Riemannian curvature measures

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If M is a compact submanifold of euclidean space, then the volume of a small tubular neighborhood is a polynomial in the radius r. As quantities depending on M, the coefficients of this polynomial can be viewed as a natural extension of the intrinsic volumes of convex bodies in $\mathbb{R}^n$ and are, by a famous theorem of Weyl, expressible as integral invariants of the curvature tensor of M.

It is natural to interpret this phenomenon in terms of curvature measures and smooth valuations, in the sense of Alesker, canonically associated to the riemannian structure of M. We achieve this by localizing the individual summands of the tube coefficients. We study the behavior of the elements of the resulting space $\mathcal{R}(M)$ of riemannian curvature measures under isometric immersions and show that the Lipschitz-Killing curvatures arise as the unique elements invariant under all immersions. We then give an explicit description of the action of the Lipschitz-Killing algebra $\mathcal{LK}(M)$ on $\mathcal{R}(M)$ with respect to Alesker multiplication.

If M admits a group of isometries acting transitively on the sphere bundle SM, then Alesker multiplication on $\mathcal{V}(M)^G, \mathcal{C}(M)^G$, the spaces of invariant valuations and curvature measures, is intimately related to the array of kinematic formulas on (M, G). Since $\mathcal{LK}(M) \subset \mathcal{V}(M)^G, \mathcal{R}(M) \subset \mathcal{C}(M)^G$, the action of $\mathcal{LK}(M)$ on $\mathcal{R}(M)$ represents a universal piece of any such array. We illustrate this principle in precise terms in the case where M is a complex space form.