Interval sequences and the combinatorial encoding of planar families of pairwise disjoint convex sets

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Abstract

We extend a combinatorial encoding of families of pairwise disjoint convex sets in the plane recently introduced by J. E. Goodman and R. Pollack to the case of families not in general position. This encoding generalizes allowable sequences, which encode finite planar point sets. Further we prove several results on realizability questions, and discuss a number of different combinatorial properties that are captured by this encoding, including a theorem of Helly type and a generalization of a separation result of Tverberg.

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1 Introduction

In [7], J. E. Goodman and R. Pollack introduced the notion of an "allowable sequence" of permutations of the set $[n] = \{1, \ldots, n\}$, and used it to encode the combinatorial properties of a finite configuration $\{P_1, \ldots, P_n\}$ of points in $\mathbb{R}^2$. This encoding has since proven useful in solving various problems concerning configurations, as well as arrangements of lines and pseudolines (see, e.g., [6], [8], [15], and [16]).

In a recent paper, the same authors have extended this combinatorial encoding to families of mutually disjoint compact convex sets whose pairwise tangent lines form a simple arrangement (i.e., no two coincide or are parallel), have generalized the encoding to apply to families of *connected* sets with *pseudoline* pairwise double tangents forming an arrangement, have used this encoding to give a new combinatorial proof of the Edelsbrunner-Sharir theorem on the maximum number of "geometric permutations" of a family of n pairwise disjoint compact convex sets in the plane, and have shown that this theorem extends to a family of n pairwise disjoint compact connected sets with pseudoline pairwise tangents forming an arrangement. In that paper, the authors considered only the *simple* case, where no two pairwise tangents are identical or parallel; in the present paper, we generalize their construction, and derive a number of properties of these generalized sequences, which we call *allowable interval sequences*: a "local" condition for a sequence of switches to give an interval sequence, the nonrealizability of allowable interval sequences by special families of convex sets such as circular disks and line segments, and their realizability by families of polygons.

We also show how interval sequences can be used to give a simple proof of an old result of Tverberg's on $(1,2)$-separation of convex sets, and — at the same time — we generalize this result; and we prove a Helly-type theorem for allowable interval sequences with the help of a further generalization of these sequences to the case where the underlying sets may not be pairwise disjoint. Finally, we discuss the question of generating allowable interval sequences algorithmically.

For convenience we recall the definition of the circular sequence of permutations associated to a configuration S of n points in the plane, and its generalization to an "allowable sequence of permutations."

As in Figure 1, we project the points $P_1, \ldots, P_n$ onto a directed line L, which gives a permutation of $1, \ldots, n$, and we let L rotate counterclockwise. The permutation changes whenever L becomes orthogonal to a line connecting two of the points, and we obtain a sequence of permutations of $[n]$,

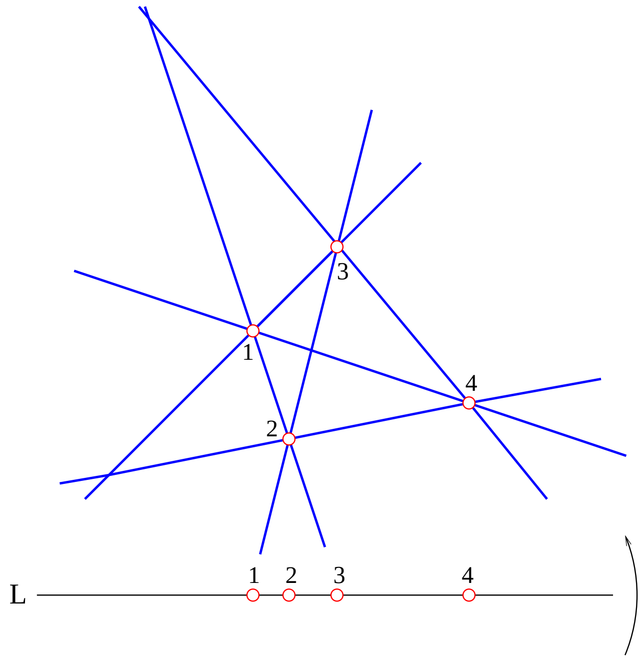

Figure 1: A configuration of points and its circular sequence of permutations

$$\ldots \underline{12}34 - 21\underline{34} - 2\underline{14}3 - \underline{24}13 - 42\underline{13} - 4\underline{23}1 -$$
$$43\underline{21} - \underline{43}12 - 3\underline{41}2 - 31\underline{42} - \underline{31}24 - 1\underline{32}4\ (-\underline{12}34) \ldots$$

which is called the *circular sequence of permutations* of the configuraton $P_1, \ldots, P_n$, which has the following properties:

1. The sequence is periodic, and each period breaks up into two half-periods, in which each term in the second half is just the reverse of the corresponding term in the first half.

2. Each ordered pair i, j switches exactly once in each period. (In the example, we have underlined each ordered pair just before it switches.)

Any sequence of permutations of $[n]$ with these two properties is called an *allowable sequence of permutations*.

