# Fluid dynamics of swimming & active particles, from one to many

**Schedule**:

1. A primer on some basic fluid mechanics; basic properties of the Stokes equations – reversibility, reciprocal theorem, boundary integral formulation.
2. Some useful exact solutions; slender-body theory; swimming at low Reynolds number
3. Swimming at low Reynolds number continued, many-body problems and coarse-graining of active-particle suspensions
4. Collective behavior at low Reynolds numbers – basic instabilities and dynamics of active fluids, extensions
5. Some work on swimming and flying at higher Reynolds number; flapping flight, bifurcation to flapping flight; fluid-mediated collective behaviors in "ordered" schools.
6. Current applications in cellular biophysics and active materials.

**Some useful reference texts**:

- Incompressible Fluid Dynamics:

  George Batchelor – *An Introduction to Fluid Dynamics*

  D.J. Acheson – *Elementary Fluid Dynamics*

  C. Pozrikidis – *Boundary Integral and Singularity Methods for Linearized Viscous Flow*

- Complex Fluids and Solids

  R.G. Larson – *The Structure and Rheology of Complex Fluids*

  G.A. Holzapfel – *Nonlinear Solid Mechanics*

  Doi & Edwards – *The Theory of Polymer Dynamics*

- Bio Mechanics, Fluids, Locomotion

  S. Childress – *Mechanics of Swimming and Flying*

  S. Vogel – *Life in Moving Fluids*

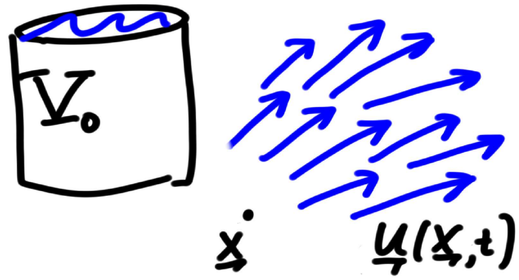
  S. Vogel – *Comparative Biomechanics*

  R.M. Alexander – *Principles of Animal Locomotion*

- Active Fluids / Materials

  S. Spagnolie, *Complex Fluids in Biological Systems*

# Lecture 1: Basic concepts of fluid and continuum mechanics




**The volume of fluid $V$ and the velocity field**.

Consider a volume $V$ filled with a fluid or continuous material. At each time $t$ and at each point $\mathbf{x}$ the fluid has a velocity $\mathbf{u}(\mathbf{x},t)$ and density $\rho(\mathbf{x},t)$. Here we describe the fluid flow as passing through a fixed (lab) coordinate frame, traditionally called the *Eulerian frame.*

## *Notation:*

$$\mathbf{x} = (x,y,z) = (x_1,x_2,x_3)$$

$$\mathbf{u} = (u,v,w) = (u_1,u_2,u_2)$$

## The basic constituents of the velocity field – translation, deformation, rotation

Consider a steady velocity field $\mathbf{u}(\mathbf{x})$, fixing a point $\mathbf{x}$ and considering a nearby point $\mathbf{x}+\mathbf{r}$. Then,

$$\begin{aligned}\mathbf{u}(\mathbf{x}+\mathbf{r}) &= \mathbf{u}(\mathbf{x}) + \nabla\mathbf{u}(\mathbf{x})\,\mathbf{r} + O(|\mathbf{r}|^2)\\ &\approx \mathbf{u}(\mathbf{x}) + \frac{1}{2}(\nabla\mathbf{u}+\nabla\mathbf{u}^\top)\mathbf{r} + \frac{1}{2}(\nabla\mathbf{u}-\nabla\mathbf{u}^\top)\mathbf{r}\\ &\approx \mathbf{u}(\mathbf{x}) + \mathbf{E}\,\mathbf{r} + \mathbf{W}\,\mathbf{r}\end{aligned}$$

where $(\nabla\mathbf{u})_{ij} = \partial u_i/\partial x_j$ called the *rate-of-strain tensor*, which we have re-expressed in terms of its symmetric and anti-symmetric parts $E_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$, the *symmetric rate-of-strain tensor*, and $W_{ij} = (\partial u_i/\partial x_j - \partial u_j/\partial x_i)/2$, the *rotation tensor*. The velocity field can be decomposed as

1. A translation $\mathbf{u}(\mathbf{x})$.
2. A pure straining flow: Note that $\mathbf{E}$ is a symmetric matrix, and thus has 3 real eigenvalues $\lambda_i$ with 3 associated, mutually orthogonal eigenvectors $\mathbf{p}_i$. Here,

   the $\lambda_i$ are called the principal rates-of-strain, and

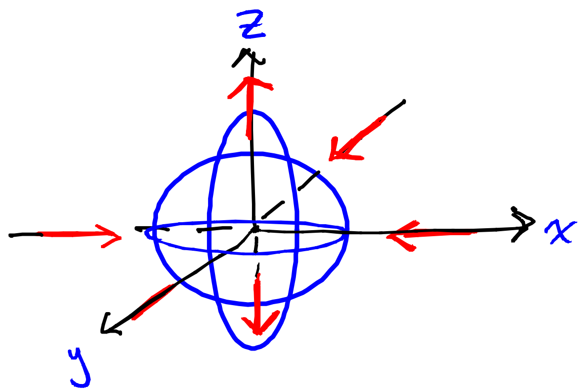
   the $\mathbf{p}_i$ are called the principal axes (or directions) of strain

   The local effect of $\mathbf{E}$ is to deform, through compression and expansion, a ball centered at $\mathbf{r} = \mathbf{0}$ into an ellipsoid whose principal axes are the principal axes of strain. The velocity $\mathbf{Er}$ is called a *pure straining flow*.

   **Example**: If

$$\mathbf{E} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

then the sketch below illustrates the induced material deformation.




**An incompressible straining flow**

**3**. The *rotation tensor* **W** is anti-symmetric with purely imaginary eigenvalues, and can be expressed as

$$\mathbf{W} = \frac{1}{2}\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is called the *vorticity*. Vorticity is a fundamental quantity in incompressible fluid dynamics.

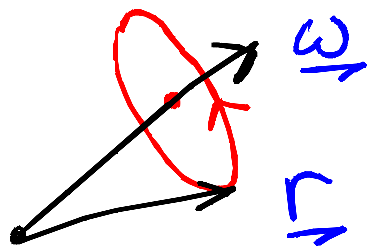
$$\mathbf{Wr} = \frac{1}{2}\begin{pmatrix} \omega_2 r_3 - \omega_3 r_2 \\ \omega_3 r_1 - \omega_1 r_3 \\ \omega_1 r_2 - \omega_2 r_1 \end{pmatrix} = \frac{1}{2}\boldsymbol{\omega} \times \mathbf{r}$$

The velocity field **Wr** is a rigid-body rotation (and is divergence free), w. angular velocity $\frac{1}{2}\boldsymbol{\omega}$. Note

$$\dot{\mathbf{r}} = \frac{1}{2}\boldsymbol{\omega} \times \mathbf{r} \Rightarrow$$

$$\frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) = 0 \ \text{ fixed length}$$

$$\frac{d}{dt}(\mathbf{r} \cdot \boldsymbol{\omega}) = 0 \ \text{ fixed angle}$$



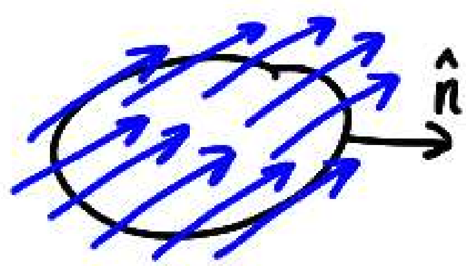

**ω generates a cone upon which r moves**.

*In summary:* The local flow is composed of (i) a translation; (ii) a pure straining flow, itself decomposable into an incompressible part, and an isotropic expansion or compression; (iii) a rigid body rotation.

