated by a pseudoline L_{ij} and provided with a set of four double tangents, two internal and two external. Suppose further that these $5\binom{n}{2}$ pseudolines form an arrangement $\mathcal{A} = \mathcal{A}_T \cup \mathcal{A}_S$ ($\mathcal{A}_T$ being the tangents, $\mathcal{A}_S$ the separators). Then $(\mathcal{C}, \mathcal{A})$ has no more than $2n-2$ geometric permutations.*

Definition 2. A *geometric permutation* of $(\mathcal{C}, \mathcal{A})$ is an ordering of the sets $C_1, \ldots, C_n$ such that there exists a pseudoline L compatible with A meeting them in that order (and its reverse).

Notice that for a particular (directed) pseudoline transversal compatible with $(\mathcal{C}, \mathcal{A})$, the order in which it meets the sets in $\mathcal{C}$ is well-defined; this follows from its compatibility with $\mathcal{A}_S$. More precisely, a directed transversal cannot meet C_i and C_j in the order (say) C_i, C_j, C_i, since then it would cross L_{ij} twice.

Theorem 1 generalizes the Edelsbrunner-Sharir theorem: Notice that there is no assumption there that every pair of compact convex sets has four *distinct* tangents, no two parallel. But it is a simple matter to blow up each set slightly, by taking its Minkowski sum with a sufficiently small disk of generic radius, to insure this; if the original collection had more than $2n-2$ geometric permutations, then so would the new one. This works equally well with an arbitrary family of compact connected sets with pseudoline separators.

In Section 2, we establish basic properties of tangent pseudolines to compact connected sets. Section 3 is devoted to the definition and essential features of our encoding, both for convex sets and more generally for connected sets. The proof of Theorem 1 is completed in Section 4.

For the remainder of the paper, $\mathcal{C}$ and $\mathcal{A}$ will be as in the hypothesis of Theorem 1.

For recent surveys in geometric transversal theory, see **[3, 9, 18]**; for recent work on pseudoline arrangements, see **[2,** Chap. 6**]** and **[6]**.

2 Connected sets, separators, and tangents

Remark 3. We use several facts about pseudoline arrangements in what follows, which we collect here for the convenience of the reader:

(i) Any arrangement of no more than eight pseudolines is stretchable, i.e., there is a homeomorphism of the interior of the disk Δ to the affine plane that maps the pseudolines in the arrangement to straight lines **[8]**.

(ii) Every pseudoline arrangement can be embedded in a topological plane, i.e., a 2-parameter family of pseudolines forming an (infinite) arrangement such that every pair of points are joined by (exactly) one of the pseudolines, which varies continuously with the points **[10]**.

In several arguments below, the stretchability of a small pseudoline arrangement is used, more for convenience than for necessity. In each case the argument could be carried out by referring to various cells in the original arrangement, but the use of stretchability allows us to simply draw a figure with straight lines and carry out the argument with reference to the latter.

Definition 4. The directed pseudoline L is a *right tangent* to a compact connected set C if L meets C and C is contained in the closed left halfplane of L. (Note: This is called a “left tangent” in **[4]**.) We will call a directed pseudoline that is a right tangent to C_i and a left tangent to C_j, and that meets them in the order C_i, C_j, a “right-left ij tangent”; similarly for the terms “left-right ij tangent,” “right-right ij tangent,” and “left-left ij tangent”.

Lemma 5. *Suppose T_i is a right tangent to C_i that meets C_i before C_j, and that is compatible with the arrangement $\mathcal{A}$. Then T_i points into the halfplane to the left of the right-right ij tangent T_{ij}. (See Figure 3(a).)*

Proof. Suppose T_i pointed into the right halfplane of T_{ij}. For convenience, using Remark 3(i), stretch T_i, T_{ij}, and the separator L_{ij}. Then the situation would be as in Figure 2(a), (b), or (c), depending on where T_i met T_{ij} with respect to L_{ij}. Case (a) is impossible since C_i must lie in the shaded region, hence T_{ij} cannot be tangent to it. Case (b) is impossible for the same reason (recall that L_{ij} is a *strict* separator). Finally, in Case (c), C_j would have to lie in the shaded region, hence T_i could not meet it. Thus T_i cannot point into the halfplane to the right of T_{ij}, and so it points into the left halfplane. □

