Homework 3. Due on December 19th

1. Consider a vortex sheet separating two ideal fluids filling the 2D plane. The fluid below has density ρ_1 and that above ρ_2. Gravity points downwards, here $\mathbf{F} = -g\hat{\mathbf{y}}$. For the case of $\rho_1 = \rho_2$, we derived in class the evolution equation for the vortex sheet strength in the average velocity frame ($\gamma_t = 0$ if the pressure is continuous). If $\rho_1 \neq \rho_2$, its more complicated, and easiest done in a mix of real and complex variables notation, but one can show that

$$\gamma_t = -2A\left[\operatorname{Re}(z_\alpha Q_t^*) + \frac{1}{8}\left(\frac{\gamma}{|z_\alpha|}\right)^2 - |z_\alpha|\hat{\mathbf{s}} \cdot \mathbf{F}\right]$$

where $z(\alpha, t)$ is the sheet position parametrized in the average velocity frame, $\hat{\mathbf{s}}$ is the sheet tangent vector, $A = (\rho_1 - \rho_2)/(\rho_1 + \rho_2)$ is the *Atwood number*, and the average velocity is given by the principal value integral (denoted $P\int$)

$$Q^*[z(\alpha)] = \frac{1}{2\pi i}\left(P\int\right)\frac{\gamma(\alpha', t)d\alpha'}{z(\alpha, t) - z(\alpha', t)}$$

The relevant steady state here is that the fluid is at rest at infinity so that the mean vortex sheet strength is zero, and the surface is flat, i.e. $z = \alpha$. Perturb the interface from being flat (and the sheet strength from zero) and derive the linearized dynamics equations. Discuss the Rayleigh-Taylor instability when $A < 0$. Discuss the classical case of dispersive "water waves" when $\rho_2 = 0$.

2. A viscous fluid is carried between two parallel, flat walls located at $y = 0$ and $y = d$. For $t < 0$, both the walls and the fluid are moving at speed U in the x direction. At $t = 0$ the walls are impulsively stopped. Calculate the resulting unidirectional flow between the walls as the fluid slows down. Calculate the time for the velocity midway between the walls to reach $U/2$. Speculate on what might be observed if walls were not flat, but wavy.

3. Applications of the Circle Theorem: Consider two vortices of equal and opposite circulations $\pm\Gamma$ moving within an impermeable disk of radius a. If their initial positions are $(\pm b, 0)$ with $b < a$, calculate their subsequent dynamics. Compare this to the case in an open fluid. Make a plot of the stream-lines at $t = 0$. Now consider the case of two vortices (again, or equal and opposite circulations) approaching the disk from the exterior through their mutually induced flows. Calculate how the pair is deflected by the presence of the disk.

4. Compute the flows arising from the superposition of a 2D mass source and a uniform flow. What if the mass source is replaced with a mass dipole aligned with the uniform flow direction?

5. Consider viscous flows flows of the form $\mathbf{U} = (u, v, w) = (\gamma x, \beta y, (\gamma - \beta)z) + (\tilde{u}(x, y, t), \tilde{v}(x, y, t), \tilde{w}(x, y, t))$. Show that the $\tilde{w}$ does not couple to the dynamics of $(\tilde{u}, \tilde{v})$ and is transported and diffused as a scalar field by the 2D flow $\mathbf{V} = (\gamma x + \tilde{u}, \beta y + \tilde{v})$. Show that the out of plane vorticity $\omega = v_x - u_y$ obeys the equation

$$\omega_t + (\mathbf{V} \cdot \nabla)\omega = (\alpha - \beta)\omega + \nu\Delta\omega$$

Find the form of this equation when the $(\tilde{u}, \tilde{v})$ flows are axisymmetric in the xy –plane.

6. Work through, and give transcribed notes on, the posted material on using conformal mapping to find the flows around, and lift upon, a flat-plate airfoil.