Conformal Mapping. Find new flow given a known flow.

Suppose \( w(z) \) is the complex potential of a flow in some part of the \( z \)-plane.

If \( z = f(\zeta) \), \( f = \zeta + i\eta \), \( f \) analytic, then \( w \) is also an analytic function of \( \zeta \).

\[
\frac{w(\zeta)}{f(\zeta)} = \frac{w(z)}{z}
\]

Thus, \( w(f(\zeta)) \) is the complex potential of some region of the \( \zeta \)-plane.

The flow in the \( z \)-plane is said to be folded into flow in the \( \zeta \)-plane.

Equi-potential lines \& streamlines in \( z \)-plane remain the same in \( \zeta \), \( \eta \) except at singular points of the \( f \).

Velocity components:

\[
\begin{align*}
\eta &= -i\eta \\
&= \frac{dw}{ds} = \frac{dw}{dz} \frac{dz}{ds}
\end{align*}
\]

If \( w(z) = \frac{m-i\eta}{2\pi} \log(z-z_0) \) near \( z = z_0 \)

then \( w(\zeta) = \frac{m-i\eta}{2\pi} \log(\zeta-\zeta_0) \) near \( \zeta = \zeta_0 \)

(similarly with modification for other flow singularities).
An elliptic cylinder in translational motion.
(Batchelor, p. 427)

Conformal th. of the exterior of an ellipse, in the $z$-plane, to the exterior of a circle in the $s$-plane:

(1) \[ Z = s + \frac{x^2}{s} \]

\[ \Rightarrow x = s \left(1 + \frac{x^2}{s^2}\right), \quad y = s \left(1 - \frac{x^2}{s^2}\right) \]

If $|s| = c$ (circle of radius $c$ centered at origin)

\[ \frac{x^2}{c^2(1 + \frac{x^2}{c^2})^2} + \frac{y^2}{c^2(1 - \frac{x^2}{c^2})^2} = 1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} \]

\[ c = \frac{1}{2}(a+b), \quad \lambda = \frac{1}{2}(a^2 - b^2)^{1/2}, \quad a > b \]

\[ \frac{b}{a} = \frac{1/2 \left(1/(\lambda/c)^2\right)}{1 + (\lambda/c)^2}, \quad \text{choice of } \lambda/c \text{ determines aspect ratio.} \]
Fixed $x$
\[
\frac{b}{a} \to 1 \text{ as } c \to \infty
\]
circles
\[
\delta \to 0 \text{ as } c \to a, \ b \to 0, \text{ the "flat" plate}
\]

Inverse $T$s:
\[
\psi = \frac{1}{2} z + \frac{i}{2} (z^2 - 4x^2)^{1/2}
\]

branch cut at $y = 0$. Take branch which gives positive value of $(z^2 - 4x^2)^{1/2}$ for $x > 2a, y = 0$

Other branch maps $z$-plane outside

$z$ ellipse into interior of $|\psi| = c$

Check this.

$T$s (1) & (2) have been used in more complicated

transfom $z$ circles to more complicated

shapes.
Flow in the $S$-plane:

A circular cylinder of radius $c$ held in a stream of uniform velocity $(-U, -V)$ at infinity, with circulation $\Gamma$.

\[
W(S) = -\frac{(U-iV)S - (U+iV)\overline{S}}{\overline{S}} + \frac{\Gamma}{\pi i} \ln \frac{S}{\overline{S}}
\]

the circle through

Eqs (3) + (1) gives (parametrically) the flow around the elliptic cylinder with velocity $(-U, -V)$ at $\infty$, with circulation $\Gamma$.

Let $S = \Gamma e^{i\alpha}$, $U+iV = \Gamma 1 \frac{e^{-i\alpha}}{2}$

\[
x = \Gamma \left( 1 + \frac{\alpha^2}{\Gamma^2} \right) \cos \gamma, \quad y = \Gamma \left( 1 - \frac{\alpha^2}{\Gamma^2} \right) \sin \gamma
\]