If there are no parallel connecting lines and no three points are collinear, we obtain a *simple* allowable sequence (one in which each move from one term to the next consists of just a single switch); more generally, several switches of substrings of a permutation may occur simultaneously.

We can also derive such a sequence starting with a *generalized configuration*, i.e., a configuration of points with each pair provided with a "connecting pseudoline," these pseudolines forming an *arrangement*, i.e., any two meeting just once and crossing there. The basic representation theorem for allowable sequences says that any allowable sequence comes from such a generalized configuration (see [3] or [9]).

2 Interval sequences

Basic definitions

Consider a set of n closed intervals on a line l. If we direct l it makes sense to speak of the starting and ending point of each interval, i and i', say, so a simple encoding of the set of intervals is a permutation of $[n] \cup [n']$, where i precedes i' for every $1 \leq i \leq n$. We call this a *double permutation* of $[n]$.

Let $P = \{p_k\}$ be a sequence, finite or infinite, of double permutations of $[n]$ such that p_k and p_{k+1} differ by a reversal of one or more pairwise disjoint substrings. We call P an *interval sequence*. A *switch* is an ordered k-tuple, $x_1 \ldots x_k$, where $\{x_l, x_m\} \neq \{i, i'\}$ for $k \neq m$, of $[n] \cup [n']$. A *switch sequence* is a sequence $S = \{s_k\}$ where each s_k is a nonempty set of switches, each switch in s_k involving a subset of $[n] \cup [n']$ disjoint from the one involved in every other switch in s_k. Every interval sequence P defines a switch sequence S_P in the following way: let $s_i \in S_P$ if every switch in s_i occurs in this order in p_i and in the opposite order in p_{i+1}. (This is slightly different from the notation used in [10]: what we call s_i here is referred to as s_{i+1} in that paper.) Thus P can be uniquely reconstructed from p_0 and S_P. There are two types of switches, which we will refer to as the *separating switches* (those of the form xy' or $x'y$) and the *supporting switches* (those of the form xy or $x'y'$). For instance, for

$$\begin{aligned} p_k &= 122'31'43'4' \\ p_{k+1} &= 212'41'34'3' \end{aligned}$$

we have

$$s_k = \{3'4', 12, 31'4\}$$

Hence a double permutation p_0 and a switch sequence S can determine an interval sequence P. The following shows that they will do so if and only if they do when restricted to every triple:

Theorem 1. *Let p_0 be a double permutation of $[n]$ and let S be a switch sequence. S and p_0 determine an interval sequence if and only if the restriction of S and p_0 to $\{i, j, k, i', j', k'\}$ determine an interval sequence for every $1 \leq i < j < k \leq n$.*

Proof. Start applying the given switch sequence to the double permutation p_0. If the entire sequence goes through we are done, so assume that at some point we get stuck, i.e., we have reached the double permutation p_m but are unable to perform the next switch, xy. Then it is easy to see that we must have one of these two cases: either $p_m = \cdots x \cdots z \cdots y \cdots$ or $p_m = \cdots y \cdots x \cdots$. But in either case the restriction of p_0 and S to $\{x,y,z,x',y',z'\}$ would also get stuck, which we have assumed is not the case. ☐

Allowable interval sequences

We will focus our attention, here and in most of the sequel, on a special type of periodic interval sequence. Here we describe its construction. The *reverse* $r(p)$ of a double permutation p is the reversed permutation, with primed indices in p unprimed in $r(p)$, and vice versa. Notice that $r(p)$ is also a double permutation. The reverse of a switch is defined the same way; e.g., $r(xyz') = zy'x'$. We also use this notation for the terms of a switch sequence S, i.e. if $s \in S$ then $r(s)$ is obtained by reversing every switch belonging to s.

Now let $P' = \{p_1, \ldots, p_{N+1}\}$ be an interval sequence where $p_{N+1} = r(p_1)$, with switch sequence $S' = \{s_1, \ldots, s_N\}$. Then the switch sequence $S = \{s_1, \ldots, s_N, r(s_1), \ldots, r(s_N), s_1, \ldots, s_N, r(s_1), \ldots\}$ will, together with p_1, determine an interval sequence of period $2N$. We can extend this sequence backward as well, using the periodicity, to give a doubly-infinite periodic sequence. We call this a *cyclic interval sequence*, and if each s_i of the switch sequence consists of a single ordered pair it is also called *simple*.

Our goal is to obtain an encoding of finite families of pairwise disjoint convex sets in the plane analogous to allowable sequences, which encode finite planar point sets. Here is the main definition:

A cyclic interval sequence is called an *allowable interval sequence* if, for any $x \neq y$ in $[n]$, each of the separating switch xy' and the supporting switch xy occurs exactly once in a full period. (This implies, of course, by the definition of a cyclic interval sequence, that each $x'y'$ also occurs exactly once in a full period; as remarked next, it also follows that each $x'y$ does as well.)