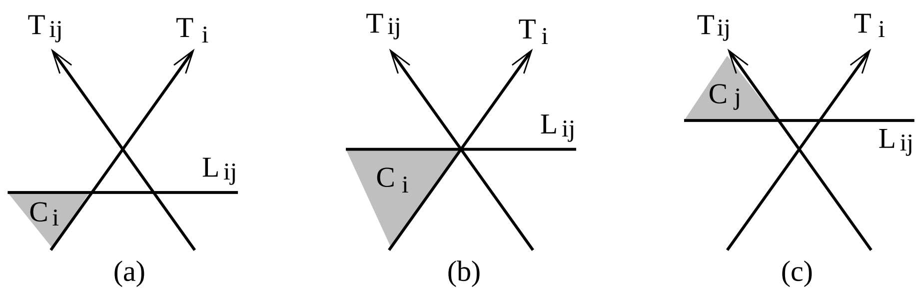

Figure 2

Corollary 6. *(The remaining three cases:)*

(i) If T_j is a right tangent to C_j that meets C_i before C_j, then T_j points into the halfplane to the right of the right-right ij tangent T_{ij}.

(ii) If T_i is a left tangent to C_i that meets C_i before C_j, then T_i points into the halfplane to the right of the left-left ij tangent T_{ij}.

(iii) If T_j is a left tangent to C_j that meets C_i before C_j, then T_j points into the halfplane to the left of the left-left ij tangent T_{ij}.

Proof. These statements follow either by the analogous arguments to the proof of Lemma 5, or else by reflecting appropriately (on) the statement of Lemma 5. □

Figure 3 illustrates the four possibilities—of Lemma 5, and of the three parts of Corollary 6, respectively.

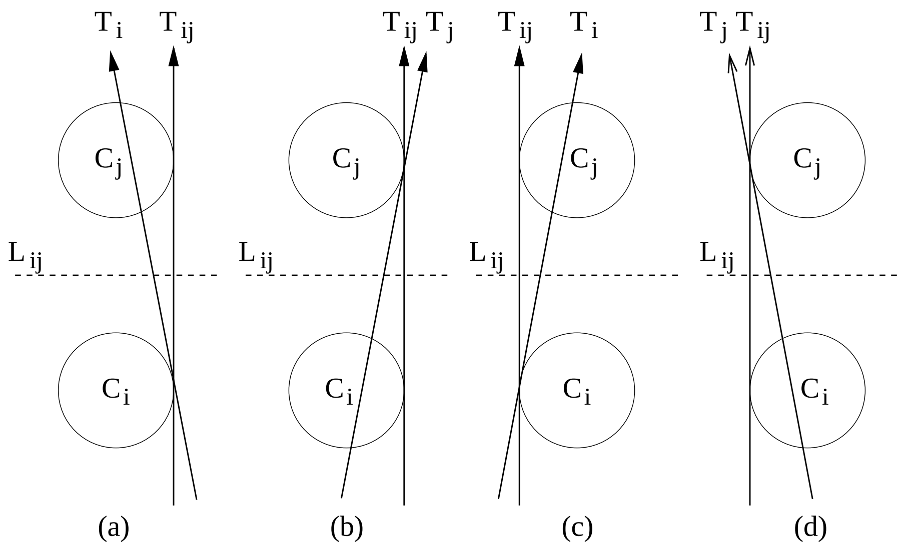

Figure 3

Corollary 7. *If there is a directed right tangent T_i to C_i meeting C_j, and—counter-clockwise from it—a directed right tangent T_j to C_j meeting C_i, and if T_i and T_j are each compatible with $\mathcal{A}$, then the right-right ji tangent T_{ji} points between T_i and T_j.*

Proof. There are several cases, depending on the order in which T_i and T_j meet the two sets. The result in each case follows immediately from Lemma 5 or Corollary 6. □

3 The sequence of double permutations

Definition 8. If $\mathcal{C} = \{C_1, \ldots, C_n\}$ is a family of mutually disjoint compact convex sets, we associate to the family $\mathcal{C}$ a circular sequence of permutations of the symbols $1, 1', 2, 2', \ldots, n, n'$, as follows. Project the sets in $\mathcal{C}$ orthogonally onto a directed line L, and record the order in which the endpoints of the intervals constituting the projections of the sets C_i occur. This gives a permutation of the indices $1, 1', \ldots, n, n'$ (where the primed indices correspond to the right-hand endpoints of the intervals). If L rotates counterclockwise, the permutation changes every time L becomes orthogonal to an (undirected) double tangent, and we obtain a periodic sequence of permutations of $1, 1' \ldots, n, n'$ which we call the *circular sequence of double permutations of* $C_1, \ldots, C_n$. The *move* from each term to the next is simply the switch of two adjacent indices such as i, j or i', j or i, j' (but not i, i').

