AN ASYMPTOTICALLY TIGHT BOUND ON THE NUMBER OF CONNECTED COMPONENTS OF REALIZABLE SIGN CONDITIONS

SAUGATA BASU, RICHARD POLLACK, AND MARIE-FRANÇOISE ROY

Abstract. In this paper we prove an asymptotically tight bound (asymptotic with respect to the number of polynomials for fixed degrees and number of variables) on the number of connected components of the realizations of all realizable sign conditions of a family of real polynomials. More precisely, we prove that the number of connected components of the realizations of all realizable sign conditions of a family of s polynomials in $\mathrm{R}[X_1, \ldots, X_k]$ whose degrees are at most d, is bounded by

$$\frac{(2d)^k}{k!} s^k + O(s^{k-1}).$$

This improves the best upper bound known previously, which was

$$\frac{1}{2}\frac{(8d)^k}{k!} s^k + O(s^{k-1}).$$

The new bound matches asymptotically the lower bound obtained for families of polynomials each of which is a product of generic polynomials of degree one.

1. Introduction

Let K be a field and $\mathcal{P} \subset \mathrm{K}[X_1, \ldots, X_k]$ be a finite family of polynomials with $\#\mathcal{P} = s$. Then, $\mathcal{P}$ naturally induces a partition of K^k into constructible subsets, such that over each subset belonging to the partition, each polynomial $P \in \mathcal{P}$ either vanishes at every point of the subset or is non-zero at every point of the subset and each such set is maximal with respect to this property. If moreover K is an ordered field, we can consider a refined partition, such that each $P \in \mathcal{P}$ maintains its *sign* over each element of the partition. Notice that the number of sets in the partition can be at most 2^s in the unordered and 3^s in the ordered case. However, much tighter bounds are known in both cases [2, 13]. In fact, in case the field K is $\mathbb{R}$ or $\mathbb{C}$, the number of connected components of the partition of K^k induced by $\mathcal{P}$ is bounded by a polynomial function of s of degree k.

In this paper, we prove asymptotically tight bounds on the number of connected components of sets in the partitions described above. Our bounds are asymptotic in s, with the number of variables and the degrees of the polynomials in $\mathcal{P}$ considered fixed. Asymptotics with respect to the number of polynomials (with the degrees and

Key words and phrases. Betti numbers, Semi-algebraic sets.

Supported in part by an NSF Career Award 0133597 and a Alfred P. Sloan Foundation Fellowship.

Supported in part by NSA grant MDA904-01-1-0057 and NSF grants CCR-9732101 and CCR-0098246.

2000 MATHEMATICS SUBJECT CLASSIFICATION 14P10, 14P25

the number of variables considered fixed) is considered important in applications in discrete and computational geometry (see [10]).

The proofs of all previous results proving that the number of connected components of the partition of K^k induced by $\mathcal{P}$ is bounded by a polynomial function of s of degree k (see [16, 1, 2, 3]), used a common technique of replacing the given family of polynomials, $\mathcal{P}$ by another family $\mathcal{P}^\star$ obtained by infinitesimal perturbations of polynomials in $\mathcal{P}$. The partition induced by the family $\mathcal{P}^\star$ is closely related to that of $\mathcal{P}$, and at the same time the family $\mathcal{P}^\star$ has useful properties which makes it easier to bound the topological complexity of the partition induced by it. However, the cardinality of $\mathcal{P}^\star$ is larger (often by a factor of 4) than that of $\mathcal{P}$, and this fact introduces an extra factor of 4^k in the upper bound. In this paper we use a new technique (see Section 4 below) that avoids replacing $\mathcal{P}$ by a larger family of polynomials and thus we are able to obtain a tight asymptotic bound.

2. Preliminaries

We begin with a few definitions.

Let R be a real closed field and C its algebraic closure. For $x \in \mathrm{R}$, we define

$$\mathrm{sign}(x) = \begin{cases} 0 & \text{if and only if } \ x = 0, \\ 1 & \text{if and only if } \ x > 0, \\ -1 & \text{if and only if } \ x < 0. \end{cases}$$

Similarly for $x \in \mathrm{C}$, we define

$$\mathrm{zero}(x) = \begin{cases} 0 & \text{if and only if } \ x = 0, \\ 1 & \text{if and only if } \ x \neq 0. \end{cases}$$

Let $\mathcal{P}$ be a finite subset of $\mathrm{R}[X_1, \ldots, X_k]$. A *sign condition* (resp. *zero pattern*) on $\mathcal{P}$ is an element of $\{0, 1, -1\}^{\mathcal{P}}$ (resp. $\{0, 1\}^{\mathcal{P}}$).

Given $\mathcal{P}$, we fix $\Omega > 0$ to be a sufficiently large element of R, and denote by

$$B_k(0, \Omega) = \{x \in \mathrm{R}^k \mid |x|^2 \leq \Omega\}.$$

In our arguments we will often consider subsets of $\mathrm{C}^k \cong \mathrm{R}^{2k}$ and we will denote the unit ball in C^k also by $B_{2k}(0, \Omega)$, that is

$$B_{2k}(0, \Omega) = \{x \in \mathrm{C}^k \mid |x|^2 \leq \Omega\}.$$

We will restrict our attention to realizations of sign conditions (resp. zero patterns) inside the ball $B_k(0, \Omega)$ (resp. $B_{2k}(0, \Omega)$).

The *realization* of the sign condition σ, is the semi-algebraic set

$$\mathcal{R}(\sigma, \mathrm{R}^k) = \{x \in \mathrm{R}^k \mid \bigwedge_{P \in \mathcal{P}} \mathrm{sign}(P(x)) = \sigma(P)\} \cap B_k(0, \Omega).$$

We call a connected component C of $\mathcal{R}(\sigma, \mathrm{R}^k)$ to be bounded if $C \cap \partial B_k(0, \Omega) = \emptyset$ and unbounded otherwise.

The *realization* of the zero pattern ρ, is the set

$$\mathcal{R}(\rho, \mathrm{C}^k) = \{x \in \mathrm{C}^k \mid \bigwedge_{P \in \mathcal{P}} \mathrm{zero}(P(x)) = \rho(P)\} \cap B_{2k}(0, \Omega).$$

It is a consequence of Hardt's triviality theorem that the homeomorphism type of $\mathcal{R}(\sigma, \mathrm{R}^k)$ (resp. $\mathcal{R}(\rho, \mathrm{C}^k)$) is constant for all sufficiently large $\Omega > 0$.