Remark 2. If we restrict an allowable interval sequence to only two elements, x and y, a full period of the switch sequence contains exactly 8 switches and it is not hard to see that the interval sequence must contain the following half period (note that $r(p_{k+2}) = p_{k+2}$):

$$\begin{aligned} p_k &= xyx'y' \\ p_{k+1} &= xx'yy' \\ p_{k+2} &= xyx'y' \\ p_{k+3} &= xyy'x' \text{ or } yxx'y' \\ p_{k+4} &= r(p_k) = yxy'x' \end{aligned}$$

The corresponding switch sequence is:

$$\begin{aligned} s_k &= yx' \\ s_{k+1} &= x'y \\ s_{k+2} &= x'y' \text{ or } xy \text{ (resp.)} \\ s_{k+3} &= xy \text{ or } x'y' \text{ (resp.)} \\ s_{k+4} &= r(s_k) = xy' \end{aligned}$$

3 Realizability

Let F be a finite family of pairwise disjoint compact convex sets in the plane. For simplicity, let us assume that each set contains interior points. Let l be a directed line in direction d. The orthogonal projection of the members of F onto l will be a family of segments, which we record as a double permutation p. We proceed to rotate l counterclockwise until its direction becomes orthogonal to some common tangent of a pair of sets in F. As l passes this direction, at least one pair of elements in p will reverse its order, and we will get a new double permutation p'. Clearly when we reach the direction $d+\pi$, we will arrive at the permutation $r(p)$. We have the following.

Theorem 3. *A family of n pairwise disjoint compact convex sets in the plane determines an allowable interval sequence of period* $2N$*, where* $N \leq 4\binom{n}{2}$.

Proof. The fact that we get an interval sequence follows from the discussion above. It also follows that $p_i = r(p_{i+N})$, so that the sequence is cyclic. Each pair of members of F determines exactly four common tangents, having at most eight directions, so that all together the length of the period is at most $2N$. (For more details we refer the reader to [10].) □

We denote the interval sequence associated to the family F by $P(F)$. See Figure 2 for an example of three convex sets giving rise to the following interval sequence:

$$\dots 12\underline{1'3}2'3' - 123\underline{1'2'}3' - \underline{12}32'1'3' - 21\underline{32'}1'3' - 21\underline{2'3}1'3' - 2132'\underline{1'3'} -$$

$$-2\underline{13}2'3'1' - 23\underline{12'}3'1' - 232'\underline{13'}1' - 23\underline{2'3'}11' - \underline{23}32'2'11' - 323'\underline{2'1}1' -$$

$$-32\underline{3'1}2'1' - 3\underline{21}3'2'1' - 3123'\underline{2'1'} - 31\underline{23'}1'2' - 31\underline{3'2}1'2' - \underline{31}23'1'2' -$$

$$-132\underline{3'1'}2' - 13\underline{21'}3'2' - 1\underline{31'}23'2' - 11'\underline{32}3'2' - 11'23\underline{3'2'} - 1\underline{1'2}32'3'\ (-12\underline{1'3}2'3')\dots$$

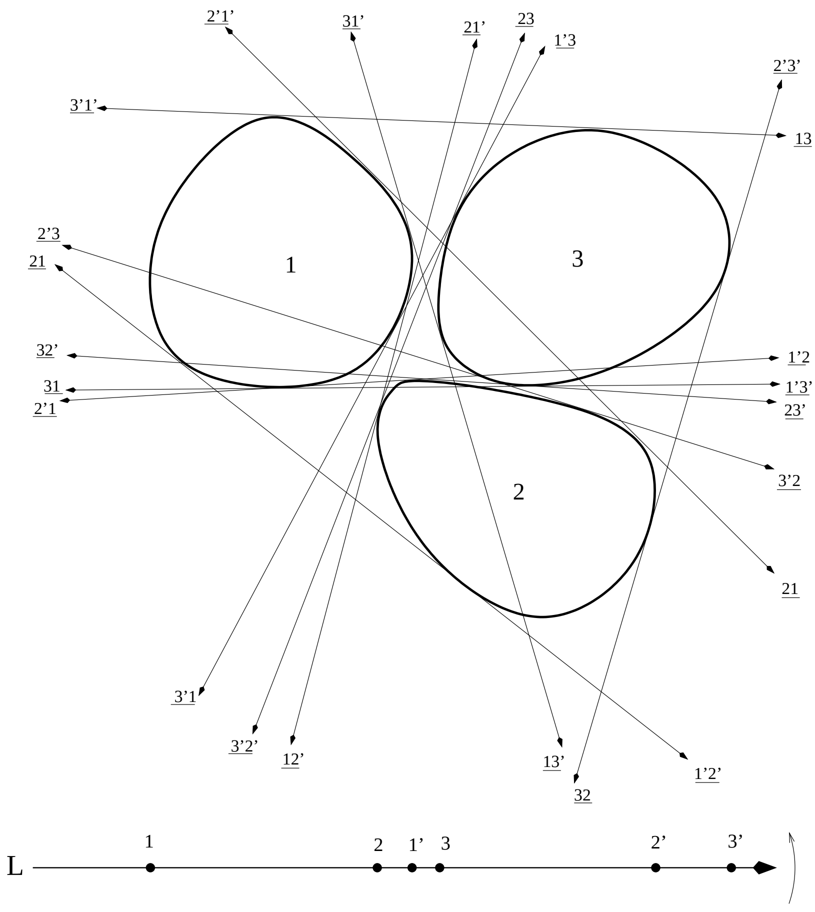

Figure 2: Three compact convex sets and their pairwise common tangents

A family of pairwise disjoint convex sets in the plane is said to be in *general position* if the pairwise common tangents determine distinct directions.

