Homework 1.

Each worked problem has equal weight.

1. Consider the two-dimensional unidirectional flows

$$\mathbf{u}(x,y) = h(y)(1,0) \text{ and } \mathbf{u}(r,\theta) = f(r)(-\sin\theta, \cos\theta)$$

Show that both are incompressible, and calculate their associated symmetric rate-of-strain tensors $\mathbf{E}$, and rotation tensors $\mathbf{W}$. Interpret the deformation of the flowing fluid in terms of these tensors (i.e. the principal direction and eigenvectors of $\mathbf{E}$, and the vorticity ω).

2. Given a velocity field $\mathbf{u}(\mathbf{x} \in \mathbb{R}^{\mathbf{n}}, t) \in \mathbb{R}^n$, the Lagrangian flow map is defined by

$$\frac{\partial \mathbf{X}}{\partial t}(\boldsymbol{\alpha}, t) = \mathbf{u}(\mathbf{X}(\boldsymbol{\alpha}, t), t) \text{ with } \mathbf{X}(\boldsymbol{\alpha}, 0) = \boldsymbol{\alpha}.$$

The *deformation tensor* is defined as $(\mathbf{F})_{ij} = \frac{\partial \mathbf{X}_i}{\partial \boldsymbol{\alpha}_j}$.

(**i**) Show that

$$\frac{\partial \mathbf{F}}{\partial t} = \nabla_x \mathbf{u}|_{\mathbf{x}=\mathbf{X}(\boldsymbol{\alpha},t)} \mathbf{F}$$

(**ii**) Consider the so-called *Finger* tensor $\mathbf{A} = \mathbf{F}\mathbf{F}^T$. Show that $\mathbf{A} \geq \mathbf{0}$, i.e. $\mathbf{A}$ is a symmetric, positive semi-definite tensor. Show that in the Eulerian frame:

$$\frac{D\mathbf{A}}{Dt} - (\nabla \mathbf{u} \cdot \mathbf{A} + \mathbf{A} \cdot \nabla \mathbf{u}^T) = \mathbf{0}$$

The operator $\mathbf{A}^\nabla = \frac{D\mathbf{A}}{Dt} - (\nabla \mathbf{u} \cdot \mathbf{A} + \mathbf{A} \cdot \nabla \mathbf{u}^T)$ is called the *upper convected derivative*, and arises in the modeling of elastic solids. What is the analogous expression for the SPSD tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$?

(**iii**) Consider a tensor $\mathbf{B}$ satisfying $\mathbf{B}^\nabla = 0$. Prove that if $\mathbf{B}(\mathbf{x}, t = 0) \geq \mathbf{0}$ then $\mathbf{B}(\mathbf{x}, t \geq 0) \geq \mathbf{0}$. **Hint**: Recall the method of proof for the result of Cauchy.

(**iv**) Let $J = \det \mathbf{F}$. Prove Liouville's Formula:

$$\frac{\partial J}{\partial t} = (\nabla_x \cdot \mathbf{u})|_{\mathbf{x}=\mathbf{X}(\boldsymbol{\alpha},t)} J$$

3. Consider an incompressible viscous fluid governed by the Navier-Stokes equations:

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \Delta \mathbf{u} + \mathbf{F}$$

$$\frac{D\rho}{Dt} = 0 \text{ and } \nabla \cdot \mathbf{u} = 0$$

moving in a closed volume Ω with the boundary condition $\mathbf{u}|_\Omega = \mathbf{0}$. The kinetic energy of the system is given by

$$\mathsf{E} = \int_\Omega dV \, \frac{1}{2} \rho \, |\mathbf{u}|^2$$

(i) Show that

$$\dot{\mathsf{E}} = -\mu \int_\Omega dV \, \mathbf{E} : \mathbf{E} + \int_\Omega dV \, \mathbf{u} \cdot \mathbf{F} \quad (1)$$

where $\mathbf{A} : \mathbf{B} = A_{ij} B_{ij}$. The first negative-definite term is the so-called *rate of* *viscous dissipation* and relates the loss of kinetic energy to the magnitude of viscous stress. The second term is the rate-of-work done upon the fluid by the body force. **Note**: if ρ is constant and the body force $\mathbf{F}$ is given by a potential, i.e. $\mathbf{F} = \nabla\phi$, then the body force performs no work and $\dot{E} \leq 0$.

(ii) Rather than having a domain of fixed shape as above, consider Ω as time-dependent (i.e., shape-changing) with surface velocity $\mathbf{V}$. The no-slip condition is then modified to $\mathbf{u}|_{\Omega} = \mathbf{V}$. Are there any constraints upon the specification of $\mathbf{V}$? Let $\mathbf{F} = \mathbf{0}$, and derive an expression analogous to Eq. (1) where there is again a rate-of-work term but now involving the stress the boundary Ω exerts upon the flow.

(iii) Now consider Ω as a droplet of fluid moving within a vacuum. In this case (neglecting surface tension) the drop can deform and no-slip condition is replaced by the condition of *zero stress*: $\boldsymbol{\sigma}|_{\Omega} \cdot \mathbf{n} = \mathbf{0}$ (recall that $\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\mathbf{E}$). Prove the same identity for $\dot{E}$ as above.