## Conservation of Mass

Consider a *fixed* subvolume $V_0 \subseteq V$ with outward normal $\hat{\mathbf{n}}$. The mass of $V_0$ at time $t$ is:

$$M[V_0,t] = \int_{V_0} dV_x \rho(\mathbf{x},t)$$




**The flux of mass through an Eulerian volume $V_0$**

The rate of change of $M[V_0,t]$ is balanced by the flux of mass through its boundary $\partial V_0$, or

$$\frac{d}{dt}\int_{V_0} dV_x \rho = -\int_{\partial V_0} dS_x \left(\rho\,\mathbf{u}\right)\cdot\hat{\mathbf{n}} \qquad \#$$

This is the *integral form* of mass conservation. Using the divergence theorem we can write

$$\int_{V_0} dV_x \left[\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho\,\mathbf{u}\right)\right] = 0$$

As $V_0$ was arbitrary, this gives the *continuity equation:*

$$\frac{\partial \rho}{\partial t} + \nabla_x \cdot \left(\rho\,\mathbf{u}\right) = 0$$

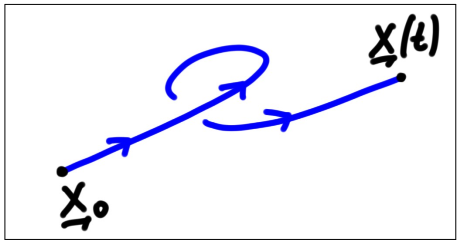
which is a PDE governing the evolution of material density in a moving fluid or continuous material, and is called the *differential form* of mass conservation. $\mathbf{j} = \rho\,\mathbf{u}$ is called the *mass density flux*.

## The Lagrangian formulation

The quantities $\mathbf{u}$ and $\rho$ have been expressed in the *Eulerian frame*, e.g., $\rho$ is measured at a fixed point $\mathbf{x}$ in the lab frame. In the *Lagrangian frame* a quantity, say $\rho$, is measured in the frame of moving fluid. Let $\mathbf{X}(t)$ satisfy

$$\frac{d\mathbf{X}}{dt} = \mathbf{u}(\mathbf{X}(t), t) \text{ with } \mathbf{X}(0) = \mathbf{X}_0$$

The function $\mathbf{X}(t)$ is called the *Lagrangian*, *material*, or *particle path.* Consider $\rho(\mathbf{X}(t), t)$, that is, the evolution of fluid density along a Lagrangian path.




**A Lagrangian path**

Then

$$\frac{d}{dt}\rho(\mathbf{X}(t), t) = \left[ \frac{\partial \rho}{\partial t}(\mathbf{x}, t) + \dot{\mathbf{X}} \cdot \nabla_x \rho(\mathbf{x}, t) \right]_{\mathbf{x}=\mathbf{X}(t)}$$

$$= \left[ \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla_x \rho \right]_{\mathbf{x}=\mathbf{X}(t)}$$

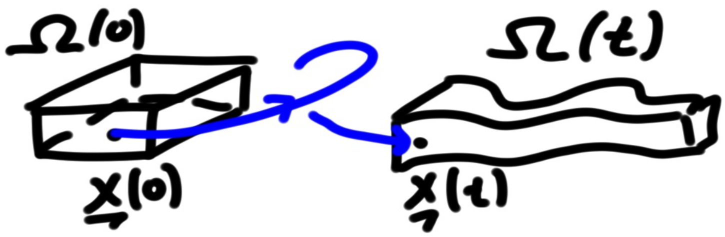
The operator $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla_x$ is called the *Lagrangian, material,* or *substantial derivative*. It is the Eulerian expression for the time-rate-of-change of quantities along Lagrangian paths. And so

$$\frac{D\rho}{Dt} = -\rho\, (\nabla_x \cdot \mathbf{u})$$

*Some properties of the substantial derivative:*

- $\frac{D}{Dt}(fg) = f\frac{Dg}{Dt} + g\frac{Df}{Dt}$ plus other usual aspects of a derivative
- $\frac{Df}{Dt} = 0 \Leftrightarrow f(\mathbf{X}(t), t) = f(\mathbf{X}_0, 0)$, i.e., $f$ is conserved along particle paths.

Previously we had considered a fixed, or Eulerian volume $V_0$. Now, let $\Omega(t)$ be a time dependent volume moved by the flow from $\Omega(0)$:




**The deformation of the Ω under the flow**

That is, solve

$$\frac{d\mathbf{X}}{dt} = \mathbf{u}(\mathbf{X}(t), t) \text{ with } \mathbf{X}(0) = \mathbf{X}_0 \ \forall \mathbf{X}_0 \subseteq \Omega_0$$

$\Omega(t)$ is the set of all consequent $\mathbf{X}(t)$, and is called a *Lagrangian* or *material volume.*

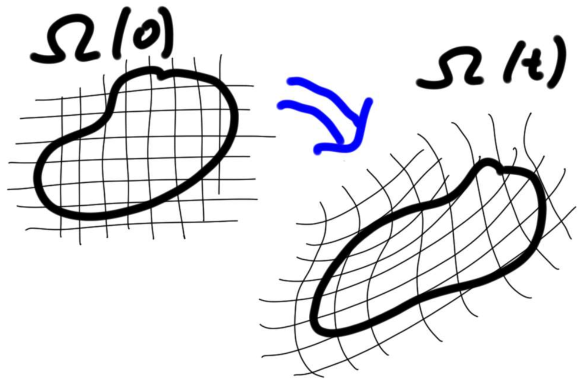
*Lagrangian flow-map:* A *Lagrangian variable* is one that stays constant along a Lagrangian path. The key idea of the Lagrangian formulation is to use the set of initial coordinates $\mathbf{X}_0 \subseteq \Omega_0$ as independent spatial coordinates. So, consider the time-dependent transformation of spatial coordinates

$$\boldsymbol{\alpha} \mapsto \mathbf{X}(\boldsymbol{\alpha}, t)$$

found by solving

$$\frac{\partial \mathbf{X}}{\partial t}(\boldsymbol{\alpha}, t) = \mathbf{u}(\mathbf{X}(\boldsymbol{\alpha}, t), t) \text{ with } \mathbf{X}(\boldsymbol{\alpha}, 0) = \boldsymbol{\alpha}$$

(i.e., $\boldsymbol{\alpha} = \mathbf{X}_0$). $\mathbf{X}(\boldsymbol{\alpha}, t)$ is the *Lagrangian flow-map* and $\boldsymbol{\alpha}$ is the *Lagrangian variable.*




**The evolution of the Lagrangian flow-map**.

The Lagrangian flow map has many important properties:

**(1)**

$$\begin{aligned}\frac{\partial}{\partial t} f(\mathbf{X}(\boldsymbol{\alpha}, t), t) &= \left[ \frac{\partial f}{\partial t} + \frac{\partial \mathbf{X}}{\partial t} \cdot \nabla_x f \right]_{\mathbf{x} = \mathbf{X}(\alpha, t)} \\ &= \left[ \frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla_x f \right]_{\mathbf{x} = \mathbf{X}(\alpha, t)} \\ &= \left[ \frac{Df}{Dt} \right]_{\mathbf{x} = \mathbf{X}(\alpha, t)}\end{aligned}$$

Hence, the substantial derivative relates the Eulerian and Lagrangian frames.

(2) A fundamental object defined by the Lagrangian flow-map is the deformation tensor or matrix $\mathbf{F}$, defined as the Jacobian of the flow-map:

$$\mathbf{F} = \frac{\partial \mathbf{X}}{\partial \boldsymbol{\alpha}} \text{ or } F_{ij} = \frac{\partial X_i}{\partial \alpha_j}$$

$F$ encodes the deformations of the Lagrangian flow-map relative to the initial state. Let $\mathbf{V}(\boldsymbol{\alpha}, t) = \mathbf{u}(\mathbf{X}(\boldsymbol{\alpha}, t), t)$. Then $\mathbf{F}$ evolves by

$$\begin{aligned}\frac{\partial F_{ij}}{\partial t} &= \frac{\partial}{\partial t}\frac{\partial X_i}{\partial \alpha_j} = \frac{\partial}{\partial \alpha_j}\frac{\partial X_i}{\partial t} = \frac{\partial V_i}{\partial \alpha_j} \\ &= \frac{\partial}{\partial \alpha_j} u_i(\mathbf{X}(\boldsymbol{\alpha}, t), t) = \frac{\partial u_i}{\partial X_k}\frac{\partial X_k}{\partial \alpha_j}\end{aligned}$$

or

$$\boxed{\frac{\partial \mathbf{F}}{\partial t} = \nabla_\alpha \mathbf{V} = (\nabla_x \mathbf{u})|_{\mathbf{X}}(\boldsymbol{\alpha}, t)\mathbf{F} \text{ with } \mathbf{F}(\boldsymbol{\alpha}, 0) = \mathbf{I}}$$

or in Eulerian variables:

$$\boxed{\frac{D\mathbf{F}}{Dt} = (\nabla_x \mathbf{u})\mathbf{F} \text{ with } \mathbf{F}(\mathbf{x},0) = \mathbf{I}}$$

This introduces $\mathbf{D} = \nabla_x \mathbf{u}$, the rate-of-strain tensor. A related tensor is $\mathbf{E} = (\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^\top)/2$, the symmetric rate-of-strain tensor.