Corollary 4. *If F is a family of n pairwise disjoint compact convex sets in general position in the plane, then $P(F)$ is a simple allowable interval sequence of period $2N$, where $N = 4\binom{n}{2}$.*

Notice that each term of the sequence can also be obtained by sweeping a directed line (perpendicular to the line onto which we project) from left to right across the sets; the order in which the sweepline enters and then leaves the sets determines the double permutation in the corresponding direction.

An allowable interval sequence is called *stretchable* if it comes from some collection of pairwise disjoint compact convex sets. It is shown in [10], using the corresponding fact about pseudoline arrangements, that not every allowable interval sequence is stretchable. If we restrict our attention to those that are, however, we may ask whether these can always be realized by families of special convex sets, such as disks, line segments, or polygons. The following three theorems answer this question.

Theorem 5. *There are stretchable simple allowable interval sequences that cannot be realized by families of circular disks.*

Proof. For a pair of disjoint compact convex sets, X and Y, in general position, we can define an order relation $\prec$ as follows; the two common separating tangents to X and Y partition the plane into quadrants, with X and Y lying in opposite quadrants. If the two common supporting tangents cross in the quadrant containing X, say, we set $X \prec Y$ in the ordering. The reader will easily verify that this is encoded in the interval sequence by the existence of a term $\cdots y \cdots x \cdots x' \cdots y' \cdots$. Now notice that if X and Y are circular disks of radius x and y, respectively, then $x < y$ if and only if $X \prec Y$. So for a family of disks the following can never occur: $X \prec Y \prec Z \prec X$. For more general families of convex sets, however, this can occur, as illustrated in Figure 3. □

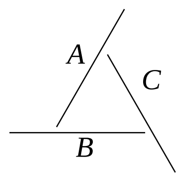

Figure 3: $A \prec B \prec C \prec A$

Moreover, we also have

Theorem 6. *There are stretchable simple allowable interval sequences that cannot be realized by families of line segments.*

NB: Segments in general position give four tangents per pair, but some projections do not give intervals; then just "thicken" each segment slightly to separate i from i' when this happens.

Proof. Given a set $s_1, \ldots, s_4$ of segments, notice that any two can be separated by a line parallel to one of the two. Choose such a separating line for each pair, and mark a direction on it. Then we have a total of at most eight directions, i.e., at most four pairs of opposite directions. Choose four lines through the origin O in these four (pairs of) directions. They divide the unit circle centered at O into eight arcs. Any two directed lines whose directions are those pointing from O into the same arc must cross the separating lines in the same order, hence must induce the same ordering on the four segments. Thus there are no more than eight different orderings on the segments induced by transversals, which shows that there can be no more than four geometric permutations (see Section 4 for the definition of a "geometric permutations").

But it is well-known [12] that a family of four compact convex sets may have as many as six geometric permutations, and — as shown in [10] — the geometric permutations of a family of compact convex sets are encoded by its interval sequence; hence not all interval sequences can arise from families of segments □

On the other hand, we do have

Theorem 7. *Every stretchable simple allowable interval sequence can be realized by a family of polygons.*

Proof. Consider the interval sequence $P(F)$ of a family F of pairwise disjoint compact convex sets. For every two sets $C_i, C_j \in F$ there are 4 tangent lines $C_{ij}^1, \ldots, C_{ij}^4$. For i fixed, consider the arrangement $\{C_{ij}^k \mid j \in [n] \setminus \{i\}, k = 1, \ldots, 4\}$. C_i lies in some (bounded or unbounded) cell C_i' of this arrangement. If C_i' is unbounded, we can add a line one side of which, when intersected with the cell, gives a bounded cell still containing C_i. It is then easy to see that the collection of all the resulting polygonal cells has the same system of pairwise tangents as F, hence defines the same interval sequence as $P(F)$. See Figure 4. □