Eq (3) becomes

\[
W(S) = -(U^2+V^2)^{1/2} \left\{ \Gamma e^{i(\alpha+\gamma)} + \frac{C^2}{\alpha} e^{-i(\alpha+\gamma)} \right\} + \frac{\Gamma}{2\pi i} \ln \left( \frac{\Gamma}{c} + i\alpha \right)
\]
\[ \phi = -\left( U^2 + V^2 \right)^{\frac{1}{2}} \left( \sigma + \frac{c^2}{\rho} \right) \cos (\nu + \alpha) \quad \Psi + \frac{\pi}{2n} \nu \]

\[ \Psi = -\left( U^2 + V^2 \right)^{\frac{1}{2}} \left( \sigma - \frac{c^2}{\rho} \right) \sin (\nu + \alpha) - \frac{\pi}{2n} \log \frac{\nu}{c} \]

If \( \Psi = k \) along a streamline in the \( \theta \)-plane

\[ k = \epsilon R(\nu, \sigma), \] locally \( \nu = \text{Diff}(\nu; k) \)

In the \( z \)-plane

\[ (x = (\nu, \nu), \quad y(\nu, \nu)) \] traces the streamline.

**Ex:** If \( (\nu, \nu) \neq 0 \), i.e. only circulatory flows, \( \Gamma \neq 0 \).

In \( \theta \)-plane, streamlines are circles.

Streamlines: \( \nu = k \)

\[ x = \theta K \left( 1 + \frac{a^2}{k^2} \right) \cos \nu \]

\[ y = K \left( 1 - \frac{a^2}{k^2} \right) \sin \nu \]

Circulation \( \Gamma \) of the ellipse

\[ C = \oint \frac{d\nu}{ds} ds = \oint \frac{d\nu}{ds} ds = \Gamma. \]
\( T = 0 \)

\( s - \text{plane} \)

\( z - \text{plane}, \ c > A \)

\( c = 2 \) The flat plate.
width \( \frac{1}{2} \)
Velocity on the cylinder surface

\[ U - i \, \bar{U} = \left( \frac{d\gamma}{ds} \bigg/ \frac{d\bar{z}}{ds} \right) \]

\[ \bar{U} = c \]

\[ = -i \left( U^2 + V^2 \right)^{1/2} \left( a + b \right) \sin(V + \alpha) - i \frac{T}{2\pi} \]

\[ i \left( a \sin V + b \cos V \right) \]

\[ W = 0 : \quad V = -\alpha, \quad V = \pi - \alpha \quad \text{stagnation points} \]

\[ T \neq 0 \quad \text{As before, the stagnation points shift, with positions on the elliptic cylinder calculable.} \]

\[ b = 0, \ c = \alpha \quad \text{The flat plate} \]

\[ \frac{\alpha}{2} = \frac{\alpha}{2} \]

\[ U - i \, \bar{U} = -i \left( U^2 + V^2 \right)^{1/2} 2 \alpha \sin(V + \alpha) - i \frac{T}{2\pi} \]

\[ 2i \alpha \sin V \quad \text{Note: } T = 0 \]

Generally an infinite plate velocity for

\[ V = 0, \quad \pi \]

A flow around a tip or sharp corner generally produces a divergence.
However, note that $T$ can be chosen so as to move the stagnation point to the ends, making the flow smooth at the "trailing edge" $V = \pi$.

Reg. that $u$ is finite:

$$u = -\left(U^2 + V^2\right)^{1/2} \frac{a \sin (V + x) - \pi/2}{\pi}, \quad V = \pi.$$ 

$$u' = \left(U^2 + V^2\right)^{1/2} \frac{a \sin V}{\pi}$$

If we choose

$$T = 2\pi a \left(U^2 + V^2\right)^{1/2} \sin \alpha > 0$$

we have cancelling zeros:

$$u = -\frac{a}{\pi} \left(U^2 + V^2\right)^{1/2} \left[ \frac{\sin (V + x) + \sin \alpha}{\sin V} \right]$$

$$\lim_{V \to \pi} \frac{\sin (V + x) + \sin \alpha}{\sin V} = \lim_{V \to \pi} \frac{\cos (V + x)}{\cos \pi} = \frac{\cos (x + \pi)}{\cos \pi} = \cos x$$

$$u \to -\left(U^2 + V^2\right)^{1/2} \cos \alpha$$
The point: Asking that the flow depart smoothly at the trailing edge generates a circulation, and hence lift. Called the **Toukowskii condition**.

In practice: Viscosity acting in the thin layer generates the circulation around the "airfoil" necessary to remove the velocity singularity at the trailing edge, and to thereafter ignore the effects of viscosity. Called **Toukowskii's Hypothesis**.

Justified (at least qualitatively) by boundary layer theory.