(3) Let $J$ be the Jacobian determinant of the flow-map, that is,

$$J(\boldsymbol{\alpha},t) = \det[\mathbf{F}] = \det\left[\ \mathbf{F}_1,\ldots,\mathbf{F}_n\ \right] = \det\left[\ \nabla_\alpha X_1, \nabla_\alpha X_2, \nabla_\alpha X_3\ \right]$$

*Note:* $J(\boldsymbol{\alpha},0) \equiv 1$. We have the following important and standard result from dynamical systems theory for its evolution: *Louiville's Formula:*

$$\frac{\partial}{\partial t} J(\boldsymbol{\alpha},t) = (\nabla_x \cdot \mathbf{u})|_{\mathbf{X}}(\boldsymbol{\alpha},t)(\mathbf{X}(\boldsymbol{\alpha},t),t) \cdot J(\boldsymbol{\alpha},t)$$

*Proof:* In $\mathbf{R}^n$

$$\mathbf{F} = \left[\nabla_\alpha X_1,\ldots,\nabla_\alpha X_n\ \right] = \left[\ \mathbf{F}_1,\ldots,\mathbf{F}_n\ \right]$$

The Jacobian can be expressed in terms of the multi-linear operator, the *wedge product*:

$$J = \mathbf{F}_1 \wedge \mathbf{F}_2 \wedge \ldots \wedge \mathbf{F}_n$$

which has the properties:

1. $\mathbf{F}_1 \wedge \ldots \wedge (\alpha\mathbf{U} + \boldsymbol{\beta}\mathbf{W}) \wedge \ldots \wedge \mathbf{F}_n = \alpha(\mathbf{F}_1 \wedge \ldots \wedge \mathbf{U} \wedge \ldots \wedge \mathbf{F}_n) + \beta(\mathbf{F}_1 \wedge \ldots \wedge \mathbf{W} \wedge \ldots \wedge \mathbf{F}_n)$
2. $\mathbf{F}_i \subseteq span[\mathbf{F}_j, j \neq i] \Rightarrow J = 0$
3. $\frac{d}{dt} J = (\dot{\mathbf{F}}_1 \wedge \mathbf{F}_2 \wedge \ldots \wedge \mathbf{F}_n) + (\mathbf{F}_1 \wedge \dot{\mathbf{F}}_2 \wedge \ldots \wedge \mathbf{F}_n) + \ldots + (\mathbf{F}_1 \wedge \mathbf{F}_2 \wedge \ldots \wedge \dot{\mathbf{F}}_n)$

Now,

$$\begin{aligned}
\mathbf{F}_i &= \frac{\partial}{\partial t} \nabla_\alpha X_i = \nabla_\alpha u_i(\mathbf{X}(\boldsymbol{\alpha},t),t) \\
&= \frac{\partial u_i}{\partial X_1}(\nabla_\alpha X_1) + \frac{\partial u_i}{\partial X_2}(\nabla_\alpha X_2) + \ldots + \frac{\partial u_i}{\partial X_i}(\nabla_\alpha X_i) + \ldots + \frac{\partial u_i}{\partial X_n}(\nabla_\alpha X_n) \\
&= \frac{\partial u_i}{\partial X_i}\mathbf{F}_i + \mathbf{T}_i \text{ with } \mathbf{T}_i \subseteq span[\mathbf{F}_j, j \neq i]
\end{aligned}$$

Then

$$\begin{aligned}
\frac{d}{dt} J &= \left( \left( \frac{\partial u_1}{\partial X_1}\mathbf{F}_1 + \mathbf{T}_1 \right) \wedge \mathbf{F}_2 \wedge \ldots \wedge \mathbf{F}_n \right) \\
&+ \left( \mathbf{F}_1 \wedge \left( \frac{\partial u_2}{\partial X_2}\mathbf{F}_2 + \mathbf{T}_2 \right) \wedge \ldots \wedge \mathbf{F}_n \right) + \ldots \\
&+ \left( \mathbf{F}_1 \wedge \mathbf{F}_2 \wedge \ldots \wedge \left( \frac{\partial u_n}{\partial X_n}\mathbf{F}_n + \mathbf{T}_n \right) \right) \\
&= \frac{\partial u_1}{\partial X_1} J + \frac{\partial u_2}{\partial X_2} J + \ldots + \frac{\partial u_n}{\partial X_n} J \\
&= (\nabla_x \cdot \mathbf{u}) J
\end{aligned}$$

(**4**) The effect of change in geometry: Consider two nearby Lagrangian points $\boldsymbol{\alpha}$ and $\boldsymbol{\beta} = \boldsymbol{\alpha} + d\boldsymbol{\alpha}$. Now consider the displacement of these points in the Eulerian frame under the flow of the material:

$$d\mathbf{X} = \mathbf{X}(\boldsymbol{\beta},t) - \mathbf{X}(\boldsymbol{\alpha},t) \approx \mathbf{F}(\boldsymbol{\alpha},t)d\boldsymbol{\alpha}$$

Then, $|d\mathbf{X}|^2 = d\boldsymbol{\alpha}^T\mathbf{F}^T\mathbf{F}d\boldsymbol{\alpha} = d\boldsymbol{\alpha}^T\mathbf{C}d\boldsymbol{\alpha}$. Hence $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ controls the relative stretching of Lagrangian line elements by the flow. $\mathbf{C}$ is the *right Cauchy-Green tensor*, which is symmetric and positive definite (spd), satisfying $\det\mathbf{C} = (\det\mathbf{F})^2 = J^2 > 0$. $\mathbf{C}$ satisfies the dynamics equation:

$$\mathbf{C}_t = \mathbf{F}_t^T\mathbf{F} + \mathbf{F}^T\mathbf{F}_t = \mathbf{F}^T\,\nabla\mathbf{u}^T\,\mathbf{F} + \mathbf{F}^T\,\nabla_x\mathbf{u}\,\mathbf{F} = 2\mathbf{F}^T\,\mathbf{E}\,\mathbf{F}$$

We can also write $d\boldsymbol{\alpha} = \mathbf{F}^{-1}d\mathbf{X}$, or $|d\boldsymbol{\alpha}|^2 = d\mathbf{X}^T\mathbf{F}^{-T}\mathbf{F}^{-1}d\mathbf{X} = d\mathbf{X}^T(\mathbf{F}\mathbf{F}^T)^{-1}d\mathbf{X} = d\mathbf{X}^T\mathbf{b}^{-1}d\mathbf{X}$. Here

$$\boxed{\mathbf{b} = \mathbf{F}\mathbf{F}^T}$$

is the *left Cauchy-Green* (or *Finger*) tensor, which is also spd, and satisfies $\det\mathbf{b} = (\det\mathbf{F})^2 = J^2 > 0$. This tensor arises very naturally in the theory of rubber elasticity. This has the far more attractive dynamics:

$$\frac{D\mathbf{b}}{Dt} = \nabla_x\mathbf{u}\,\mathbf{F}\,\mathbf{F}^T + \mathbf{F}\,\mathbf{F}^T\nabla_x\mathbf{u}^T = \nabla_x\mathbf{u}\,\mathbf{b} + \mathbf{b}\,\nabla_x\mathbf{u}^T$$

**Note**: (i) $\mathbf{C}$ and $\mathbf{b}$ have the same invariants. Let $\mathbf{F} = \mathbf{U}\mathbf{D}\mathbf{V}^T$ be the singular value decomposition of $\mathbf{F}$, so that $\mathbf{D}$ contains the singular values, and $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices. Then,

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{V}\mathbf{D}^2\mathbf{V}^T \text{ and } \mathbf{b} = \mathbf{F}\mathbf{F}^T = \mathbf{U}\mathbf{D}^2\mathbf{U}^T$$

and so $\mathbf{C}$ and $\mathbf{b}$ have the same eigenvalues, $\lambda_l^2$, and hence have the same invariants. (ii) The evolution for $\mathbf{b}$ is closed, and does not require knowledge of $\mathbf{F}$. This is not so for $\mathbf{C}$.