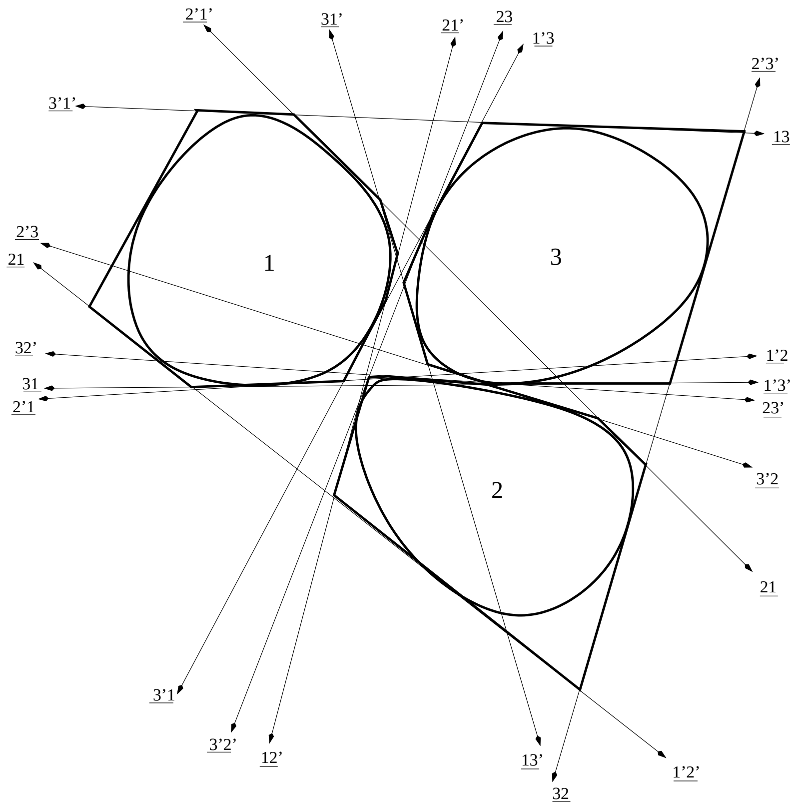

Figure 4: Polygonal cells surrounding the given compact convex sets

4 Combinatorial properties

One may think of the allowable interval sequence simply as a combinatorial encoding of families of pairwise disjoint compact convex sets in the plane. As shown in [10] this encoding can be useful in attacking combinatorial problems concerning such families of sets. In this section we will discuss some of the combinatorial properties that are captured by the interval sequence encoding.

Common and partial transversals

Let P be an allowable interval sequence. We call $p \in P$ a *transversal term* if it is of the form $p = i_1 \dots i_n j_1' \dots j_n'$. Notice that if P came from a family of convex sets, any common transversal would give rise to a transversal term since when we reach it in a sweep of lines parallel to the transversal, we have entered all the sets before leaving any. For a general term of P, if we delete some of the pairs, i and i', so that the remaining subpermutation has the form of a transversal term, this will correspond to a partial transversal.

Geometric permutations

A common transversal to a family F of pairwise disjoint convex sets induces a pair of orderings on the family of sets called a *geometric permutation*. Let $p_k \in P(F)$ be a transversal term. If $p_{k+1} \in P$ is also a transversal term, simple continuity arguments show that a transversal corresponding to p_{k+1} must induce the same geometric permutation on the members of F as a transversal corresponfing to p_k. However, if p_{k+1} is not a transversal term then there must have occured a separating switch $ij' \in s_k$, and then — within the same half period — there is a later transversal term $p_{k'}$, for which the corresponding geometric permutation will be different. To see this consider two transversal terms p_k and p_l $(1 < l - k)$ in the same half period and suppose $ij' \in s_k$. Then the switch $j'i$ must occur within the time we reach p_l. There must also be separating switches of the form xy' that have not yet occured or else it can be shown that p_k and p_l will induce the same geometric permutation. This means the sets i and j have switched order in the geometric permutation while x and y have not, so that the geometric permutations are different. This shows that the interval sequence $P(F)$ encodes all the geometric permutations of F.

Separations

Let F be a family of pairwise disjoint convex sets in the plane. A subset $A \subset F$ is said to be *separated* from $B \subset F$ if there is a straight line l such that the members of A and the members of B lie in opposite open halfplanes determined by l. The interval sequence $P(F)$ will encode all the separations of F. To see this, consider a term $p \in P(F)$. The elements $j_1', \dots, j_k'$ that precede $i_1, \dots, i_m$ in p correspond to members $j_1, \dots, j_k \in F$ that lie to the left of some (sweep)line l with $i_1, \dots, i_m \in F$ lying to the right of l. Clearly all separations of F will be recorded in this way. We use this to give a simple proof of the following observation by Tverberg [14].

Theorem 8. *Let F be a family of n pairwise disjoint convex sets in the plane. If $n \geq 5$, then there is a line that separates a member of F from two other members of F.*

Proof. Consider the switch sequence S of the interval sequence $P(F)$, which we may assume is simple by inflating the sets a bit at different rates if necessary. Since S is periodic we are free to assume that $s_1 = \{ij'\}$. It is easily checked that if the next separating switch is anything other than $j'i$ we have passed a direction for which some line separates a member of F from two other members of F. Therefore the switch sequence has the property that after a separating switch ij' the next one must be $j'i$. Note that this implies that p_0 must be a transversal term. And after the next two separating switches we have a new transversal term and a new geometric permutation. In a half-period of a simple allowable interval sequence there are precisely $2\binom{n}{2}$ separating switches, and if we assume there is no line separating a member of F from two others, F must then have $\binom{n}{2}$ geometric permutations. But it is well known that the maximum number of geometric permutations is $2(n-1)$ (see [5] for the case of convex sets and [10] for a generalization to simple interval sequences). It follows that $n < 5$. □