We write the set of zeros of $\mathcal{P}$ in R^k (resp. in C^k) as $\mathrm{Z}(\mathcal{P},\mathrm{R}^k) = \{x \in \mathrm{R}^k \mid \bigwedge_{P\in\mathcal{P}} P(x) = 0\}$ (resp. $\mathrm{Z}(\mathcal{P},\mathrm{C}^k) = \{x \in \mathrm{C}^k \mid \bigwedge_{P\in\mathcal{P}} P(x) = 0\}$). We let (resp.) the set

$$\mathrm{Sign}(\mathcal{P}) = \{\sigma \in \{0,1,-1\}^{\mathcal{P}} \mid \mathcal{R}(\sigma,\mathrm{R}^k) \neq \emptyset\},$$

and

$$\text{Zero-pattern}(\mathcal{P}) = \{\rho \in \{0,1\}^{\mathcal{P}} \mid \mathcal{R}(\rho,\mathrm{C}^k) \neq \emptyset\}.$$

For $\sigma \in \{0,1,-1\}^{\mathcal{P}}$ we will denote by $b_i^{BM}(\sigma,\mathrm{R}^k)$ the dimension of $H_i^{BM}(\mathcal{R}(\sigma,\mathrm{R}^k))$, the i-th Borel-Moore homology group of the locally closed set $\mathcal{R}(\sigma,\mathrm{R}^k)$ with coefficients in $\mathbb{Z}_2$. Similarly, we will denote by $b_i(\sigma,\mathrm{R}^k)$ the dimension of $H_i(\mathcal{R}(\sigma,\mathrm{R}^k))$, the i-th singular homology group of $\mathcal{R}(\sigma,\mathrm{R}^k)$ with coefficients in $\mathbb{Z}_2$. We will denote by

$$b^{BM}(\sigma,\mathrm{R}^k) = \sum_{i\geq 0} b_i^{BM}(\sigma,\mathrm{R}^k).$$

For $\rho \in \{0,1\}^{\mathcal{P}}$ we will denote by $b_i^{BM}(\rho,\mathrm{C}^k)$ the dimension of $H_i^{BM}(\mathcal{R}(\rho,\mathrm{C}^k))$, the i-th Borel-Moore homology group of the locally closed set $\mathcal{R}(\rho,\mathrm{C}^k)$ with coefficients in $\mathbb{Z}_2$. We will denote by

$$b^{BM}(\rho,\mathrm{C}^k) = \sum_{i\geq 0} b_i^{BM}(\rho,\mathrm{C}^k).$$

In general for any locally closed semi-algebraic set X, we denote by $b_i^{BM}(X)$ to be the dimension of $H_i^{BM}(X)$ with $\mathbb{Z}_2$-coefficients, and we will denote

$$b^{BM}(X) = \sum_{i\geq 0} b_i^{BM}(X).$$

It will be useful to have following notation.

$$b_{i,\mathrm{R}}(s,d,k) = \max_{\mathcal{P}\subset \mathrm{R}[X_1,\ldots,X_k], \#\mathcal{P}=s, \deg(P)\leq d, P\in\mathcal{P}} b_i(\mathcal{P},\mathrm{R}^k),$$

where

$$b_i(\mathcal{P},\mathrm{R}^k) = \sum_{\sigma\in\{0,1,-1\}^{\mathcal{P}}} b_i(\sigma,\mathrm{R}^k).$$

In some applications, for instance in bounding the number of combinatorial types of polytopes or order types of point configurations (see [6, 7]), one is only interested in the number of realizable sign conditions or zero patterns and not in any topological properties of their realizations. In these situations it is useful to obtain bounds on,

$$N_{\mathrm{R}}(s,d,k) = \max_{\mathcal{P}\subset \mathrm{R}[X_1,\ldots,X_k], \#\mathcal{P}=s, \deg(P)\leq d, P\in\mathcal{P}} \#\mathrm{Sign}(\mathcal{P}),$$

as well as

$$N_{\mathrm{C}}(s,d,k) = \max_{\mathcal{P}\subset \mathrm{R}[X_1,\ldots,X_k], \#\mathcal{P}=s, \deg(P)\leq d, P\in\mathcal{P}} \#\text{Zero-pattern}(\mathcal{P}).$$

Note that, $N_{\mathrm{R}}(s,d,k)$ is the maximum number of sign conditions realized in R^k by a set of polynomials of size s and degrees bounded by d.

3. Known Bounds

3.1. **Zero patterns.** The problem of bounding the number of realizable zero patterns of a sequence of s polynomials in k variables of degree at most d over an arbitary field was considered by Ronyai et al. in [13], where they prove

$$N_{\mathrm{C}}(s,d,k) \leq \binom{sd+k}{k} \leq \left(\frac{d^k}{k!}\right) s^k + O(s^{k-1}) \tag{3.1}$$

using linear algebra arguments. However, their method is not useful for the problem of bounding the number of realizable sign conditions over an ordered field.

3.2. **Sign conditions.** We now consider the case of sign conditions. When $d = 1$, we have $N_{\mathrm{R}}(s,1,k) = b_{0,\mathrm{R}}(s,1,k)$, since the realization of each realizable sign condition in this case is a convex polyhedron, which is clearly connected. It is also easy in this case to deduce an exact expression for $N_{\mathrm{R}}(s,1,k) = b_{0,\mathrm{R}}(s,1,k)$, namely,

$$\begin{aligned} N_{\mathrm{R}}(s,1,k) &= b_{0,\mathrm{R}}(s,1,k) \\ &= \sum_{i=0}^{k}\sum_{j=0}^{k-i}\binom{s}{i}\binom{s-i}{j} \\ &= \left(\sum_{i=0}^{k}\frac{1}{i!(k-i)!}\right) s^k + O(s^{k-1}) \qquad (3.2) \\ &= \frac{2^k}{k!} s^k + O(s^{k-1}). \qquad (3.3) \end{aligned}$$

This is the number of cells of all dimensions in an arrangement of s hyperplanes in general position in R^k. For $0 \ll k \ll s$ we have,

$$N_{\mathrm{R}}(s,1,k) = b_{0,\mathrm{R}}(s,1,k) \sim \left(\frac{2es}{k}\right)^k,$$

using Stirling's approsimation for $k!$.

For $d > 1$, Alon [1] proved a bound of

$$\left(\frac{8esd}{k}\right)^k$$

on $N_{\mathrm{R}}(s,d,k)$. Previously, Warren [16] had proved a bound of

$$\left(\frac{4esd}{k}\right)^k$$

on the number of realizable strict sign conditions (that is sign conditions σ such that $\sigma(P) \neq 0$ for all $P \in \mathcal{P}$).

The following bounds on the sums of the individual Betti numbers of the realizations of all realizable sign conditions restricted to a real variety of real dimension k' also defined by polynomials of degree at most d was proved in [3].

$$\begin{aligned} b_{i,\mathrm{R}}(s,d,k,k') &\leq \sum_{j=0}^{k'-i} \binom{s}{j} 4^j d(2d-1)^{k-1} \\ &= \left(\frac{2^{2k'+k-2i-1}d^k}{(k'-i)!} \right) s^{k'-i} + O(s^{k'-i-1}), \end{aligned} \tag{3.4}$$

(where the last parameter k' denotes the dimension of the ambient variety).

Note that, $b_{0,\mathrm{R}}(\mathcal{P})$ is the total number of semi-algebraically connected components of the realizations of all realizable sign conditions of $\mathcal{P}$ and the above result gives an asymptotic bound of

$$\frac{1}{2}\left(\frac{8esd}{k}\right)^k$$

(which is asymptotically same as that proved by Alon in [1]) on the number of connected components of the realizations of all realizable sign conditions which is a priori larger than just the number of realizable sign conditions. This distinction is important since in many applications, for instance in bounding the number of isotopy classes of point configuations, it is the number of connected components which is important. Also, note that in the case $d = 1$, the upper bound in (3.2) is actually realized by families of hyperplanes in general position and hence, the expression in (3.2) also provides a lower bound on $N_{\mathrm{R}}(s, 1, k) = b_{0,\mathrm{R}}(s, 1, k)$.