**Side Note**: The operator $\mathbf{b}^\nabla = \frac{D\mathbf{b}}{Dt} - \left(\nabla_x\mathbf{u}\,\mathbf{b} + \mathbf{b}\,\nabla_x\mathbf{u}^T\right)$ is called the *upper convected derivative* and is intimately related to conservation principles in the Lagrangian frame. First, a simple proof of the result of Cauchy. For the incompressible 3D Euler equations, vorticity transport is given by (in the Lagrangian frame)

$$\boldsymbol{\omega}_t = \nabla_x\mathbf{u}\,\boldsymbol{\omega} = \nabla_x\mathbf{u}\,\mathbf{F}\mathbf{F}^{-1}\,\boldsymbol{\omega} = \mathbf{F}_t\,\mathbf{F}^{-1}\boldsymbol{\omega} = -\mathbf{F}(\mathbf{F}^{-1})_t\,\boldsymbol{\omega}$$

or

$$\left(\mathbf{F}^{-1}\,\boldsymbol{\omega}\right)_t = \mathbf{0} \text{ giving } \boldsymbol{\omega} = \mathbf{F}\,\boldsymbol{\omega}_0$$

which is the result. Hence, any vector or matrix satisfying

$$\mathbf{W}_t = \nabla_x\mathbf{u}\,\mathbf{W} \text{ also satisfies } \mathbf{W} = \mathbf{F}\,\mathbf{W}_0$$

Consider now the dyadic matrix $\mathbf{Z} = \mathbf{W}\mathbf{W}^T = \mathbf{F}\,\mathbf{W}_0\mathbf{W}_0^T\mathbf{F}^T$. Then

$$\mathbf{Z}_t = \nabla_x\mathbf{u}\,\mathbf{Z} + \mathbf{Z}\,\nabla_x\mathbf{u}^T$$

*In general*, we have the result that $\mathbf{Z}$ satisfies the conservation law

$$(\mathbf{F}^{-1}\mathbf{Z}\mathbf{F}^{-T})_t = \mathbf{0} \text{ or } \mathbf{Z} = \mathbf{F}\,\mathbf{Z}_0\mathbf{F}^T$$

if and only if

$$\mathbf{Z}^\nabla \equiv \mathbf{Z}_t - \left(\nabla_x\mathbf{u}\,\mathbf{Z} + \mathbf{Z}\,\nabla_x\mathbf{u}^T\right) = \mathbf{0}$$

## Mass Conservation in the Lagrangian frame

The mass of a Lagrangian volume does not change in time, that is

$$M[\Omega(t)] = M[\Omega(0)]$$

where $M$ can be expressed as:

$$M[\Omega(t)] = \int_{\Omega(t)} dV_x\,\rho(\mathbf{x},t) \quad \text{tf. to Lagrangian coords}$$
$$= \int_{\Omega(0)} dV_\alpha\,\rho(\mathbf{X}(\boldsymbol{\alpha},t),t)J(\boldsymbol{\alpha},t)$$

Then

$$0 = M[\Omega(t)] - M[\Omega(0)] = \int_{\Omega(0)} dV_\alpha \, [\rho(\mathbf{X}(\boldsymbol{\alpha},t),t)J(\boldsymbol{\alpha},t) - \rho(\boldsymbol{\alpha},0)]$$

As $\Omega(0)$ was arbitrary we have

$$\rho(\mathbf{X}(\boldsymbol{\alpha},t),t)J(\boldsymbol{\alpha},t) = \rho(\boldsymbol{\alpha},0)$$

This is one Lagrangian form of mass conservation. As $J$ also represents the measure of an infinitesmal volume, it says that if the volume increases, then the density must decrease. This now gives sense to incompressibility of a fluid or material. Incompressibility means that material volumes, infinitesimal or otherwise, do not change their volume. That is, $J(\boldsymbol{\alpha},t) \equiv 1$, and consequently $\rho(\mathbf{X}(\boldsymbol{\alpha},t),t) = \rho(\boldsymbol{\alpha},0)$. This has two consequences, following Liouville's formula:

$$(\nabla_x \cdot \mathbf{u}) = 0 \text{ and } \frac{D\rho}{Dt} = 0$$

Hence, incompressible fluids are have divergence free velocity fields, and the density is conserved along Lagrangian paths.

To recover the Eulerian form, take a time-derivative of Eq. (1''') and use the relation with the substantial derivative and Liouville's formula:

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}(\rho(\mathbf{X}(\boldsymbol{\alpha},t),t)J(\boldsymbol{\alpha},t)) \\ &= \left[ \frac{D\rho}{Dt}(\mathbf{X}(\boldsymbol{\alpha},t),t)\, J(\boldsymbol{\alpha},t) + \rho(\mathbf{X}(\boldsymbol{\alpha},t),t)(\nabla_x \cdot \mathbf{u})(\mathbf{X}(\boldsymbol{\alpha},t),t)\, J(\boldsymbol{\alpha},t) \right] \\ &= \left[ \frac{D\rho}{Dt}(\mathbf{X}(\boldsymbol{\alpha},t),t) + \rho(\mathbf{X}(\boldsymbol{\alpha},t),t)(\nabla_x \cdot \mathbf{u})(\mathbf{X}(\boldsymbol{\alpha},t),t) \right] J(\boldsymbol{\alpha},t) \end{aligned}$$

If the flow is smooth, then $J \neq 0$, and we have

$$\frac{D\rho}{Dt}(\mathbf{x},t) = -\rho(\mathbf{x},t)(\nabla_x \cdot \mathbf{u})(\mathbf{x},t)$$

which we have already proved.

The Lagrangian statement of mass conservation yields the following fundamental result:

*The Transport Theorem:* For any smooth $f(\mathbf{x},t)$

$$\frac{d}{dt}\int_{\Omega(t)} dV_x \rho f = \int_{\Omega(t)} dV_x \, \rho \frac{Df}{Dt}$$

*Proof:* Use that $\partial_t(\rho(\mathbf{X}(\boldsymbol{\alpha},t),t)J(\boldsymbol{\alpha},t)) = 0$ and that $\partial_t(f(\mathbf{X}(\boldsymbol{\alpha},t),t)) = (Df/Dt)(\mathbf{X}(\boldsymbol{\alpha},t),t)$:

$$\begin{aligned} \frac{d}{dt}\int_{\Omega(t)} dV_x \rho f &= \frac{d}{dt}\int_{\Omega(0)} dV_\alpha \, J\rho f = \int_{\Omega(0)} dV_\alpha \, \frac{\partial}{\partial t}(J\rho f) \\ &= \int_{\Omega(0)} dV_\alpha \left[ f\frac{\partial}{\partial t}(J\rho) + (J\rho)\frac{\partial f}{\partial t} \right] \\ &= \int_{\Omega(0)} dV_\alpha \, J\rho \frac{\partial f}{\partial t} = \int_{\Omega(t)} dV_x \, \rho \frac{Df}{Dt} \end{aligned}$$

*Side Note:* If the flow is incompressible, then the deformation tensor $\mathbf{F}$ satisfies $\nabla \cdot \mathbf{F}^T = \mathbf{0}$ at all times.

*Proof:* In Eulerian coordinates, using that $\partial u_i/\partial x_i = 0$, $\mathbf{F}$ satisfies

$$\partial_t F_{ij} + u_k \frac{\partial F_{ij}}{\partial x_k} = \frac{\partial u_i}{\partial x_k} F_{kj}$$

$$\Rightarrow \partial_t \frac{\partial F_{ij}}{\partial x_i} + \underline{\frac{\partial u_k}{\partial x_i} \frac{\partial F_{ij}}{\partial x_k}} + u_k \frac{\partial}{\partial x_k} \frac{\partial F_{ij}}{\partial x_i} = \underline{\frac{\partial u_i}{\partial x_k} \frac{\partial F_{kj}}{\partial x_i}}$$

The underlined terms are identical under interchange of $k$ and $i$. Hence

$$\partial_t \frac{\partial F_{ij}}{\partial x_i} + u_k \frac{\partial}{\partial x_k} \frac{\partial F_{ij}}{\partial x_i} = 0 \quad \text{or}$$

$$\frac{D}{Dt}(\nabla \cdot \mathbf{F}^T) = \mathbf{0} \text{ where } \mathbf{F}_0 = \mathbf{I}$$

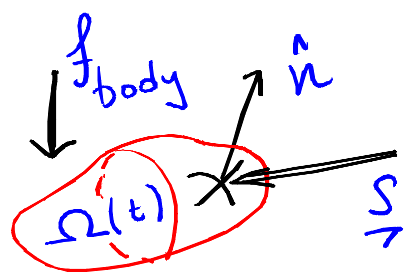
## Balance of momentum and forces in a fluid or deformable material

The acceleration of a fluid particle is given by

$$\mathbf{a}(t) = \frac{d^2}{dt^2}\mathbf{X}(t) = \frac{d}{dt}\mathbf{u}(\mathbf{X}(t), t)$$

$$= \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla_x)\mathbf{u} \right) = \left( \frac{D\mathbf{u}}{Dt} \right)(\mathbf{X}(t), t)$$

where notationally $[(\mathbf{u} \cdot \nabla_x)\mathbf{u}]_i = u_j \partial_{x_j} u_i$.