Corollary 9. *Let F be a family of n pairwise disjoint compact connected sets in the plane, each pair provided with four pairwise tangent pseudolines and a separating pseudoline, all of these pseudolines together forming an arrangement. If $n \geq 5$, then there is a pseudoline that separates a member of F from two other members of F and which is compatible with the given arrangement.*

Proof. As shown in [10], such a family of sets, together with the given pseudoline arrangement, can also be encoded by an allowable interval sequence. Since the proof above uses only the combinatorial properties of the sequence, and since the bound of $2n-2$ is shown in [10] to hold for families of connected sets with pseudolinear tangents, it applies as well in this more general case. Hence the result follows as in the proof of Theorem 8. □

A question of Kalai

Let F be a family of pairwise disjoint convex sets and let $F' = \{X \mid X = \mathrm{conv}(A \cup B)$ for $A, B \in F\}$. What can the interval sequence $P(F)$ say about F' ? G. Kalai [11] asked if it could distinguish between $\cap F'$ being empty or not, and here we answer this question in the negative.

Theorem 10. *There exist families F_1, F_2 of pairwise disjoint compact convex sets for which $P(F_1) = P(F_2)$, yet $\cap_{X \in F_1'} X = \emptyset$ and $\cap_{X \in F_2'} X \neq \emptyset$.*

Proof. The familes of sets shown in Figure 5 have this property. □

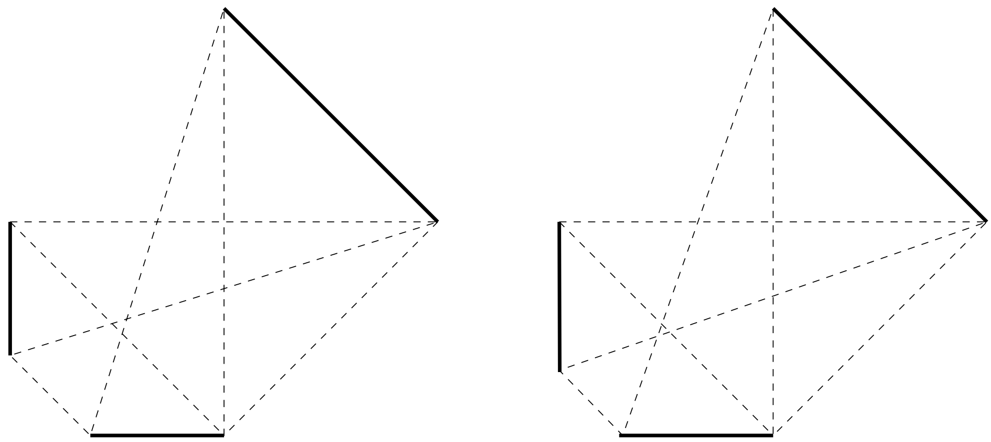

Figure 5: Two triples of convex sets with the same interval sequence

5 A Helly-type theorem for interval sequences

We now go back to our more general notion, as given in Section 2, of interval sequences. Our goal is to prove a Helly-type theorem for such sequences.

Since the Helly theorem we are going to prove is concerned with a point common to several sets, we can no longer assume that the sets we consider are pairwise disjoint. Nevertheless, the definition given at the beginning of Section 3 works just as well in this more general case.

Definition 11. *If F is a family of n compact convex sets in the plane and l is a directed line, orthogonal projection of the members of F onto l defines a family of n intervals. As l rotates counterclockwise, these intervals, hence their endpoints, give us a sequence of double permutations. (Actually a cyclically ordered set of double permutations — see Example 12 below), which we call the associated interval sequence $P(F)$ of F.*

Example 12. If $F = \{C_1, C_2\}$ with each C_i a polygon, say, $P(F)$ may have arbitrarily many terms in it (see Figure 6(a)). By choosing the convex sets C_i appropriately, we may even get infinitely many terms (see Figure 6(b)).

Definition 13. *We say that the (generalized) interval sequence P has a common point if every term of P is a transversal term, i.e., has the form $\pi_1, \ldots, \pi_n, \phi_1, \ldots, \phi_n$, where π is a permutation of $[n]$ and ϕ of $[n']$.*