In this paper we consider the problem of proving an asymptotic upper bound on $b_{0,\mathrm{R}}(s, d, k)$ (that is the number of connected components of all realizable sign conditions of a family of s polynomials in $\mathrm{R}[X_1, \ldots, X_k]$ of degrees bounded by d), for fixed d, k and large s. It follows from (3.4), that $b_{0,\mathrm{R}}(s, d, k)$ is bounded from above by a polynomial in s of degree k, namely,

$$b_{0,\mathrm{R}}(s,d,k) \leq \left(\frac{2^{3k-1}d^k}{k!}\right) s^k + O(s^{k-1}).$$

The leading coefficient of the above bound is $\sim \left(\frac{8ed}{k}\right)^k$.

On the other hand by taking s polynomials each a product of d generic linear polynomials, we have by (3.2) a lower bound of

$$\begin{aligned} b_{0,\mathrm{R}}(s,d,k) &\geq \sum_{i=0}^{k}\sum_{j=0}^{k-i} \binom{sd}{i}\binom{sd-i}{j} \\ &= \left(\sum_{i=0}^{k} \frac{d^k}{i!(k-i)!}\right) s^k + \Theta(s^{k-1}) \\ &= \frac{(2d)^k}{k!} s^k + \Theta(s^{k-1}). \end{aligned}$$

The leading coefficient of the lower bound is,

$$\frac{(2d)^k}{k!} \sim \left(\frac{2ed}{k}\right)^k.$$

(In the above bounds we provide the asymptotic estimates on the right for aid in comparison.)

Clearly, the upper and lower estimates differ by a factor of 4^k. As mentioned in the introduction, this gap is due to the fact that the technique used in proving the upper bound (3.4) involves replacing the given family of polynomials by another family obtained by infinitesimal perturbations of the original polynomials. The realizations of a certain subset of *strict* sign conditions of this new family are then shown to be in one to one correspondence with the realizations of all realizable sign conditions of the original family. Moreover, the corresponding realizations have the same homotopy type. However, the cardinality of the new family is four times that of the original family, and this fact introduces an extra factor of 4^k in the upper bound.

4. Brief Summary of our method

In this paper we use a new technique for proving an upper bound on $b_{0,\mathrm{R}}(s,d,k)$ that avoids replacing the given family of polynomials by a larger family of polynomials. The new idea is to consider the *zero patterns* of the given family of polynomials over C^k, where we are able to use degree theory of complex varieties and certain generalized Bezout inequalities, as well as inequalities derived from the Mayer-Vietoris exact sequence, to directly obtain an asymptotically tight bound on the sum of the Borel-Moore Betti numbers of the realizations of all realizable zero patterns of the original family of polynomials in C^k. This in turn provides an asymptotically tight bound on the sum of the Borel-Moore Betti numbers of the realizations of all realizable sign conditions of the same family in R^k using Smith inequalities (see Proposition 6.5 below). Note that even though we are interested in bounding the number of connected components of sign conditions over R^k, in order to use Smith inequality we need to bound the higher Betti numbers of the zero patterns in C^k.

The reason for using Borel-Moore homology instead of singular homology is that Borel-Moore homology has an useful additivity property (see Proposition 6.3 below) not satisfied by singular homology. This property plays an important role in our proofs. However, in order to obtain bounds on the number of connected components of the realizations of sign conditions, we need to relate the Borel-Moore Betti numbers of the realizations of sign conditions, with the number of connected components. We prove that the number of bounded connected components of any fixed sign conditions is bounded by the sum of the Borel-Moore Betti numbers of the realization of the sign condition (see Lemma 7.9 below). This allows us to bound the number of bounded connected components in terms of the Borel-Moore Betti numbers. We bound the number of unbounded components separately using (3.4).

5. Main Results

We obtain the following bound on the number of connected components of the realizations of all realizable sign conditions of a family of polynomial.

Theorem 5.1.

$$\begin{aligned} b_{0,\mathrm{R}}(s,d,k) &\leq \sum_{0\leq \ell\leq k}\left(\binom{s}{k-\ell}\binom{s}{\ell}d^k+\sum_{1\leq i\leq \ell}\binom{s}{k-\ell}\binom{s}{\ell-i}d^{O(k^2)}\right)\\ &= \sum_{\ell=0}^{k}\binom{s}{k-\ell}\binom{s}{\ell}d^k+O(s^{k-1}) \end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{\ell=0}^{k} \frac{d^k}{\ell!(k-\ell)!}\right) s^k + O(s^{k-1}) \\
&= \frac{(2d)^k}{k!} s^k + O(s^{k-1}).
\end{aligned}
$$

For $0 < d, k \ll s$, this gives

$$b_{0,\mathrm{R}}(s,d,k) \sim \left(\frac{2esd}{k}\right)^k.$$

This matches asymptotically the lower bound obtained by taking s polynomials each of which is a product of d linear polynomials in general position.

In the process of proving Theorem 5.1, we slightly improve the bound on $N_{\mathrm{C}}(s,d,k)$ proved by Ronyai et al. [13] mentioned above.

Denoting by $N_{\mathrm{C},\ell}(s,d,k)$ the maximum number of realizable zero patterns whose realizations are of (complex) dimension ℓ, we have

Theorem 5.2.

$$N_{\mathrm{C},\ell}(s,d,k) \le \binom{s}{k-\ell} d^{k-\ell}.$$

This yields immediately the bound,

$$N_{\mathrm{C}}(s,d,k) \le \sum_{0 \le \ell \le k} \binom{s}{k-\ell} d^{k-\ell}. \tag{5.1}$$

Notice that,

$$\sum_{0 \le \ell \le k} \binom{s}{k-\ell} d^{k-\ell} \le \binom{sd+k}{k}$$

for all values of $s, d, k \ge 0$, and thus the bound in (5.1) is slightly better than that in (3.1).

Remark 5.3. We would like to point out that it is tempting to try to prove Theorem 5.1 by first proving the bounds for sufficiently generic polynomials $\mathcal{P} \subset \mathrm{R}^k$, and then proving that generic families of polynomials actually represents the worst case. While it is indeed easy to prove the bound in Theorem 5.1 for generic families, it is not at all clear how to prove the second statement.

In some special case, for instance, for families of linear polynomials, one can prove that infinitesimal perturbations of the input polynomials can only increase the number of cells in the arrangement (see for instance [10]). Hence, for the problem of bounding the number of cells in an arrangement of hyperplanes it suffices to consider hyperplanes in general position. But for polynomials of degree greater that one, there is no guarantee that an arbitrary infinitesimal perturbation of any given family of polynomials will not increase the Betti numbers of the realizations of the realizable sign conditions.

The rest of the paper is organized as follows. In Section 6, we state some results we will need for the proof of the main theorems with appropriate references. In Section 7 we prove the main results of the paper.

6. Mathematical Background

In this section we state without proof some well known results regarding the topology of complex affine varieties which we will need in the proof of the main theorem. We also point the reader to appropriate references for the proofs of these results.

6.1. **Homotopy type of complex affine varieties.** Let $V \subset \mathrm{C}^k$ be an affine variety of complex dimension ℓ. If we consider V as a real semi-algebraic set in R^{2k}, then its real dimension is 2ℓ. However, following theorem due to Karchyauskas [9], tells us that V has the homotopy type of a CW-complex of (real) dimension ℓ. A proof which uses stratified Morse theory can be found in [8] (Section 5.1, page 198).

Theorem 6.1. *Let $V_1, \ldots, V_n$ be affine sub-varieties of C^k and $V = V_1 \cup \cdots \cdot V_n$. Let $\ell = \dim_{\mathrm{C}} V = \max_{1 \leq i \leq n} \dim_{\mathrm{C}} V_i$. Then, V is homotopy equivalent to a CW-complex of (real) dimension ℓ and each V_i is homotopy equivalent to a sub-complex of dimension $\dim_{\mathrm{C}} V_i$. In particular, $H_i(V) = 0$ for $i > \ell$.*

6.2. **Generalized Bezout Inequality.** In our proof we will need to use the following generalized form of Bezout's theorem (see for instance Example 8.4.6 (page 148) in [5]).