**A Lagrangian subvolume Ω being acted upon by body forces ($\mathbf{f}_{body}$) and forces of stress $\left(\mathbf{s}\, \Delta A\right)$.**

Now, let's develop Newton's 2$^{nd}$ Law for balance of forces in a fluid. The momentum carried by a Lagrangian volume of fluid $\Omega(t)$ is

$$\mathbf{m}(t) = \int_{\Omega(t)} dV_x\, \rho \mathbf{u}$$

Generally, forces come in two flavors, *body forces* and *stresses:*

- *Body forces* – externally imposed forces such as gravity or electro-magnetic fields, that exert a force/unit mass. Let $\mathbf{g}(\mathbf{x}, t)$ be such a force/unit mass. The total body force exerted upon $\Omega(t)$ is:

$$\mathbf{f}_{body} = \int_{\Omega(t)} dV_x\, \rho \mathbf{g}$$

- *Forces of stress* – Forces arising from the mechanical contact of the volume $\Omega$, across $\partial\Omega$, with the rest of the fluid or material. According to Cauchy, the (Cauchy) stress $\mathbf{s}$ (units of force/unit area) across a surface with outward normal $\hat{\mathbf{n}}$, at a point $\mathbf{x}$, has the form

$$\mathbf{s} = \boldsymbol{\sigma}\hat{\mathbf{n}} \quad \text{or} \quad s_i = \sigma_{ij} n_j$$

$\boldsymbol{\sigma}$ is called the *Cauchy stress tensor* and it is a central focus of nearly all modeling of complex fluids and deformable materials. Conservation of angular momentum implies that the stress tensor is symmetric. The total force of stress exerted upon $\Omega(t)$ is:

$$\mathbf{f}_{stress} = \int_{\partial\Omega(t)} dS_x \, \boldsymbol{\sigma}\hat{\mathbf{n}}$$

Newton's $2^{nd}$ law then gives

$$\frac{d}{dt}\mathbf{m} = \mathbf{f}_{body} + \mathbf{f}_{stress}$$

Applying the transport theorem to the expression for $(d/dt)\mathbf{m}$ and the divergence theorem to the expression of $\mathbf{f}_{stress}$ gives:

$$\int_{\Omega(t)} dV_x \, \rho \frac{Du_i}{Dt} = \int_{\Omega(t)} dV_x \, \rho g_i + \int_{\Omega(t)} dV_x \, \frac{\partial}{\partial x_j}\sigma_{ij}$$

or

$$\rho \frac{D\mathbf{u}}{Dt} = \nabla_x \cdot \sigma + \rho\mathbf{g}$$

Consider stresses written in the form

$$\sigma = -p\mathbf{I} + \mathbf{d} \text{ with } tr(\mathbf{d}) = 0$$

*Newtonian fluids* are those that have a linear relation between the *deviatoric stress* $\mathbf{d}$ and the rate-of-strain tensor $\nabla\mathbf{u}$. All others are termed non-Newtonian.

*Classical examples – (1) the Euler equations.* $\mathbf{d} = \mathbf{0}$. Take the stress to be only in the direction of the normal, that is:

$$\boldsymbol{\sigma} = -p(\mathbf{x}, t)\mathbf{I}$$

$p$ is called the (*mechanical) pressure*, and is compressive for $p > 0$. Hence,

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla_x p + \rho\mathbf{g} \quad \text{(L. Euler, 1755)}$$

When the fluid is incompressible, then we have a closed set of evolution equations

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla_x p + \rho\mathbf{g}$$
$$\frac{D\rho}{Dt} = 0 \text{ and } \nabla_x \cdot \mathbf{u} = 0$$

*Very Important Note:* Here the pressure plays the role of a Lagrange multiplier that enforces incompressibility, adjusting itself at each time to ensure that velocity remains divergence free. This system is *nonlinear* due to the nature of the substantial derivative, but also *nonlocal* as the divergence free condition yields an elliptic character to the equations.

*(2) the isotropic Navier-Stokes equations for an incompressible fluid.*

$$\boldsymbol{\sigma} = -p(\mathbf{x}, t)\mathbf{I} + 2\mu\mathbf{E}$$

yields the N-S equations

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla_x p + \mu\Delta\mathbf{u} + \rho\mathbf{g}$$
$$\frac{D\rho}{Dt} = 0 \text{ and } \nabla_x \cdot \mathbf{u} = 0$$

$\mu$ is *bulk* or *shear viscosity.*

*(3) a different example: Neo-Hookean elastic solid* – $\boldsymbol{\sigma} = -p\mathbf{I} + GJ^{-1}\mathbf{b}$ gives the simplest model of a perfectly elastic solid that dissipates no energy. If the material is incompressible, then when

combined with

$$\frac{D\mathbf{b}}{Dt} = \nabla_x \mathbf{u}\,\mathbf{b} + \mathbf{b}\,\nabla_x \mathbf{u}^T \text{ and } \nabla_x \cdot \mathbf{u} = 0 \ \ (J \equiv 1)$$

this system is closed.

## Momentum balance in the Lagrangian frame

What is the analogous expression for momentum balance in the Lagrangian frame? For this, we need *Nanson's formula.* This crucial identity allows a change of surface variables between Eulerian and Lagrangian descriptions (Holzapfel, Eq. 2.55):

$$\hat{\mathbf{n}}\, dS_x = J\,\mathbf{F}^{-T}\,\hat{\mathbf{N}}\, dS_\alpha$$

Here $\hat{\mathbf{n}}$ is the normal to a patch of surface of area $dS_x$ in the Eulerian frame, while $\hat{\mathbf{N}}$ is the surface normal to the originating Lagrangian surface of size $dS_\alpha$. Now, here is the proof *not* given by Holzapfel of Nanson's equality in differential form:

$$\nabla_x \cdot \boldsymbol{\sigma} = J^{-}1\nabla_\alpha \cdot (\boldsymbol{\sigma} \cdot \mathbf{F}^{-T} J)$$

*Proof:* Consider the (stress) tensor $\boldsymbol{\sigma}$ as a set of vectors indexed by $i$, the row index. i.e., $\sigma_{ij} \to \sigma_j^i$. Then, $\nabla_x \cdot \boldsymbol{\sigma}^i = \frac{\partial \sigma_j^i}{\partial x_j}$. Now

$$\frac{\partial}{\partial \alpha_p}\sigma_j^i = \frac{\partial \sigma_j^i}{\partial x_q}\frac{\partial x_q}{\partial \alpha_p} = \frac{\partial \sigma_j^i}{\partial x_q} F_{qp} \text{ or } \nabla_\alpha \boldsymbol{\sigma}^i = \nabla_x \boldsymbol{\sigma}^i \cdot \mathbf{F}$$

$$\text{and hence, } \nabla_x \boldsymbol{\sigma}^i = \nabla_\alpha \boldsymbol{\sigma}^i \cdot \mathbf{F}^{-1} \text{ or taking a trace: } \frac{\partial \sigma_j^i}{\partial x_j} = \frac{\partial \sigma_j^i}{\partial \alpha_p} F_{pj}^{-1}$$

We now make use of the following identity:

$$\frac{\partial}{\partial \alpha_p}(F_{pj}^{-1} J) = [\nabla_\alpha \cdot (\mathbf{F}^{-T} J)]_j = 0$$

(Do this as an exercise.) And so,

$$[\nabla_x \cdot \boldsymbol{\sigma}]_i = J^{-1}\frac{\partial}{\partial \alpha_p}(\sigma_{ij} F_{pj}^{-1} J) = J^{-1}\frac{\partial}{\partial \alpha_p}(\sigma_{ij} F_{jp}^{-T} J) = J^{-1}[\nabla_\alpha \cdot (\boldsymbol{\sigma} \cdot \mathbf{F}^{-T} J)]_i$$

Hence, we have

$$\int_{\Omega(t)} \nabla_x \cdot \boldsymbol{\sigma}\, dV_x = \int_{\Omega(0)} \nabla_\alpha \cdot (\boldsymbol{\sigma} \cdot \mathbf{F}^{-T} J)\, dV_\alpha$$

or setting $\mathbf{P} = \boldsymbol{\sigma} \cdot \mathbf{F}^{-T} J \Leftrightarrow \boldsymbol{\sigma} = J^{-1}\mathbf{P}\mathbf{F}^T$ we have

$$\int_{\Omega(t)} \nabla_x \cdot \boldsymbol{\sigma}\, dV_x = \int_{\Omega(0)} \nabla_\alpha \cdot \mathbf{P}\, dV_\alpha \text{ or}$$

Here $\boldsymbol{\sigma}$ is the symmetric Cauchy stress tensor, and $\mathbf{P}$ is called the *first Piola-Kirchoff* stress tensor. Since $\boldsymbol{\sigma}$ is symmetric,

$$\mathbf{P}\mathbf{F}^T = \mathbf{F}\mathbf{P}^T$$

The tensor $\mathbf{P}$ is itself not generally symmetric. We also have

$$\int_{\partial\mathbf{\Omega}(\mathbf{t})} \boldsymbol{\sigma}\,\hat{\mathbf{n}}\, dS_x = \int_{\partial\mathbf{\Omega}(\mathbf{0})} \mathbf{P}\,\hat{\mathbf{N}}\, dS_\alpha$$

which defines the two stress vectors:

$$\mathbf{s}(\mathbf{x}, t, \hat{\mathbf{n}}) = \boldsymbol{\sigma}(\mathbf{x}, t)\hat{\mathbf{n}} \text{ and } \mathbf{S}(\boldsymbol{\alpha}, t, \hat{\mathbf{N}}) = \mathbf{P}(\boldsymbol{\alpha}, t)\hat{\mathbf{N}}$$

where $\mathbf{s}$ is the (Cauchy) stress relative to the current configuration, and $\mathbf{S}$ is the (first Piola-Kirchoff) stress relative to the reference configuration.