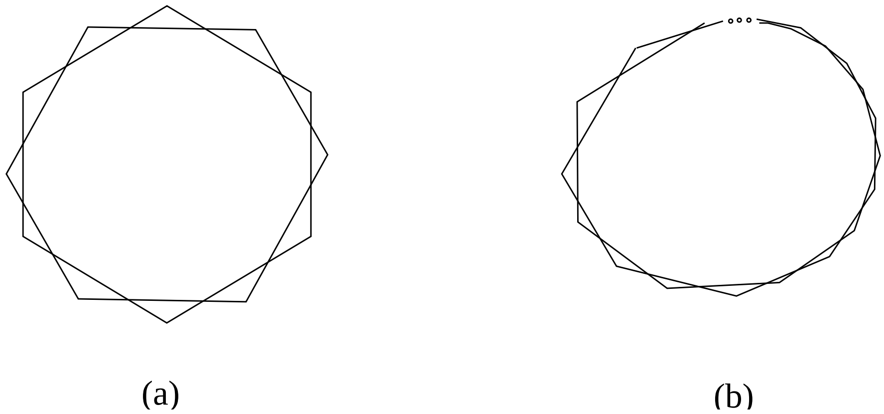

Figure 6: Many (resp. infinitely many) terms in $P(F)$

Theorem 14. *If every restriction $P_{i,j}$ of an interval sequence P to indices $i, j, i', j' \in [n] \cup [n']$ has a common point, then P has a common point.*

Proof. The hypothesis means that in every term of P, the indices $\{i, j\}$ precede $\{i', j'\}$ for every i, j. Hence no term can look like $\ldots i' \ldots j' \ldots$, since otherwise when restricted to $\{i, j, i', j'\}$ the same term would become $ii' j j'$, so that P contains a common point. □

Notice that Theorem 14 says that the combinatorial Helly number of our interval sequences is 2, as opposed to the *geometric* planar Helly number, which is 3. The point whose existence is guaranteed by Theorem 14 is only a *virtual* point, however, in a sense allied to that of [1], i.e., a "transversal in every direction."

6 Examples and computer experiments

In contrast to allowable sequences of (ordinary) permutations of $[n]$ — see, e.g., [9] — there is no known procedure for generating allowable interval sequences that is guaranteed not to "get stuck." (Of course we can then backtrack to an earlier stage and continue down another path, eventually obtaining all allowable interval sequences in this way.) In this section, we report on a few computer experiments on generating simple allowable interval sequences. Allowable sequences can be generated using a simple greedy algorithm: letting the initial permutation be labeled $1, \ldots n$, we repeatedly switch the leftmost pair of indices ij such that $j > i$.

The algorithm for generating allowable interval sequences, however, is a little more involved.

The algorithm discussed below was implemented in `Python`. The source code can be obtained at [13].

Given a double permutation P_0 (of $[n]$), we investigate how a simple allowable interval sequence containing P_0 can be found by applying a sequence of switches to P_0. A natural (greedy) method would be to make the leftmost switch possible. More precisely, we obtain P_i from P_{i-1} by making the leftmost switch in P_{i-1} such that in the associated switch sequence, the series of switches involving any two given indices $i, j \in [1..n]$ follow the order given in Remark 2 above.

As shown in Table 1, this algorithm does not always succeed in obtaining an allowable interval sequence containing P_0. The algorithm fails if every possible switch in some term P_l violates the order given in Remark 2. Note that if a sequence of length equal to the half period $(4\binom{n}{2})$ can be obtained then the complete allowable sequence can be obtained from this. Hence, if the algorithm does fail, it must do so before $4\binom{n}{2}$ terms are produced.

n	Total terms	Works	Fails
3	90	72	18
4	2520	1272	1248
5	113400	29280	84120

Table 1: The result of the greedy algorithm given above.

A few examples for $n = 3$ are shown below.

- An allowable sequence found by the greedy algorithm. A half period is shown:

  $\underline{12}31'2'3' \to 2\underline{13}1'2'3' \to \underline{23}11'2'3' \to 321\underline{1'2'}3' \to 32\underline{12'}1'3' \to 32\underline{2'1}1'3' \to 3212'\underline{1'3'} \to 321\underline{2'3'}1' \to 32\underline{13'}2'1' \to 3\underline{23'}12'1' \to 3\underline{3'2}12'1' \to 32\underline{3'1}2'1' \to 3213'\underline{2'1'}\ldots$

- The unique (up to renumbering) double permutations for which the algorithm fails to find a complete sequence: $1231'3'2'$, $1233'1'2'$, and $1233'2'1'$. (The other double permutations for which this happens can be obtained from one these three by renaming the sets.)

- An example where the greedy algorithm fails:

  $\underline{12}31'3'2' \to 2\underline{13}1'3'2' \to \underline{23}11'3'2' \to 321\underline{1'3'}2' \to 32\underline{13'}1'2' \to 3\underline{23'}11'2' \to$

$3\underline{3'2}11'2' \rightarrow \underline{32}3'11'2' \rightarrow 23\underline{3'1}1'2' \rightarrow 2313'\underline{1'2'} \rightarrow 23\underline{13'2'}1' \rightarrow\ ?$

> In a half period 4 switches involving each pair of indices are possible. While all 4 switches involving $\{1,3\}$ and $\{2,3\}$ occur in the above sequence, only two switches involving $\{1,2\}$ do so. The other two switches (in order) are $12'$ and $2'1$. Note that the first switch ($12'$) cannot be performed on the last term as 1 and $2'$ are separated by $3'$.

For $n = 3$, there are a total of 90 possible terms, 36 of which are transversal terms. Of these only 6 are unique up to renumbering. The greedy algorithm succeeds in finding allowable sequences for all nontransversal terms and exactly 18 of the transversal terms. Hence all the terms for which the greedy algorithm fails are transversal terms, and of these only three (given above) are unique up to renumbering.