Recall that the degree of an irreducible complex projective variety $V \subset \mathbb{P}^k_{\mathrm{C}}$ of complex dimension ℓ, denoted $\deg(V)$, is the number of points in the interesction $V \cap L$, where L is a generic linear subspace $\mathbb{P}^k_{\mathrm{C}}$ of dimension $k - \ell$.

If $V = \cup_{1 \leq i \leq n} V_i$ is a union of irreducible varieties each of dimension ℓ, then we define $\deg(V) = \sum_{1 \leq i \leq n} V_i$.

Theorem 6.2. *Let $V_1, \ldots, V_s$ be complex projective varieties, such that each V_i is a union of irreducible varieties of the same dimension, and $Z_1, \ldots, Z_t$ be the irreducible components of the intersection $V_1 \cap \cdots \cap V_s$. Then,*

$$\sum_{1 \leq i \leq t} \deg(Z_i) \leq \prod_{1 \leq i \leq s} \deg(V_i).$$

Note that there is no assumption on the dimensions of the Z_i's or those of V_i's.

6.3. **Exact Sequence for Borel-Moore Homology.** We will assume that the reader is familiar with the definition of Borel-Moore homology groups for locally closed semi-algebraic sets (see for instance [4], Definition 11.7.13). Let $W \subset C^k$ be a closed set and $A \subset W$ a closed subset of W. Then the following exact sequence relates the Borel-Moore homology groups of W, A and $W \setminus A$ (see Proposition 11.7.15 in [4]).

$$\cdots \rightarrow H_p^{BM}(A) \rightarrow H_p^{BM}(W) \rightarrow H_p^{BM}(W \setminus A) \rightarrow H_{p-1}^{BM}(A) \rightarrow \cdots$$

It follows that,

Proposition 6.3. For each $p > 0$,

$$b_p^{BM}(W \setminus A) \leq b_{p-1}^{BM}(A) + b_p^{BM}(W).$$

In particular,

$$\sum_{p \geq 0} b_p^{BM}(W \setminus A) \leq \sum_{p \geq 0} b_p^{BM}(A) + \sum_{p \geq 0} b_p^{BM}(W).$$

Now suppose that $A = \cup_{1\le i\le n} A_i$, where each A_i are closed and bounded subsets of C^k. The following inequality is a consequence of the Mayer-Vietoris exact sequence for simplicial homology groups, noting that the Borel-Moore homology are isomorphic to the simplicial homology groups for closed and bounded sets.

Proposition 6.4. For $p \ge 0$,

$$b_p^{BM}(A) \le \sum_{0\le i\le p} \sum_{1\le\alpha_0<\cdots<\alpha_i\le n} b_{p-i}^{BM}(A_{\alpha_0}\cap\cdots\cap A_{\alpha_i}).$$

6.4. **Smith Inequalities.** Let X be any compact topological space and $c : X \to X$ an involution. Let $F \subset X$ denote the set of fixed points of c, and let X' denote the quotient space $X/(x = cx)$. The projection map $X \to X'$ maps F homeomorphically onto a subset of X' (which we also denote by F). The following exact sequence called the Smith exact sequence is well known (see for instance [15], page 131).

$$\cdots \to H_p(X', F)\oplus H_p(F) \to H_p(X) \to H_p(X', F) \to H_{p-1}(X', F)\oplus H_{p-1}(F) \to \cdots$$

(here as elsewhere in this paper all homology groups are taken with coefficients in $\mathbb{Z}_2$). As an immediate consequence we have that for any $q \ge 0$,

$$\sum_{i\ge q} b_i(F) \le \sum_{i\ge q} b_i(X).$$

We are going to apply Smith theory in the case where $X \subset \mathrm{C}^k$ is the intersection with $B_{2k}(0,\Omega)$ of the basic constructible subset, defined by a formula,

$$P_1 = \cdots P_\ell = 0, P_{\ell+1} \ne 0, \ldots, P_s \ne 0, \tag{6.1}$$

where $P_i \in \mathrm{R}[X_1, \ldots, X_k]$. Let c denote the complex conjugation and $F \subset X$ its set of fixed points. Clearly, $F \subset B_k(0,\Omega)$ is defined by the same formula as in (6.1).

The Borel-Moore Betti numbers of X (as well as F) are by definition the ordinary simplicial homology groups of the compact pairs $(\overline{X}, \overline{X}\setminus X)$ (resp. $(\overline{F}, \overline{F}\setminus F)$) where $\overline{X}$ (resp. $\overline{F}$) denotes the closure (in the Euclidean topology) of X (respectively F). It is easy to see that the action of conjugation c extends to $\overline{X}$ and the Smith exact sequence yields the following inequality.

Proposition 6.5. For any $q \ge 0$,

$$\sum_{i\ge q} b_i^{BM}(F) \le \sum_{i\ge q} b_i^{BM}(X).$$

6.5. **Oleinik-Petrovsky-Thom-Milnor Inequalities.** We will also need the following inequality proved separately by Oleinik and Petrovsky [12], Thom [14] and Milnor [11], bounding the sum of the Betti numbers of a real algebraic set. In particular, it also gives a bound on the sum of the Betti numbers of a complex algebraic set considered as a real algebraic set in an affine space twice the dimension of the complex one. Moreover, since Borel-Moore homology agrees with singular homology for compact sets, we have

Proposition 6.6. Let $\mathcal{Q} \subset \mathrm{R}[X_1, \ldots, X_k]$, $\deg(Q) \le d, Q \in \mathcal{Q}$. Let $V_{\mathrm{R}} = \mathrm{Z}(\mathcal{Q}, \mathrm{R}^k) \cap B_k(0,\Omega)$ and $V_{\mathrm{C}} = \mathrm{Z}(\mathcal{Q}, \mathrm{C}^k) \cap B_{2k}(0,\Omega)$. Then,

(1)

$$\sum_{0\le i\le k} b_i(V_{\mathrm{R}}) = \sum_{0\le i\le k} b_i^{BM}(V_{\mathrm{R}}) \le d(2d-1)^{k-1},$$

(2)
$$\sum_{0 \le i \le k} b_i(V_{\mathrm{C}}) = \sum_{0 \le i \le k} b_i^{BM}(V_{\mathrm{C}}) \le d(2d-1)^{2k-1}.$$

7. Proofs of the Main Results

As mentioned before, we first prove a bound on the sum of Borel-Moore Betti numbers of the realizations of all realizable zero patterns of a given family of polynomials over C^k. We will then apply this result in the real case via the Smith inequality. The first part of this section is devoted to the proof of the following proposition.