Now, reconsidering balance of momentum,

$$\frac{d}{dt}\int_{\Omega(t)} dV_x \rho(\mathbf{x},t)\,\mathbf{u}(\mathbf{x},t) = \int_{\partial\Omega(t)} dS_x \boldsymbol{\sigma}(\mathbf{x},t)\,\hat{\mathbf{n}} + \int_{\Omega(t)} dV_x \rho(\mathbf{x},t)\mathbf{g}(\mathbf{x},t)$$

or, rewriting everything in Lagrangian variables, i.e. letting $\mathbf{V}(\boldsymbol{\alpha},t) = \mathbf{u}(\mathbf{X}(\boldsymbol{\alpha},t),t)$, $\mathbf{G}(\boldsymbol{\alpha},t) = \mathbf{g}(\mathbf{X}(\boldsymbol{\alpha},t),t)$, and using $\rho(\boldsymbol{\alpha},t)\,J(\boldsymbol{\alpha},t) = \rho_0(\boldsymbol{\alpha})$:

$$\frac{d}{dt}\int_{\Omega(0)} dV_\alpha \rho(\boldsymbol{\alpha},t)\,\mathbf{u}(\mathbf{X}(\boldsymbol{\alpha},t),t)\,J(\boldsymbol{\alpha},t) = \int_{\Omega(0)} dV_\alpha\, \rho_0(\boldsymbol{\alpha})\,\frac{\partial \mathbf{V}}{\partial t}(\boldsymbol{\alpha},t)$$

$$= \int_{\partial\Omega(0)} dS_\alpha\, \mathbf{P}(\boldsymbol{\alpha},t)\,\hat{\mathbf{N}}(\boldsymbol{\alpha}) + \int_{\Omega(0)} dV_\alpha\, \rho_0(\boldsymbol{\alpha})\,\mathbf{G}(\boldsymbol{\alpha},t)$$

And applying the divergence theorem yields:

$$\int_{\Omega(0)} dV_\alpha\, \rho_0(\boldsymbol{\alpha})\,\mathbf{V}_t(\boldsymbol{\alpha},t) = \int_{\Omega(0)} dV_\alpha\, \nabla_\alpha \cdot \mathbf{P}(\boldsymbol{\alpha},t) + \int_{\Omega(0)} dV_\alpha\, \rho_0(\boldsymbol{\alpha})\mathbf{G}(\boldsymbol{\alpha},t)$$

Now using the arbitrariness of $\Omega(0)$, we have

$$\rho_0 \frac{\partial \mathbf{V}}{\partial t} = \nabla_\alpha \cdot \mathbf{P} + \rho_0 \mathbf{G}$$

Very nice.

And so, back to the Neo-Hookean solid:

$$\boldsymbol{\sigma} = -p\mathbf{I} + GJ^{-1}\mathbf{b} \Leftrightarrow \mathbf{P} = -p\mathbf{F}^{-T} + G\mathbf{F}$$

Ignoring incompressibility for the moment, in the Lagrangian frame this yields:

$$\rho_0 \frac{\partial \mathbf{V}}{\partial t} = G\,\nabla_\alpha \cdot \mathbf{F}$$

$$\frac{\partial \mathbf{F}}{\partial t} = \nabla_a \mathbf{V}$$

that is, two coupled *linear* PDEs.

For an incompressible material ($J \equiv 1$), we would have:

$$\rho_0 \frac{D\mathbf{u}}{Dt} = -\nabla_x p + G\nabla_x \cdot \mathbf{b}$$

$$\frac{D\mathbf{b}}{Dt} = \nabla_x \mathbf{u}\,\mathbf{b} + \mathbf{b}\,\nabla_x \mathbf{u}^T$$

$$\nabla \cdot \mathbf{u} = 0$$

For small displacements: $\mathbf{u} \to \varepsilon\mathbf{u}$, $\mathbf{b} = \mathbf{I} + \varepsilon\mathbf{c}$, and expanding to first-order in $\varepsilon$:

$$\rho_0 \mathbf{u}_t = -\nabla_x p + G\,\nabla_x \cdot \mathbf{c}$$

$$\mathbf{c}_t = \nabla_x \mathbf{u} + \nabla_x \mathbf{u}^T = 2\mathbf{E}$$

$$\nabla \cdot \mathbf{u} = 0$$

Taking a time derivative of the first equation and setting $q = p_t$, we have

$$\rho_0 \mathbf{u}_{tt} = -\nabla_x q + G\,\nabla_x \cdot (\nabla_x \mathbf{u})$$

$$= -\nabla_x q + G\Delta\mathbf{u}$$

$$\nabla \cdot \mathbf{u} = 0$$

That is, an "incompressible" wave equation.

## Conservation of Energy

We will come back to this if necessary. Usually associated with non-isothermal situations, which get quite ugly.

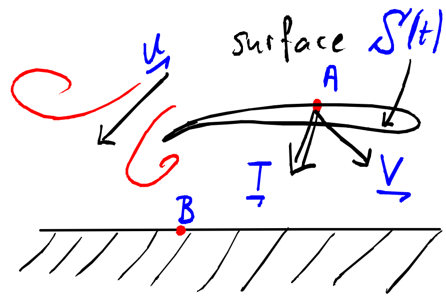
# Back to the Navier-Stokes Eqns and its properties.

Let us assume constant density $\rho$, so that

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla_x p + \mu \Delta \mathbf{u} + \rho \mathbf{g}$$

$$\nabla_x \cdot \mathbf{u} = 0$$

1. Again, the pressure $p$ is determined through satisfying the divergence free condition.
2. Boundary conditions on the N-S equations:




**A body of time-dependent (surface $S(t)$) moves through a fluid above a solid wall.**

- On a solid boundary, as at point $B$ above, we require for a viscous fluid the *no-slip condition:* $\mathbf{u}|_B = \mathbf{0}$. For an inviscid fluid, $\mathbf{u}|_B \cdot \hat{\mathbf{n}}_{wall} = 0$ so that no fluid penetrates the wall.
- On an impenetrable time-dependent body with surface $S(t)$ which has velocity $\mathbf{V}$ and which exerts a stress $\mathbf{T}$ on the fluid, we require that $\mathbf{u}|_{B \in S} = \mathbf{V}$ and $\boldsymbol{\sigma}|_{B \in S}\,\hat{\mathbf{n}} = -\mathbf{T}$.
- At an interface $S(t)$ between two fluids, or at least two continuum materials, with stress tensors $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$, we require $\boldsymbol{\sigma}_1\,\hat{\mathbf{n}} - \boldsymbol{\sigma}_2\,\hat{\mathbf{n}} = \mathbf{T}$ where $\mathbf{T}$ is the surface traction. Typically, $\mathbf{T} = \gamma \kappa \hat{\mathbf{n}}$ for surface tension.

3. The N-S equations have a symmetric stress tensor: $\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$. This guarantees conservation of angular momentum.
4. If no work is done on the system, then N-S has a decaying kinetic energy: Let $\Omega$ be either a fixed closed domain upon whose boundaries the no-slip condition is applied, or all of $R^3$. The kinetic energy is given by

$$K = \frac{1}{2}\int_\Omega \rho \mathbf{u}^2 dV_x$$

and satisfies

$$\dot{K} = \underbrace{-\mu\int_\Omega |\nabla \mathbf{u}|^2 dV_x}_{\text{viscous dissipation to heat}} + \underbrace{\int_\Omega \rho \mathbf{g} \cdot \mathbf{u}}_{\text{work done on the system}}$$

The latter term is zero if the body force arises from a potential. Work can also be done by the time-dependent motion of boundaries in the fluid.