(Notice that instead of thinking of each term of the sequence as defined by projection onto L, we can think of it as arising by sweeping a line orthogonal to L from left to right, and recording the order in which the sweepline enters and then leaves the various sets.)

If we adopt the convention that each switch is written in the order in which the indices appear *after* they have switched, then it is easy to see that

(i) the switch ij corresponds to a left-left ij tangent,

(ii) the switch $i'j'$ corresponds to a right-right ij tangent,

(iii) the switch $i'j$ corresponds to a left-right ij tangent,

(iv) the switch ij' corresponds to a right-left ij tangent,

It is also easy to see that, knowing the sequence of ordered switches, we can reconstruct the double-permutation sequence itself: Any term can be reconstructed from the half-period of switches following it (since these are compatible with only one possible order among the indices $1, 1', \ldots, n, n'$), and the remaining terms can be then be written down by successively applying the switches that follow.

This allows us to define the double-permutation sequence in greater generality:

Definition 9. For a family $\mathcal{C}$ of connected sets, together with a given arrangement $\mathcal{A} = \mathcal{A}_T \cup \mathcal{A}_S$ of double tangent pseudolines and separators, we generate the circular sequence $\mathcal{S}(\mathcal{C}, \mathcal{A})$ of double permutations as follows. We first record the periodic sequence of ordered switches (which is read off from the directions of the $8\binom{n}{2}$ ordered double tangents), and we then (re)construct the double permutations themselves by the method described just above. We must show that no cycle is possible among the pairwise orderings. The easiest way to see this is to use Remark 3(ii), which says that a pseudoline arrangement can always be extended to a full topological plane. If we do this, we can then mimic the construction described above by sweeping a pseudoline with a fixed terminal point on $\partial\Delta$ continuously across all the sets C_i. (If desired, we can even replace each connected set C_i by its "convex hull" in the topological plane, i.e., the intersecton of all the (pseudo)halfplanes containing it, to arrive at a situation completely analogous to the one described above.) Since the resulting sequence of double permutations has the $8\binom{n}{2}$ ordered double tangent directions, in counterclockwise order, as its sequence of ordered switches, each double permutation must be consistent with the switches following it, so that no cycle is possible.

The following is immediate from the definitions, with the help of Lemma 5 and Corollary 6.

Proposition 10. *The sequence of ordered switches of* $(\mathcal{C},\mathcal{A})$ *has the following properties:*

(i) Each period is composed of two half-periods. Each switch in the second half is the reversal of the corresponding switch in the first half, with the primed and unprimed indices interchanged; for example, if the switch $\underline{32'}$ *occurs somewhere, then the switch* $\underline{23'}$ *will occur exactly a half-period later.*

(ii) In any half-period, for i, j *distinct,*

(a) one but not both of the switches $\underline{ij}$ *and* $\underline{j'i'}$ *occur;*

(b) one but not both of the switches $\underline{ij'}$ *and* $\underline{ji'}$ *occur;*

(c) one but not both of the switches $\underline{i'j}$ *and* $\underline{j'i}$ *occur.*

Hence in any full period every switch of two distinct indices chosen from the list $1,\ldots,n,1',\ldots,n'$ *occurs, with the exception of pairs of the form* i,i' *(or* i',i*); thus the full period consists of precisely* $8\binom{n}{2}$ *switches.*

(iii) For i, j *distinct, the* i, j *switches in which* i *precedes* j *occur in the order* ij', $\{ij, i'j'\}$, $i'j$ *in some half-period.*

And from this we obtain

Corollary 11. *The double-permutation sequence* $\mathcal{S}(\mathcal{C},\mathcal{A})$ *has the following properties:*

(i) Each term is a permutation of the symbols $1,1',\ldots,n,n'$*, with each pair* i,i' *appearing in that order in every term.*

(ii) The move from each term to the next consists of a switch between two successive indices other than i,i'.