Proposition 7.1. Let $\mathcal{P} \subset \mathrm{C}[X_1, \ldots, X_k]$, be a family of polynomials, with $\#\mathcal{P} = s$ and $\deg(P) \le d$ for all $P \in \mathcal{P}$. Then,

$$\begin{aligned}\sum_{\rho \in \text{Zero-pattern}(\mathcal{P})} b^{BM}(\rho, \mathrm{C}^k) &\le \sum_{0 \le \ell \le k} \binom{s}{k-\ell}\binom{s}{\ell} d^k + \sum_{1 \le i \le \ell} \binom{s}{k-\ell}\binom{s}{\ell-i} d^{O(k^2)} \\ &= \sum_{0 \le \ell \le k} \binom{s}{k-\ell}\binom{s}{\ell} d^k + O(s^{k-1}).\end{aligned}$$

For $\rho \in \text{Zero-pattern}(\mathcal{P})$, let $\mathcal{P}_\rho = \{P \in \mathcal{P} \mid \rho(P) = 0\}$ and let $V_\rho = \mathrm{Z}(\mathcal{P}_\rho, \mathrm{C}^k)$. Let $d_\rho = \dim_{\mathrm{C}} \mathcal{R}(\rho, \mathrm{C}^k)$. Let $V_\rho = V_{\rho,1} \cup \cdots \cup V_{\rho,n(\rho)}$ denote the decomposition of V_ρ into irreducible components.

We first prove a bound on the number of isolated points occuring as realizations of zero patterns.

Lemma 7.2.
$$\#\{(\rho, i) \mid \dim_{\mathrm{C}} V_{\rho,i} = 0\} \le \binom{s}{k} d^k.$$

Proof. Let $\mathcal{P} = \{P_1, \ldots, P_s\}$ (the implicit ordering of the polynomials in $\mathcal{P}$ will play a role in the proof).

First consider a sequence of k polynomials of $\mathcal{P}$, $P_{i_1}, \ldots, P_{i_k}$, with $1 \le i_1 \le \cdots i_k \le s$. Let $\{W_{\alpha_1}\}_{\alpha_1=1,2..}$ denote the irreducible components of $\mathrm{Z}(P_{i_1}, \mathrm{C}^k)$. Similarly, let $\{W_{\alpha_1,\alpha_2}\}_{\alpha_2=1,2...}$ denote the irreducible components of $W_{\alpha_1} \cap \mathrm{Z}(P_{i_2}, \mathrm{C}^k)$ and so on. We prove by induction that for $1 \le \ell \le k$,

(7.1)
$$\sum_{\alpha_1,\alpha_2,\ldots,\alpha_\ell} \deg(W_{\alpha_1,\ldots,\alpha_\ell}) \le d^\ell.$$

The claim is trivially true for $\ell = 1$. Suppose it is true upto $\ell - 1$. Then using Theorem 6.2 we get that

$$\sum_{\alpha_1,\ldots,\alpha_{\ell-1}} \deg(W_{\alpha_1,\ldots,\alpha_{\ell-1}} \cap \mathrm{Z}(P_\ell, \mathrm{C}^k)) \le d^{\ell-1} \cdot d = d^\ell$$

Now let $\dim_{\mathrm{C}} V_{\rho,i} = 0$ and we denote the corresponding point in C^k by p.

We first claim that, there must exist $P_{i_1}, \ldots, P_{i_k} \in \mathcal{P}_\rho$, with $1 \le i_1 < i_2 < \cdots < i_k \le s$, and irreducible affine algebraic varieties $V_0, V_1, \ldots, V_k$ satisfying:

1. $\mathrm{C}^k = V_0 \supset V_1 \supset V_2 \supset \cdots \supset V_k = \{p\}$, and
2. for each j, $1 \le j \le k$, V_j is an irreducible component of $V_{j-1} \cap \mathrm{Z}(P_{i_j}, \mathrm{C}^k)$ which contains p.

Note that in the sequence, $V_0, V_1, \ldots, V_k$, $\dim_{\mathrm{C}}(V_i)$ is necessarily equal to $k-i$. To prove the claim, let

$$\mathcal{P}_\rho = \{P_{j_1}, \ldots, P_{j_m}\},$$

with $1 \leq j_1 < j_2 < \cdots < j_m \leq s$. Let $W_0 = \mathrm{C}^k$ and for each $h, 1 \leq h \leq m$ define inductively W_h to be an irreducible component of $W_{h-1} \cap \mathrm{Z}(P_{j_h}, \mathrm{C}^k)$ containing the point p. By definition, $\mathrm{C}^k = W_0 \supset W_1 \supset \cdots \supset W_m = \{p\}$ and each W_h is irreducible. Hence, there must exist precisely k indices, $1 \leq \alpha_1 < \cdots < \alpha_k \leq m$ such that $\dim_{\mathrm{C}} W_{\alpha_i} = \dim_{\mathrm{C}} W_{\alpha_{i-1}} - 1$. If $\beta \notin \{\alpha_1, \ldots, \alpha_k\}$, then $W_\beta = W_{\beta-1}$. Now let $i_1 = j_{\alpha_1}, \ldots, i_k = j_{\alpha_k}$, and $V_j = W_{i_j}$ for $0 \leq j \leq k$. It is clear from construction that the sequence $V_0, \ldots, V_k$ satisfies the properties stated above, thus proving the claim.

The number of sequences of polynomials of length k, $P_{i_1}, \ldots, P_{i_k}$, with $1 \leq i_1 < \cdots < i_k \leq s$, is $\binom{s}{k}$. The lemma now follows from the bound in (7.1). □

The following lemma is a generalization of the above lemma to positive dimensional realizations of zero patterns.

Lemma 7.3. *For* $0 \leq \ell \leq k$,

$$\sum_{\rho \in \text{Zero-pattern}(\mathcal{P})} \sum_{1 \leq i \leq n(\rho), \dim_{\mathrm{C}} V_{\rho,i} = \ell} \deg V_{\rho,i} \leq \binom{s}{k-\ell} d^{k-\ell}.$$

Proof. In order to bound the degree of $V_{\rho,i}$ with $\dim_{\mathrm{C}} V_{\rho,i} = \ell$, it suffices to count the isolated zeros of the intersection of $V_{\rho,i}$ with a generic affine subspace L of dimension $k-\ell$. Since there are only a finite number of $V_{\rho,i}$'s to consider we can assume that,

(1) L is generic for all of them,
(2) the sets $V_{\rho,i} \cap L$ are pairwise disjoint,
(3) for each $V_{\rho,i}$ with $\dim_{\mathrm{C}} V_{\rho,i} = \ell$, $L \cap V_{\rho,i} \cap \mathrm{Z}(P, C^k) = \emptyset$ for each $P \in \mathcal{P} \setminus \mathcal{P}_\rho$.

Now restrict to the subspace L and apply Lemma 7.2, noting that an isolated point of $V_{\rho,i} \cap L$ is an isolated point of the family $\mathcal{P}$ restricted to L. □

For each $\rho \in$ Zero-pattern($\mathcal{P}$), let $W_{\rho,1}, \ldots, W_{\rho,m(\rho)}$ be the irreducible components of V_ρ of dimension d_ρ having the property that no polynomial in $\mathcal{P} \setminus \mathcal{P}_\rho$ vanishes identically on on any of them. This implies that each $W_{\rho,j} \cap \mathcal{R}(\rho, \mathrm{C}^k)$ is non-empty and of complex dimension d_ρ. (Thus, the union of the sets $W_{\rho,j} \cap \mathcal{R}(\rho, \mathrm{C}^k)$ is the full dimensional part of the constructible set $\mathcal{R}(\rho, \mathrm{C}^k)$.) Let $W_\rho = \cup_{1 \leq j \leq m(\rho)} W_{\rho,j}$. We first note that, $b_\ell^{BM}(W_\rho)$ can be bounded in terms of d and k independent of s as follows. The exact nature of the dependence of this bound on d and k is not important for the asymptotic result that we prove in this paper. However, we are able to show the following.