We next show that the switch sequence obtained in the case where the greedy algorithm fails is not compatible with any initial double permutation, *i.e.,* the switches cannot be applied in that order to *any* double permutation and not just to $1231'3'2'$ as in the example given above. In fact, the largest number of switches (in the order given above) can be applied to the double permutation $1231'3'2'$. A similar result holds for the switch sequences obtained from the other two double permutations, $1233'1'2'$ and $1233'2'1'$, for which the greedy algorithm fails to produce an allowable interval sequence.

The following observation will prove useful.

Remark 15. If a switch of the form "ij", is applied to a double permutation P to obtain P', it follows that i must be the predecessor of j in P and j must be the predecessor of i in P'; similarly for the switches $i'j$, ij', and $i'j'$.

We show the relative order of the indices in initial double permutation P_0 imposed by the order of the switches, in Table 2.

We next show that the following modified version of the greedy algorithm does indeed succeed in generating allowable interval sequences for all double permutations.

> To obtain P_m, make the leftmost switch in P_{m-1} such that the restriction to $\{i, j, k, i', j', k'\}$ of P_0 and the switch sequence S is contained in some allowable interval sequence for every $1 \leq i < j < k \leq n$.

That this algorithm works follows from Theorem 1. An example of a sequence generated by this algorithm is given below. Only a half-period is shown.

Switch num.	1	2	3	4	5	6	7	8	9	10	11
Switch	12	13	23	$1'3'$	$13'$	$23'$	$3'2$	32	$3'1$	$1'2'$	$12'$
P_0	12	123	123	$1231'3'$	$1231'3'$	$1231'3'$	$1231'3'$	$1231'3'$	$1231'3'$	$1231'3'2'$	$--$

Table 2: The table is split into two for convenience. P_0 is the initial double permutation. The middle row contains the switch. In the column of switch number i, the bottom row shows the relative order of indices imposed by the switches 1 to i.

$$\begin{array}{l}
\underline{12}341'2'3'4' \to 2\underline{13}41'2'3'4' \to \underline{23}141'2'3'4' \to 32\underline{14}1'2'3'4' \to 3\underline{24}11'2'3'4' \to \\
\underline{34}211'2'3'4' \to 4321\underline{1'2'}3'4' \to 432\underline{12'}1'3'4' \to 432\underline{2'1}1'3'4' \to 43212'\underline{1'3'}4' \to \\
4321\underline{2'3'}1'4' \to 432\underline{13'}2'1'4' \to 43\underline{23'}12'1'4' \to 43\underline{3'2}12'1'4' \to 432\underline{3'1}2'1'4' \to \\
43213'2'\underline{1'4'} \to 43213'\underline{2'4'}1' \to 4321\underline{3'4'}2'1' \to 432\underline{14'}3'2'1' \to 43\underline{24'}13'2'1' \to \\
4\underline{34'}213'2'1' \to 4\underline{4'3}213'2'1' \to 43\underline{4'2}13'2'1' \to 432\underline{4'1}3'2'1' \to 43214'3'\underline{2'1'}\ldots
\end{array}$$

7 Conclusion

A number of open problems remain concerning interval sequences. Here we mention just five.

Problem 16. Theorem 8 solves a special case of the so-called $(1,k)$-separation problem posed by Tverberg [14]: How many pairwise disjoint compact convex sets in the plane are needed so that one can be separated by a line from k others? It should be possible to attack this using interval sequences, just as we have done for the case of $(1,2)$-separation.

Problem 17. Many convexity theorems concerning point sets in the plane have a natural statement in terms of interval sequences. Here is just one: If an index i is such that for every j,k,l there is a term in the interval sequence with i' preceding j, k, and l, then is it true that in some term of the sequence i' precedes *every* other index? (This would be nothing more than Carathéodory's theorem, generalized!)

Problem 18. Transversal results and questions of Gallai-type concerning convex sets can be rephrased in a natural way in terms of interval sequences, for example Eckhoff's theorem [4]: If every three compact convex sets in a family have a transversal, there exist four lines such that each set is met by at least one of them. This may be generalizable to interval sequences, hence to connected sets with pseudolinear pairwise tangents.

Problem 19. In [2], T. Bisztriczky and G. Fejes Tóth pose a generalization of the Erdős-Szekeres problem to families of convex sets in the plane, and conjecture that their problem has the same solution as the original. Since the interval sequence of a family of pairwise disjoint ovals has a simple structure, this question may be amenable to combinatorial investigation.

Problem 20. Finally, the representation question looms large as a major unsolved problem: Is there a natural geometric object that realizes every allowable interval sequence, possibly a family of connected sets equipped with pairwise tangent and separating pseudolines forming an arrangement? This would generalize the result [6, 3] that every allowable sequence of (ordinary) permutations of $[n]$ can be realized by a generalized configuration, i.e., n points joined pairwise by pseudolines forming an arrangement.

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