Homework 2. Due on November 21st

1. Consider an 2D axisymmetric flow of the form

$$\mathbf{u} = u(r,z,t)\ \hat{\mathbf{r}} + v(r,z,t)\ \hat{\mathbf{z}} + 0\ \hat{\boldsymbol{\theta}}$$

for an incompressible, homogeneous fluid under no body force.

a. Give the full equations of motion for such a flow. Show that $\boldsymbol{\omega} = \omega(r,z,t)\ \hat{\boldsymbol{\theta}}$, i.e. that the vorticity is purely azimuthal.

b. For an inviscid fluid, show that ω satisfies

$$\frac{D}{Dt}\left(\frac{\omega}{r}\right) = 0$$

Hence, a compact distribution of vorticity for an axisymmetric flow is a vortex ring.

Can you write down a simple exact solution that is analogous to a circular patch of constant vorticity for the 2D Euler equations?

2. Consider a 2D velocity field

$$\mathbf{u} = (u(x,y,t), v(x,y,t))$$

which is a solution to the 2D incompressible and homogeneous Euler equations. Consider further a function $w(x,y,t)$ which is material to the (u,v) Euler flow. That is, w satisfies

$$\frac{\partial w}{\partial t} + \mathbf{u} \cdot \nabla w = 0$$

Ignoring boundary conditions show that the 3D flow field $(u(x,y,t), v(x,y,t), w(x,y,t))$ satisfies the 3D Euler equations, and calculate the components of the 3D vorticity field.

3. **Contour Dynamics**. Consider a closed, simply connected domain $\Omega(t)$ in the plane, with smooth boundary $\Gamma(t)$. Assume that the vorticity within Ω is one, and zero outside. Using the Biot-Savart integral and the 2D Euler equations, derive an expression for the evolution of Γ, represented in terms of a Lagrangian variable, that involves only the position of Γ itself. From this formula show that Ω a unit disk is a steady-state.

4. Consider N point vortices with positions $\mathbf{X}_k = (x_k, y_k)$ and circulations Γ_k. They evolve by the system of ODEs:

$$\dot{\mathbf{X}}_j = \frac{1}{2\pi} \sum_{k \neq j} \Gamma_k \frac{(\mathbf{X}_j - \mathbf{X}_k)_\perp}{|\mathbf{X}_j - \mathbf{X}_k|^2} \quad \text{where } (x,y)_\perp = (-y,x) \quad \textbf{(1)}$$

a. Write down the stream function of the velocity field away from the point vortices.

b. Let $\Gamma = \sum_k \Gamma_k \neq 0$ be the total circulation. Prove that these three quantities are invariants:

First moments of circulation: $\mathbf{M}_1 = \frac{1}{\Gamma}\sum_k \Gamma_k \mathbf{X}_k$

Second moment of circulation: $M_2 = \frac{1}{\Gamma}\sum_k \Gamma_k |\mathbf{X}_k - \mathbf{M}_1|^2$

c. Show that system (**1**) is Hamiltonian, satisfying:

$$\Gamma_j \dot{\mathbf{X}}_j = \left(-\frac{\partial}{\partial y_p}, \frac{\partial}{\partial x_p}\right) H \quad \text{where } H = \frac{1}{2\pi}\sum_{l>k} \Gamma_l \Gamma_k \ln|\mathbf{X}_l - \mathbf{X}_k|$$

Show that this means that H is a conserved quantity. Comment on the relationship to the stream-function.

d. Assuming that $\Gamma_k > 0$ for all k, and "good" initial data $\left(\mathbf{X}_k(0) \neq \mathbf{X}_j(0) \text{ for any pair } i,j \right)$, argue that system (**1**) will have a solution for all time.

e. Are properties a & b true for the Euler equations with compact or rapidly decaying vorticity field?

5. Consider the two vortices, labeled 1 and 2, with circulations Γ_1 and Γ_2 respectively, that move in a fluid region bounded by a wall at $y = 0$. If $\Gamma_1 = -\Gamma_2$, solve for and plot the orbits of these two vortices. Do the same when $\Gamma_1 = \Gamma_2$. Comment upon, and interpret your results.