Lemma 7.4. $\sum_i b_i^{BM}(W_\rho \cap B_{2k}(0, \Omega)) \leq d^{O(k^2)}$.

Proof. We first claim that $\deg(W_\rho) \leq d^{O(k)}$. To see this first observe that $\deg(W_\rho) \leq \deg(V'_\rho)$ where $V'_\rho = \cup_{1 \leq i \leq n(\rho), \dim_{\mathrm{C}} V_{\rho,i} = d_\rho} V_{\rho,i}$. Now, $\deg(V'_\rho)$ is the number of isolated points in the intersection of V_ρ with a generic affine subspace, $L \subset \mathrm{C}^k$, of dimension $k - d_\rho$. Now, V_ρ is an affine algebraic set in C^k defined by polynomials of degree at most d. Identifying C^k with R^{2k}, and separating real and imaginary parts, we have that the intersection $L \cap V_\rho \subset \mathrm{R}^{2(k-d_\rho)}$, is a real algebraic set defined by polynomials of degree at most d. Applying Proposition 6.6 we have that, the number of isolated points of $L \cap V_\rho$ is at most $b_0(L \cap V_\rho) \leq d(2d-1)^{2k-1}$. Hence, $\deg(W_\rho) \leq d^{O(k)}$.

Since W_ρ is an affine variety of degree at most $d^{O(k)}$, it is defined by at most $k+1$ polynomials each of degree at most $d^{O(k)}$.

Now using Proposition 6.6, we obtain that

$$\sum_i b_i^{BM}(W_\rho \cap B_{2k}(0,\Omega)) \leq (d^k)^{O(k)} = d^{O(k^2)}.$$

□

If $\mathcal{Q}$ is any subset of $\mathcal{P}$ an easy extension of the above argument also yields,

Lemma 7.5. $\sum_i b_i^{BM}(W_\rho \cap \mathrm{Z}(\mathcal{Q},\mathrm{C}^k) \cap B_{2k}(0,\Omega)) \leq d^{O(k^2)}$.

We remark here that the bounds in 7.4 and 7.5 are unlikely to be tight and improving them might lead to bounds which are optimal not just asymptotically, but for all values of s, d and k.

The following lemma is key to the proof Theorem 5.1.

Lemma 7.6. *Let $\ell > 0$ and $\rho \in$ Zero-pattern$(\mathcal{P})$ with $d_\rho = \ell$. Then,*

(1)

$$\begin{aligned} b_\ell^{BM}(\rho,\mathrm{C}^k) &\leq \sum_{1\leq i\leq m(\rho)} \deg(W_{\rho,i})\binom{s}{\ell}d^\ell + \sum_{1\leq i\leq \ell}\binom{s}{\ell-i}d^{O(k^2)} \\ &= \sum_{1\leq i\leq m(\rho)} \deg(W_{\rho,i})\binom{s}{\ell}d^\ell + O(s^{\ell-1}). \end{aligned}$$

(2) *For $j < \ell$,*

$$b_j^{BM}(\rho,\mathrm{C}^k) = \sum_{1\leq i\leq \ell}\binom{s}{\ell-i}d^{O(k^2)} = O(s^{\ell-1}).$$

Proof. Let $\rho \in$ Zero-pattern$(\mathcal{P})$ with $d_\rho = \ell$, and let $A_\rho = (W_\rho \cap B_{2k}(0,\Omega)) \setminus \mathcal{R}(\rho,\mathrm{C}^k)$. Clearly,

$$A_\rho = (W_\rho \cap B_{2k}(0,\Omega)) \cap (\cup_{P\in\mathcal{P}\setminus\mathcal{P}_\rho}\mathrm{Z}(P,\mathrm{C}^k)).$$

Now, $b_\ell^{BM}(\rho,\mathrm{C}^k) = b_\ell^{BM}(W_\rho \cap \mathcal{R}(\rho,\mathrm{C}^k))$ by Theorem 6.1, since $W_\rho \cap \mathcal{R}(\rho,\mathrm{C}^k)$ is the top dimensional part of $\mathcal{R}(\rho,\mathrm{C}^k)$.

Moreover, from Proposition 6.3 it follows that

$$b_\ell^{BM}((W_\rho \cap B_{2k}(0,\Omega)) \setminus A_\rho) \leq b_\ell^{BM}(W_\rho \cap B_{2k}(0,\Omega)) + b_{\ell-1}^{BM}(A_\rho).$$

Now observe that by Lemma 7.4,

$$b_\ell^{BM}(W_\rho \cap B_{2k}(0,\Omega)) \leq d^{O(k^2)}.$$

We now bound the second term $b_{\ell-1}^{BM}(A_\rho)$ in terms of s as follows.

Expressing A_ρ as the union $\cup_{P\in\mathcal{P}\setminus\mathcal{P}_\rho}\mathrm{Z}(P,\mathrm{C}^k) \cap W_\rho \cap B_{2k}(0,\Omega)$, we obtain using Mayer-Vietoris inequalities for Borel-Moore homology groups (Proposition 6.4),

$$\begin{aligned} b_{\ell-1}(A_\rho) \leq & \sum_{\mathcal{Q}\subset\mathcal{P}\setminus\mathcal{P}_\rho,\#\mathcal{Q}=\ell} b_0(W_\rho \cap \mathrm{Z}(\mathcal{Q},\mathrm{C}^k) \cap B_{2k}(0,\Omega)) \\ &+ \sum_{1\leq i\leq \ell}\ \sum_{\mathcal{Q}\subset\mathcal{P}\setminus\mathcal{P}_\rho,\#\mathcal{Q}=\ell-i} b_i(W_\rho \cap \mathrm{Z}(\mathcal{Q},\mathrm{C}^k) \cap B_{2k}(0,\Omega)). \end{aligned}$$

Now using the generalized Bezout inequality (Theorem 6.2) and noting that for any complex affine algebraic set $X \subset \mathrm{C}^k$, $b_0(X) \leq \deg(X)$, we have that

$$b_0(W_\rho \cap \mathrm{Z}(\mathcal{Q}, \mathrm{C}^k)) = b_0(W_\rho \cap \mathrm{Z}(\mathcal{Q}, \mathrm{C}^k) \cap B_{2k}(0,\Omega)) \leq \deg(W_\rho) d^\ell,$$

since each polynomial in $\mathcal{Q}$ has degree at most d. Thus,

$$\sum_{\mathcal{Q} \subset \mathcal{P} \setminus \mathcal{P}_\rho, \#\mathcal{Q} = \ell} b_0(W_\rho \cap \mathrm{Z}(\mathcal{Q}, \mathrm{C}^k) \cap B_{2k}(0,\Omega)) \leq \deg(W_\rho) \binom{s}{\ell} d^\ell.$$

Also, using Lemma 7.5 we have that the second part of the sum,

$$\sum_{1 \leq i \leq \ell} \sum_{\mathcal{Q} \subset \mathcal{P} \setminus \mathcal{P}_\rho, \#\mathcal{Q} = \ell - i} b_i(W_\rho \cap \mathrm{Z}(\mathcal{Q}, \mathrm{C}^k) \cap B_{2k}(0,\Omega)) \leq \sum_{1 \leq i \leq \ell} \binom{s}{\ell - i} d^{O(k^2)}.$$

□

Lemma 7.7. *Let $0 < \ell \leq k$. Then,*

$$\begin{aligned}
\sum_{\rho \in \textit{Zero-pattern}(\mathcal{P}), d_\rho = \ell} b^{BM}(\rho, \mathrm{C}^k) &\leq \binom{s}{k-\ell}\binom{s}{\ell} d^k + \sum_{1 \leq i \leq \ell} \binom{s}{k-\ell}\binom{s}{\ell - i} d^{O(k^2)} \\
&= \binom{s}{k-\ell}\binom{s}{\ell} d^k + O(s^{k-1}).
\end{aligned}$$

Proof. Follows from Lemmas 7.6 and 7.3. □

Proof of Proposition 7.1. Follows directly from from Lemma 7.7, since $0 \leq d_\rho \leq k$, for each $\rho \in$ Zero-pattern($\mathcal{P}$). □

We now consider the real case directly. We first prove a few preliminary lemmas.