(iii) Each period is composed of two half-periods. Each term in the second half is the reversal of the corresponding term in the first half, with the primed and unprimed indices interchanged; for example, if the term $13543'5'1'22'4'$ *occurs somewhere, then the term* $422'1534'5'3'1'$ *will occur exactly a half-period later.*

(iv) The full period consists of precisely $8\binom{n}{2}$ *double permutations.*

The properties in Proposition 10 and Corollary 11 are reminiscent of the properties we abstracted in **[7]** from the circular sequence of permutations of a point configuration. We called such a sequence of permutations an "allowable sequence," and showed in **[5]** that every allowable sequence was, in fact, realizable by a "generalized configuration" of points, i.e., points joined by pseudolines forming an arrangement. We do not know, however, if the corresponding fact holds in this situation: does every sequence satisfying the conditions of Corollary 11 arise from a family $\mathcal{C}$ of connected sets and an arrangement $\mathcal{A}$ of pseudolinear tangents and separators?

Definition 12. A sequence of permutations of $1,\dots,n,1',\dots,n'$ satisfying the conditions of Corollary 11 is called an *allowable sequence of double permutations of* $1,\dots,n$. If the double permutation sequence comes from some collection of mutually disjoint compact convex sets and their straight-line double tangents (and some collection of separating lines), we call the sequence *stretchable*.

Notice that if an allowable double-permutation sequence can be realized by *connected* sets with straight-line tangents and separators, then it is stretchable: just replace each connected set with its convex hull, keeping the arrangement the same. Thus, stretchability depends on the nature of the double-tangent arrangement, not on the shapes of the sets themselves.

Not every allowable double-permutation sequence arising from a family of compact connected sets with pseudoline tangents and separators is stretchable: it is not difficult to construct an example based on the nonstretchable "bad pentagon" of **[7]** (see also **[6]**).

Remark 13. From the double-permutation sequence of $(\mathcal{C},\mathcal{A})$ we can identify its various geometric permutations and reconstruct the order of each one:

Call a term in the double-permutation sequence of $(\mathcal{C},\mathcal{A})$ a "transversal term" if it is of the form $i_1\dots i_n j'_1\dots j'_n$, and a "final transversal term" if it is the last member of a string of transversal terms. Any common transversal gives rise to a transversal term (since when we reach it in a sweep "parallel" to the transversal we have entered all the sets but not yet left any), and a complete string of terms of this sort will correspond to a single geometric permutation, since no switch of the type $\underline{i'j}$ takes place (for the order in which an oriented transversal meets two sets to change, its direction must pass through an internal tangent direction—again, this is most easily seen by the use of Remark 3(ii)). On the other hand, if this string is followed by a nontransversal term and then—within the same half-period—by yet another transversal term, the corresponding geometric permutation *will* be different, since a switch of the form $\underline{i'j}$ has now taken place, i.e., the direction has passed though an internal tangent direction. (One sees this by noting that the nontransversal term must contain two indices, i and j, with ii' appearing in the first half and jj' in the second; hence for the first transversal term we have $i \prec j$, but for the second, $j \prec i$ — look at the half-period following the $\dots i \dots i' \dots j \dots j' \dots$ term!) Thus *the* *number of geometric permutations of* $(\mathcal{C},\mathcal{A})$ *is the same as the number of strings of transversal terms in any half-period of* $\mathcal{S}(\mathcal{C},\mathcal{A})$.

To read off the geometric permutation corresponding to a string of transversal terms, consider the half-period that follows it. For every distinct i, j, exactly one of $\underline{i'j}$ or $\underline{j'i}$ must occur as a switch in that half-period. The former means that i precedes j in the geometric permutation, the latter that j precedes i.

Notice that since, if two directed pseudoline transversals meet the sets in different orders, there must be an internal double tangent whose direction lies between theirs, it follows that if two directed transversals are "parallel," they must meet the sets in the same order.

4 Completing the proof of Theorem 1

Suppose $i_1 \dots i_n j'_1 \dots j'_n$ is a final transversal term. Select the first primed index, j'_1, in that term. (In the convex, straight-tangent case, this corresponds to twisting a directed common transversal counterclockwise until it reaches a "final" position, in which it is necessarily a right-left internal tangent to two of the sets, and then recording the index of the set to which it is a right tangent.) If we do this for the last transversal term in each string, we obtain a periodic sequence $\dots, k'_1 \dots k'_m, k'_1 \dots k'_m, \dots$ of indices, which we call the "fundamental sequence" $F(\mathcal{C},\mathcal{A})$. We will show that the period, m, of $F(\mathcal{C},\mathcal{A})$, which (as we have seen in Remark 13 is precisely twice the number of geometric permutations in $(\mathcal{C},\mathcal{A})$), is bounded above by $4n-4$. This is done by a similar argument to the one that Edelsbrunner and Sharir used in **[4]**, although the sequence they applied it to was somewhat different from ours. As in **[4]**, we call a (possibly nonconsecutive) substring of a full period of the periodic sequence $F(\mathcal{C},\mathcal{A})$ a "scattered substring."