Lemma 7.8. *Let $C \subset \mathrm{R}^k$ be a bounded connected component having real dimension ℓ of the basic semi-algebraic set defined by,*

$$\begin{aligned}
P_1 = \ldots = P_m &= 0, \\
P_{m+1} > 0, \ldots, P_n &> 0, \\
P_{n+1} < 0, \ldots, P_s &< 0.
\end{aligned}$$

Let $h : \Delta \to C$ be a semi-algebraic triangulation of C and σ an $(\ell - 1)$-simplex of the triangulation. Then the number of ℓ-simplices η in Δ such that σ is a face of η is even.

Proof. Consider the real algebraic set $V \subset \mathrm{R}^{k+s-m}$ defined by,

$$\begin{aligned}
P_1 = \cdots = P_m &= 0, \\
T_{m+1}^2 P_{m+1} - 1 = \cdots = T_n^2 P_n - 1 &= 0, \\
T_{n+1}^2 P_n + 1 = \cdots = T_s^2 P_s + 1 &= 0,
\end{aligned}$$

where $T_{m+1}, \ldots, T_s$ are new variables. Let $\pi : \mathrm{R}^{k+s-m} \to \mathrm{R}^k$ be the projection map which forgets the new co-ordinates. Since C is bounded, $\pi^{-1}(C) \cap V$ is the disjoint union of two connected components, C_1, C_2 of V, each homemomorphic to C. Moreover, any triangulation of C can be lifted to a triangulation of $C_1 \cup C_2$.

Now, C_1 is a connected component of a real algebraic set of dimension ℓ. Thus, it suffices to prove the lemma in case C is a connected component of a real algebraic set. We now refer the reader to the proof of Theorem 11.1.1 in [4], where the same claim is proved in case C is a real algebraic set of dimension ℓ. However, the proof which is of a local character also applies in case C is only a connected component of a real algebraic set. □

Lemma 7.9. *Let $C \subset \mathrm{R}^k$ be a bounded connected components of a basic semi-algebraic set, $S \subset \mathrm{R}^k$, defined by $P_1 = \cdots = P_m = 0, P_{m+1} > 0, \ldots, P_n > 0, P_{n+1} < 0, \ldots, P_s < 0$, such that the real dimension of C is ℓ. Then, for all sufficiently large $\Omega > 0$, $b_\ell^{BM}(C \cap B_k(0,\Omega)) \geq 1$.*

Proof. Let $S \subset \mathrm{R}^k$ be the set defined by $P_1 = \cdots = P_m = 0, P_{m+1} > 0, \ldots, P_n > 0, P_{n+1} < 0, \ldots, P_s < 0$ and let $T = S \cap B_k(0,\Omega)$. For all sufficiently large $\Omega > 0$, each bounded connected component of S, is also a connected component of T. Since T is bounded, from the definition of Borel-Moore homology groups we have that,

$$H_i^{BM}(T) \cong H_i(\overline{T}, \overline{T} \setminus T).$$

Let $h : \Delta \to \overline{T}$ be a semi-algebraic triangulation of the pair $(\overline{T}, \overline{T} \setminus T)$.

Now by Lemma 7.8, and each $(\ell-1)$-simplex σ in Δ with image in C, σ is a face of an even number ℓ-simplices with image in C. Moreover, if σ is a face of an ℓ-simplex η whose image is in C and $h(\sigma) \not\subset C$, then $h(\sigma) \subset \overline{T} \setminus T$. It now follows that there exists a non-empty set of ℓ-simplices with image contained in C, having the property that each $(\ell-1)$-face of a simplex in the family is either contained in an even number of simplices of the family or has its image in $\overline{T} \setminus T$. Let $Z \in C_\ell(\Delta)$ denote the sum of these simplices. Clearly, Z represents a non-zero $\mathbb{Z}_2$-homology class in $H_\ell^{BM}(C)$. Thus,

$$b_\ell^{BM}(C) \geq 1.$$

□

Let $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$. For $\mathcal{Q} \subset \mathcal{P}$, let

$$\Sigma_{\mathcal{Q}} = \{\sigma \in \{0,1,-1\}^{\mathcal{P}} \mid \sigma(P) = 0 \ \forall P \in \mathcal{Q},\ \sigma(P) \neq 0 \ \forall P \in \mathcal{P} \setminus \mathcal{Q}\},$$

and let $\rho_{\mathcal{Q}} \in \{0,1\}^{\mathcal{P}}$ be defined by,

$$\begin{aligned} \rho_{\mathcal{Q}}(P) &= 0,\ \forall P \in \mathcal{Q}, \\ &= 1,\ \forall P \in \mathcal{P} \setminus \mathcal{Q}. \end{aligned}$$

Also, let

$$W_{\mathcal{Q},\mathrm{R}} = \cup_{\sigma \in \Sigma_{\mathcal{Q}}} \mathcal{R}(\sigma, \mathrm{R}^k).$$

Lemma 7.10. *If C is a non-empty connected component of $\mathcal{R}(\sigma, \mathrm{R}^k)$ for some $\sigma \in \Sigma_{\mathcal{Q}}$, and $i : C \hookrightarrow W_{\mathcal{Q},\mathrm{R}}$ the inclusion map, then $i_* : H_i^{BM}(C) \to H_i^{BM}(W_{\mathcal{Q},\mathrm{R}})$ is an injection if $i > 0$ and in case $i = 0$, i_* is injective if C is closed. In particular, $i_* : H_0^{BM}(C) \to H_0^{BM}(W_{\mathcal{Q},\mathrm{R}})$ is injective if C is zero dimensional.*

Proof. Follows directly from the definition of Borel-Moore homology groups. □

Proof of Theorem 5.1. We follow the notations introduced above. Let $\mathcal{Q} \subset \mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$.