Lemma 14. *No period of* $F(\mathcal{C},\mathcal{A})$ *contains a scattered substring of the form* $\dots i' \dots j' \dots i' \dots j' \dots$.

Proof. Suppose the contrary. Then the sequence $\mathcal{S}(\mathcal{C},\mathcal{A})$ of ordered switches contains in order, within one period, the four switches $\dots, \underline{i'k}, \dots, \underline{j'l}, \dots, \underline{i'm}, \dots, \underline{j'n}, \dots$. These correspond to directed double tangents $T_{i'k}, T_{j'l}, T_{i'm}, T_{j'n} \in \mathcal{A}_T$ in counterclockwise order pointing within a single traversal of $\partial\Delta$, each one a common transversal of all the sets in $\mathcal{C}$. Notice that at least three of the four arcs between successive terminal points of the four tangents are smaller than 180° in length. Suppose without loss of generality that these are from $T_{i'k}$ to $T_{j'l}$, from $T_{j'l}$ to $T_{i'm}$, and from $T_{i'm}$ to $T_{j'n}$. Consider the first of these. $T_{i'k}$ is a right tangent to C_i that meets C_j (because it is a common transversal), and likewise $T_{j'l}$ is a right tangent to C_j that meets C_i and that points $< 180^\circ$ counterclockwise from $T_{i'k}$. By Corollary 7, the right-right ji-tangent T_{ji} must point inside that arc. But by the same token T_{ji} also points inside the arc from the terminal point of $T_{i'm}$ to that of $T_{j'n}$, which is impossible. □

We now recall the definition of an $(n,2)$-Davenport-Schinzel sequence, and of a Davenport-Schinzel cycle, as given, say, in **[17]**, and the basic result about its length, which also appears in **[17]**.

Definition 15. A word on the symbols $1,\ldots,n$ is called an $(n,2)$-Davenport-Schinzel sequence if it contains no immediately repeated symbol, and if it contains no scattered substring of the form $ijij$. A (doubly infinite) periodic sequence is called an $(n,2)$-Davenport-Schinzel cycle if every period is an $(n,2)$-Davenport-Schinzel sequence.

Theorem 16. *An $(n,2)$-Davenport-Schinzel cycle has maximum length $2n-2$.*

The last step is to convert $F(\mathcal{C},\mathcal{A})$ into a Davenport-Schinzel cycle. We do this by contracting each immediate repetition of an index into a single index, and call the resulting periodic sequence $F'(\mathcal{C},\mathcal{A})$. Notice that if we amalgamate two equal indices $i'i'$ in $F(\mathcal{C},\mathcal{A})$ corresponding to switches $i'j$ and $i'k$, then j cannot equal k since $i'j$ cannot occur twice within the same period. Thus the indices $k'j'$ in $F(\mathcal{C},\mathcal{A})$ coming from switches a half-period later are *not* amalgamated. It follows that if m is the number of indices, then we can amalgamate indices at most $m/2$ times.

By Theorem 16, the period of $F'(\mathcal{C},\mathcal{A})$ has length at most $2n-2$, so that (by the comment just above) the period of $F(\mathcal{C},\mathcal{A})$ has length at most $4n-4$. Hence the number of final transversal terms in any half-period of $\mathcal{S}(\mathcal{C},\mathcal{A})$ is bounded above by $2n-2$, so that by Remark 13, Theorem 1 follows. □

5 Remarks

The combinatorial encoding described in this paper, and the use to which we have put it, were inspired by the methods used by Edelsbrunner and Sharir to prove their $2n-2$ bound for convex sets, as well as by the authors' previous work on allowable sequences of permutations. In addition to enlarging the scope of the $2n-2$ theorem, our methods also provide a means for replacing the geometric arguments in the Edelsbrunner-Sharir paper that involve moving tangents continuously by combinatorial arguments. One could, however, use similar methods and work in a topological affine plane, as described in Definition 9, and as we did in **[1]**, but that would result in a somewhat weaker theorem: it would be necessary to assume that all the transversals giving us the various geometric permutations were contained in the *same* pseudoline arrangement containing $\mathcal{A}$, rather than merely being individually compatible with $\mathcal{A}$. (Then one could use Remark 3(ii) and extend the arrangement to a topological affi ne plane, and use continuity arguments there.)

Finally, it would be interesting to see how much of this combinatorial encoding can be carried over to higher dimensions, possibly via oriented matroids **[2]**.

References

[1] S. Basu, J. E. Goodman, A. Holmsen, and R. Pollack. The Hadwiger transversal theorem for pseudolines. In J. E. Goodman, J. Pach, and E. Welzl, editors, *Combinatorial and Computational Geometry*, *Math. Sci. Res. Inst. Publ.* 52, Cambridge Univ. Press, Cambridge, 2005, pages 79–85.

[2] A. Bj¨orner, M. Las Vergnas, B. Sturmfels, N. White, and G.M. Ziegler. *Oriented Matroids*, 2nd Ed. Volume 46 of *Encyclopedia of Mathematics*, Cambridge Univ. Press, 1999.

[3] J. Eckhoff. Helly, Radon, and Carathéodory type theorems. In P. M. Gruber and J. M. Wills, editors, *Handbook of Convex Geometry*, Volume A, North-Holland, Amsterdam, 1993, pages 389–448.

[4] H. Edelsbrunner and M. Sharir. The maximum number of ways to stab n convex noninstersecting sets in the plane is $2n - 2$. *Discrete Comp. Geom.* 5 (1990), 35–42.

[5] J. E. Goodman. Proof of a conjecture of Burr, Gr¨unbaum, and Sloane. *Discrete Math.* 32 (1980), 27–35.

[6] J. E. Goodman. Pseudoline arrangements. In J. E. Goodman and J. O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, 2nd edition, CRC Press, Boca Raton, 2004, pages 97–128.

[7] J. E. Goodman and R. Pollack. On the combinatorial classification of nondegenerate configurations in the plane. *J. Combin. Theory, Ser. A* 29 (1980), 220–235.

[8] J. E. Goodman and R. Pollack. Proof of Gr¨unbaum's conjecture on the stretchability of certain arrangements of pseudolines. *J. Combin. Theory Ser. A* 29 (1980), 385–390.

[9] J. E. Goodman, R. Pollack, and R. Wenger. Geometric transversal theory. In J. Pach, editor, *New Trends in Discrete and Computational Geometry*, Volume 10 of *Algor. Combin.*, Springer-Verlag, Berlin, 1993, pages 163–198.

[10] J.E. Goodman, R. Pollack, R. Wenger, and T. Zamfirescu. Arrangements and topological planes. *Amer. Math. Monthly*, 101 (1994), 866–878.

[11] B. Grünbaum. The importance of being straight. In *Proc. Twelfth Biennial Sem. Canad. Math. Congr. on Time Series and Stochastic Processes, Convexity and Combinatorics (Vancouver, B.C., 1969)*, Canad. Math. Congr., Montreal, pages 243–254.

[12] B. Grünbaum. *Arrangements and Spreads*. Volume 10 of *CBMS Regional Conf. Ser. in Math.*, Amer. Math. Soc., Providence, 1972.

[13] M. Katchalski, T. Lewis, and J. Zaks. Geometric permutations for convex sets. *Discrete Math.* 54 (1985), 271–284.

[14] F. W. Levi. Die Teilung der projektiven Ebene durch Gerade oder Pseudogerade. *Ber. Math.-Phys. Kl. Sächs. Akad. Wiss.*, 78 (1926), 256–267.

[15] L. Lovász, K. Vesztergombi, U. Wagner, and E. Welzl. Convex quadrilaterals and *k*-sets. In J. Pach, editor, *Towards a Theory of Geometric Graphs*, *Contemp. Math.* 342, Amer. Math. Soc., Providence, 2004, pages 139–148.

[16] J. Pach and R. Pinchasi. On the number of balanced lines. *Discrete Comput. Geom.* 25 (2001), 611–628.

[17] M. Sharir and P. K. Agarwal. *Davenport-Schinzel Sequences and Their Geometric Applications.* Cambridge Univ. Press, New York, 1995.

[18] R. Wenger. Helly-type theorems and geometric transversals. In J. E. Goodman and J. O'Rourke, editors, *Handbook of Discrete and Computational Geometry*, 2nd edition, CRC Press, Boca Raton, 2004, pages 73–96.