Notice that $W_{\mathcal{Q},\mathrm{R}}$ is the fixed point of set of the action of complex conjugation on $\mathcal{R}(\rho_{\mathcal{Q}}, \mathrm{C}^k)$. It follows from Proposition 6.5 that,

$$b^{BM}(W_{\mathcal{Q},\mathrm{R}}) \leq b^{BM}(\rho_{\mathcal{Q}}, \mathrm{C}^k)). \tag{7.2}$$

Applying Proposition 7.1 we get that,

$$\begin{aligned}\sum_{\rho\in\text{Zero-pattern}(\mathcal{P})} b^{BM}(\rho,\mathrm{C}^k) &\leq \sum_{0\leq\ell\leq k}\binom{s}{k-\ell}\binom{s}{\ell}d^k+\sum_{1\leq i\leq \ell}\binom{s}{k-\ell}\binom{s}{\ell-i}d^{O(k^2)}\\ &= \sum_{0\leq\ell\leq k}\binom{s}{k-\ell}\binom{s}{\ell}d^k+O(s^{k-1}).\end{aligned}$$

$$\begin{aligned}\sum_{\mathcal{Q}\subset\mathcal{P}} b^{BM}(W_{\mathcal{Q},\mathrm{R}}) &\leq \sum_{\mathcal{Q}\subset\mathcal{P}} b^{BM}(\rho_{\mathcal{Q}},\mathrm{C}^k)\\ &= \sum_{\rho\in\text{Zero-pattern}(\mathcal{P})} b^{BM}(\rho,\mathrm{C}^k)\\ &\leq \sum_{0\leq\ell\leq k}\binom{s}{k-\ell}\binom{s}{\ell}d^k+\sum_{1\leq i\leq \ell}\binom{s}{k-\ell}\binom{s}{\ell-i}d^{O(k^2)}\\ &\leq \sum_{0\leq\ell\leq k}\binom{s}{k-\ell}\binom{s}{\ell}d^k+O(s^{k-1}). \qquad (7.3)\end{aligned}$$

We now use (7.3) to bound the number of bounded connected components of the realizable sign conditions of $\mathcal{P}$. It follows from Lemma 7.9, that if C is a bounded connected component of $\mathcal{R}(\sigma,\mathrm{R}^k)$ for $\sigma\in\mathrm{Sign}(\mathcal{P})$, then $b_\ell^{BM}(C)\geq 1$.

It now follows from Lemma 7.10 and (7.3) that the total number of bounded connected components of $\mathcal{R}(\sigma,\mathrm{R}^k)$ over all $\sigma\in\mathrm{Sign}(\mathcal{P})$, is bounded by,

$$\begin{aligned}\sum_{\mathcal{Q}\subset\mathcal{P}} b^{BM}(W_{\mathcal{Q},\mathrm{R}}) &\leq \sum_{0\leq\ell\leq k}\binom{s}{k-\ell}\binom{s}{\ell}d^k+\sum_{1\leq i\leq \ell}\binom{s}{k-\ell}\binom{s}{\ell-i}d^{O(k^2)}\\ &\leq \sum_{0\leq\ell\leq k}\binom{s}{k-\ell}\binom{s}{\ell}d^k+O(s^{k-1})\\ &= \left(\sum_{\ell=0}^{k}\frac{d^k}{\ell!(k-\ell)!}\right)s^k+O(s^{k-1})\\ &= \frac{(2d)^k}{k!}s^k+O(s^{k-1}).\end{aligned}$$

In order to bound the number of unbounded components, consider the realizations of sign conditions of the family $\mathcal{P}$ restricted to the $(k-1)$-dimensional real variety $\partial B_k(0,\Omega)$. The number of unbounded connected components of the realizable sign conditions of $\mathcal{P}$ over R^k, is bounded by the number of connected components of the realizations of sign conditions of $\mathcal{P}$ restricted to the variety $\partial B_k(0,\Omega)$. It follows from the bound in (3.4) that the number of connected components of the realizations of sign conditions of $\mathcal{P}$ restricted to the variety $\partial B_k(0,\Omega)$ is bounded by $O(s^{k-1})$. Thus, the number of unbounded connected components of the realizable sign conditions of $\mathcal{P}$ over R^k is also bounded by $O(s^{k-1})$. This proves the theorem. $\square$

Proof of Theorem 5.2. Let $\rho\in\text{Zero-pattern}(\mathcal{P})$ with $d_\rho=\ell$. Then there exists an irreducible component, $V_{\rho,i}$ of V_ρ, with $\dim_{\mathrm{C}} V_{\rho,i}=\ell$, and such that $\dim_{\mathrm{C}}(V_{\rho,i}\cap\mathcal{R}(\rho,\mathrm{C}^k))$ is also equal to ℓ. Moreover, for any $\rho'\in\text{Zero-pattern}(\mathcal{P})$, with $\rho'\neq\rho$, we must have that $\dim_{\mathrm{C}}(V_{\rho,i} \cap \mathcal{R}(\rho', \mathrm{C}^k)) < \ell$, since $\mathcal{R}(\rho, \mathrm{C}^k) \cap \mathcal{R}(\rho', \mathrm{C}^k) = \emptyset$ and $\dim_{\mathrm{C}}(V_{\rho,i} \setminus \mathcal{R}(\rho, \mathrm{C}^k))$ is clearly $< \ell$.

Thus, if we charge ρ to $V_{\rho,i}$, it is clear that $V_{\rho,i}$ cannot be charged by any zero pattern other than ρ. It follows from Lemma 7.3 the number of distinct $V_{\rho,i}$ of dimension ℓ is bounded by $\binom{s}{k-\ell} d^{k-\ell}$. □

References

[1] N. Alon Tools from higher algebra, Handbook of Combinatorics, Eds. R. Graham, M. Grotschel, L. Lovasz, MIT Press (1996).

[2] S. Basu, R. Pollack, M.-F. Roy On the number of cells defined by a family of polynomials on a variety, *Mathematika*, Vol 43, 120-126, 1996.

[3] S. Basu, R. Pollack, M.-F. Roy On the Betti Numbers of Sign Conditions, *Proc. Amer. Math. Soc.* 133 (2005), 965-974.

[4] J. Bochnak, M. Coste, M.-F. Roy *Real algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Bd. 36, Berlin : Springer-Verlag (1998).

[5] W. Fulton *Intersection Theory*, Second Edition, Springer-Verlag, (1997).

[6] J. E. Goodman, R. Pollack There are asymptotically far fewer polytopes than we thought, *Bull. Amer. Math. Soc.* 14, 127-129, (1986).

[7] J.E. Goodman, R. Pollack Upper Bounds for Configurations and Polytopes in R^d, *Discrete and Computational Geometry*, 1, 219-227, 1986.

[8] M. Goresky, R. MacPherson *Stratified Morse Theory*, Springer-Verlag (1987).

[9] K. K. Karchyauskas Homotopy properties of complex algebraic sets. Studies in Topology, Steklov Mathematical Institute (Leningrad), 1979.

[10] J. Matousek *Lectures on Discrete Geometry*, Springer-Verlag (2002).

[11] J. Milnor On the Betti numbers of real varieties, *Proc. AMS* 15, 275-280, (1964).

[12] O. A. Oleinik, I. B. Petrovskii On the topology of real algebraic surfaces, *Izv. Akad. Nauk SSSR* 13, 389-402, (1949).

[13] L. Ronyai, L. Babai, M.K. Ganapathy On the number of zero patterns of a sequence of polynomials, *Journal of the AMS* 14, 3:717-735 (2001).

[14] R. Thom Sur l'homologie des varietes algebriques reelles, Differential and Combinatorial Topology, Ed. S.S. Cairns, Princeton Univ. Press, 255-265, (1965).

[15] O.Ya. Viro, D.B. Fuchs Homology and Cohomology, Topology II, S.P. Novikov, V.A. Rokhlin (Eds) Springer-Verlag, 2004.

[16] Hugh E. Warren Lower Bounds for Approximation by Nonlinear Manifolds, *Transactions of the A.M.S.*, 133 167-178 (1968).

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, U.S.A.
E-mail address: saugata@math.gatech.edu

Courant Institute of Mathematical Sciences, New York University, New York, NY 10012, U.S.A.
E-mail address: pollack@cims.nyu.edu

IRMAR (URA CNRS 305), Université de Rennes, Campus de Beaulieu 35042 Rennes cedex FRANCE.
E-mail address: marie-francoise.roy@univ-rennes1.fr