5. The N-S equations are Galilean invariant; that is their form is conserved under the

transformation $\mathbf{u} \to \mathbf{u} + \mathbf{U}$ where $\mathbf{U}$ is a constant velocity.

6. Vorticity, $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, is a fundamental quantity for incompressible flows and has distinctly different dynamics in two and three dimensions:

   - Vorticity transport and diffusion ($\mathbf{g} = \mathbf{0}$) of 2-d fluid in the $x - y$ plane. Here, vorticity is a scalar $\boldsymbol{\omega} = \omega\,\hat{\mathbf{z}}$. Taking a curl of the momentum equation:
   $$\frac{D\omega}{Dt} = \nu\Delta\omega$$
   where $\nu = \mu/\rho$ is called the *kinematic viscosity*. This is an advection-diffusion equation.
   - In 3-d we have instead:
   $$\frac{D\boldsymbol{\omega}}{Dt} = \nabla\mathbf{u}\cdot\boldsymbol{\omega} + \nu\Delta\boldsymbol{\omega}$$
   The extra term, $\nabla\mathbf{u}\cdot\boldsymbol{\omega}$, is the so-called *vorticity stretching term*, and is the term that shows how the vorticity vector field can be amplified or diminished by the local straining flows of the fluid flow, in addition to being advected and diffused. To see this, we recall that $\nabla\mathbf{u} = \mathbf{E} + \mathbf{W}$, where $\mathbf{W}\cdot\mathbf{f} = \boldsymbol{\omega}\times\mathbf{f}$ for any vector $\mathbf{f}$. Hence we have
   $$\frac{D\boldsymbol{\omega}}{Dt} = \mathbf{E}\cdot\boldsymbol{\omega} + \nu\Delta\boldsymbol{\omega}$$
   Recall that the symmetric rate-of-strain tensor $\mathbf{E}$ is trace-free. If $\boldsymbol{\omega}$ is aligned with a principal direction of positive (negative) rate-of-strain, then the magnitude of $\boldsymbol{\omega}$ will be increased (decreased) (neglecting diffusion). What we shall see shortly is that vorticity actually induces the velocity field, and hence the straining flow in which it evolves. This coupling of vorticity stretching/depletion to the vorticity dynamics itself makes the understanding of the 3-d Navier-Stokes equations especially difficult.
   - It is worth examining vorticity transport in the Lagrangian frame in the absence of viscosity. Both equations reflect fundamental conservation laws of the Euler equations. In 2d, the statement $D\omega/Dt = 0$ simply becomes, in the Lagrangian frame:
   $$\omega_t = 0 \;\Rightarrow\; \boxed{\omega(\boldsymbol{\alpha}, t) = \omega_0(\boldsymbol{\alpha})}$$
   or that vorticity is conserved in the Lagrangian frame, that is, along Lagrangian particle paths. The 3d statement is similar, but more complicated. We manipulate the vorticity advection equation in the Lagrangian frame using the evolution equation for the deformation tensor $\mathbf{F}$:
   $$\boldsymbol{\omega}_t = \nabla_x\mathbf{u}\cdot\boldsymbol{\omega} = (\nabla_x\mathbf{u}\cdot\mathbf{F})\mathbf{F}^{-1}\cdot\boldsymbol{\omega}$$
   $$= \mathbf{F}_t\mathbf{F}^{-1}\boldsymbol{\omega} = -\mathbf{F}(\mathbf{F}^{-1})_t\boldsymbol{\omega}$$
   giving the conservation law:
   $$(\mathbf{F}^{-1}\boldsymbol{\omega})_t = 0 \quad\Rightarrow\quad \boldsymbol{\omega}(\boldsymbol{\alpha}, t) = \mathbf{F}(\boldsymbol{\alpha}, t)\boldsymbol{\omega}_0(\boldsymbol{\alpha})$$
   This is the so-called *Result of Cauchy*, which states that vorticity is stretched or depleted by the action of the deformation tensor.

7. The vorticity-stream formulation establishes the relation between velocity and vorticity.

   2D: $\nabla\cdot\mathbf{u} = 0 \Rightarrow \exists\,\psi$ such that $\mathbf{u} = \nabla^{\perp}\psi = (-\psi_y, \psi_x) \Rightarrow \omega = v_x - u_y = \psi_{xx} + \psi_{yy} = \Delta\psi$. For an open flow this then yields the Biot-Savart law.

$$\psi(\mathbf{x}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} dA_{x'} \ln|\mathbf{x}-\mathbf{x}'|\omega(\mathbf{x}') \Rightarrow$$

$$\mathbf{u}(\mathbf{x}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} dA_{x'} \frac{(\mathbf{x}-\mathbf{x}')^{\perp}}{|\mathbf{x}-\mathbf{x}'|}\omega(\mathbf{x}')$$

8. In the Lagrangian frame for an inviscid flow we have

$$\mathbf{u}(\mathbf{X}(\boldsymbol{\alpha},t),t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} dA_{\alpha'} \frac{(\mathbf{X}(\boldsymbol{\alpha},t)-\mathbf{X}(\boldsymbol{\alpha}',\mathbf{t}))^{\perp}}{|\mathbf{X}(\boldsymbol{\alpha},t)-\mathbf{X}(\boldsymbol{\alpha}',t)|}\omega(\mathbf{X}(\boldsymbol{\alpha}',t),t)$$

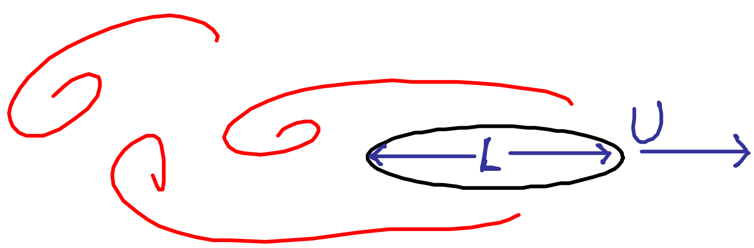
and using the definition of the Lagrangian frame, and conservation of vorticity along particle paths, we have

$$\mathbf{X}_t(\boldsymbol{\alpha},t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} dA_{\alpha'} \frac{(\mathbf{X}(\boldsymbol{\alpha},t)-\mathbf{X}(\boldsymbol{\alpha}',\mathbf{t}))^{\perp}}{|\mathbf{X}(\boldsymbol{\alpha},t)-\mathbf{X}(\boldsymbol{\alpha}',t)|}\omega_0(\boldsymbol{\alpha}')$$

which is a closed set of equations for the Lagrangian flow map (vorticity moves itself).

9. Important, special solutions. In general the nonlinearity of the NS equations, $\mathbf{u}\cdot\nabla\mathbf{u}$, prevents finding analytical solutions, and most know solutions are steady-states for which $\mathbf{u}\cdot\nabla\mathbf{u} = \mathbf{0}$. Most such solutions are *unidirectional flows*.

## The Reynolds Number




The Reynolds number is the most important (or at least the most famous; Wikipedia lists ~70) dimensionless constant in fluid dynamics. Its magnitude quantifies the relative balance of inertial and viscous forces in a fluid. Consider a body of characteristic size $L$ moving with speed $U$ through a Newtonian fluid. This also defines a characteristic time $T = L/U$. Rescale variables as

$$x \to L\,x,\ \ t \to (L/U)t,\ \ u \to U\,u,\ \text{ and } p \to P\,p$$

Then the incompressible NS eqns become:

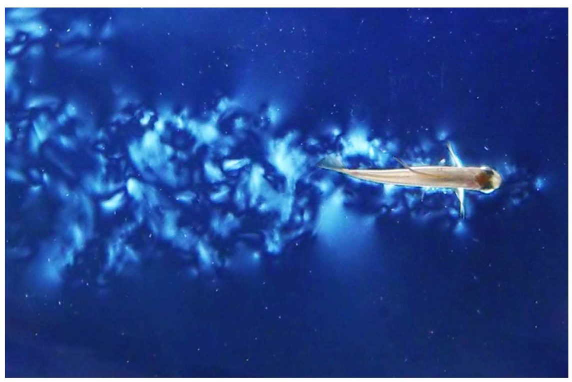
$$Re\,\frac{D\mathbf{u}}{Dt} = -\left(\frac{P/L}{\mu U/L^2}\right)\nabla p + \Delta\mathbf{u} \quad \text{and} \quad \nabla\cdot\mathbf{u} = 0$$

where all variables are without dimension. Note that the divergence free condition remains unchanged. The dimensionless constant

$$Re = \frac{\rho U^2 L^2}{\mu U L} = \frac{\text{"inertial" force}}{\text{"viscous" force}} = \frac{\rho U L}{\mu}$$

is famous Reynolds number. We have left the pressure scale to be determined. Because of the role the pressure plays in satisfying the divergence free condition it is simply scaled to keep it in the dynamics, regardless of what limiting system in Reynolds number is considered. Consider two extreme, but centrally important cases:

- $Re >> 1$, meaning that the fluid dynamics is dominated by the inertial forces of the fluid. This is typical for the locomotion of most birds, fish, whales, etc. In this case, choose $P = Re \cdot \mu U/L$, and we have
$$\frac{D\mathbf{u}}{Dt} = -\nabla p + \frac{1}{Re}\Delta\mathbf{u} \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0$$
Taking the *formal limit* $Re \to \infty$, we get the Euler equations:
$$\frac{D\mathbf{u}}{Dt} = -\nabla p \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0$$
We emphasize that this is a formal limit because in the presence of boundaries, static or dynamic, the no-slip condition is a singular perturbation and makes that limit a possibly singular one; There can be a persistent shedding of vorticity produced at the wall even in the limit of infinite Reynolds number. While the Euler equations retain the convective nonlinear of the NS equations, their lack of diffusion gives their dynamics a great deal of geometric structure that is useful in understanding the structure of solutions, as well as giving special tools, such as potential theory, for constructing special classes of solutions.



- $Re << 1$, meaning that the fluid dynamics is dominated by the viscous forces of the fluid. This is the typical situation for micro-organismal locomotion, transport of small particles of any sort, and indeed any dynamics that takes place on either a sufficiently slow time-scale, or at a sufficiently small spatial scale. In this case, we choose $P = \mu U/L$, which scales the stress tensor uniformly, $\boldsymbol{\sigma} \to (\mu U/L)\boldsymbol{\sigma}$, and gives
$$Re\frac{D\mathbf{u}}{Dt} = -\nabla p + \Delta\mathbf{u} \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0$$
and the formal limit $Re \to 0$ yields the *Stokes equations*:
$$\boxed{-\nabla p + \Delta\mathbf{u} = \mathbf{0} \ \text{ and } \nabla \cdot \mathbf{u} = 0}$$
Note that the Stokes equations are linear, constant coefficient PDEs. For the Stokes equations there is no loss of boundary conditions, unlike the Euler equations, since the highest order spatial term is retained. Unlike either the NS or Euler equations, the Stokes equations are not necessarily solved as an initial value problem as the equations do not contain any time derivatives. They are typically solved as a boundary value problem, where any dynamics devolves from time dependence in boundary data or in solution domain (e.g. as in free boundary problems).

  Note that if there free bodies in the fluid, then the low Reynolds number scaling requires that they exert zero net force and torque upon the surrounding fluid. To see this, a body in the fluid moves through Newton's 2nd law as

$$m_b \ddot{\mathbf{X}}_c = \int_\Gamma dS_x(\boldsymbol{\sigma} \cdot \mathbf{n}), \text{ or in dimensionless units}$$

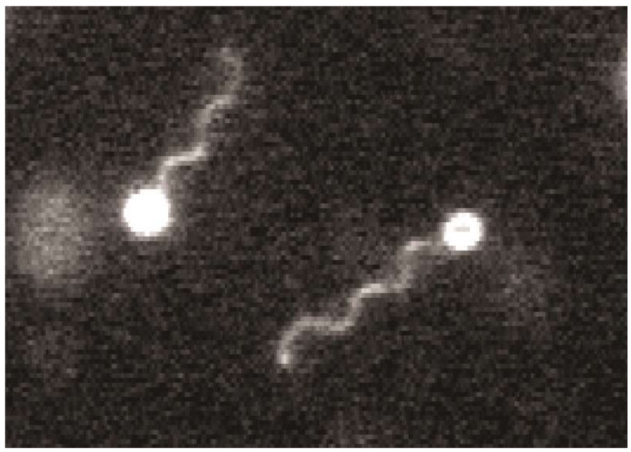
$$\frac{m_b U^2}{L} \ddot{\mathbf{X}}_c = L^2 \int_\Gamma dS_x\left(\frac{\mu U}{L} \boldsymbol{\sigma} \cdot \mathbf{n}\right), \text{ which can be rearranged to yield}$$

$$Re \frac{m_b}{m_f} \ddot{\mathbf{X}}_c = \int_\Gamma dS_x(\boldsymbol{\sigma} \cdot \mathbf{n}) \text{ where } m_f = \rho L^3$$

Hence, if $Re \ll 1$, then the inertial term can be dropped so long as $m_b/m_f$ is not large, and we will generate the constraints

$$\mathbf{F} = \int_\Gamma dS_x(\boldsymbol{\sigma} \cdot \mathbf{n}) = \mathbf{0} \text{ and}$$

$$\mathbf{T} = \int_\Gamma dS_x(\mathbf{X} - \mathbf{X}_c) \times (\boldsymbol{\sigma} \cdot \mathbf{n}) = \mathbf{0}$$



- Moderate Reynolds number. In this regime both inertial and viscous forces are important, and this is a regime that has come under increasing scrutiny, for example in studies of small insect locomotion, and efficient mixing in micro-fluidic devices. In the low and high Reynolds regimes there have been many tools – asymptotic reductions, special numerical methods – that have greatly aided in understanding the fluid dynamics. All of these tools fail in the moderate Reynolds number regime, or must be used at best perturbatively, and theoretical studies have been almost exclusively computational in nature.

| A self-propelled organism | Its Reynolds Number |
|---|---|
| A large whale swimming at 10 m/s | 300,000,000 |
| A tuna swimming at the same speed | 30,000,000 |
| A duck flying at 20 m/s | 300,000 |
| A large dragon fly going 7 m/s | 30,000 |
| A copepod in a speed burst of 0.2 m/s | 300 |
| Flapping wings of the smallest flying insects | 30 |
| An invertebrate larva, 0.3 mm long, at 1 mm/s | 0.3 |
| A sea urchin sperm advancing the species at 0.2 mm/s | 0.03 |
| A bacterium, swimming at 0.01 mm/s | 0.00001 |

Powers of 10 in swimmers.

# The Stokes Equations

The Stokes equations have considerable analytic structure. Again,

$$-\nabla p + \Delta \mathbf{u} = \mathbf{0} \ \text{ and } \ \nabla \cdot \mathbf{u} = 0$$

It is often useful to write them as:

$$\nabla \cdot \boldsymbol{\sigma} = \mathbf{0} \ \text{ and } \ \nabla \cdot \mathbf{u} = 0 \ \text{ with } \ \boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\mathbf{E}$$

Taking a divergence of the momentum equation gives that

$$\Delta p = 0$$

so the pressure is harmonic (in the absence of an external force). Taking a curl of the momentum equations gives that

$$\Delta \boldsymbol{\omega} = \mathbf{0}$$

so the vorticity is also harmonic. As before, the divergence free condition implies the existence of a vector stream function $\boldsymbol{\Psi}$ which satisfies $\Delta\boldsymbol{\Psi} = -\boldsymbol{\omega}$, and hence

$$\Delta^2 \boldsymbol{\Psi} = \mathbf{0}$$

and so the stream-function satisfies a biharmonic equation. This is actually the fundamental nature of the Stokes equations. Take two dimensions as an example. There is then a scalar stream-function and so $\Delta^2\psi = 0$. If we have velocity boundary conditions on a body then we are specifying $\partial\psi/\partial s$ and $\partial\psi/\partial n$, which determines the solution to the biharmonic equation up to a constant, and determines the gradients of $\psi$ uniquely (given appropriate boundary conditions).

## An Application – Lubrication Theory

Here I suppress a discussion thereof, and the derivation of Darcy's Law

# Some exact solutions to the Stokes equations:

## The Stokes solution for a sphere

Consider a sphere of radius $a$ moving at velocity $\mathbf{U} = U\hat{\boldsymbol{\zeta}}$. George Stokes showed that the fluid stress everywhere on the sphere is given by

$$\mathbf{s} = \boldsymbol{\sigma} \cdot \mathbf{n} = \mu \frac{3U}{2a} \hat{\boldsymbol{\zeta}}$$

where the polar axis of the sphere is taken in the $\hat{\boldsymbol{\zeta}}$ direction, and the far-field pressure has been set to zero. We then have for the force $\mathbf{F}$ on the sphere the famous Stokes formula:

$$\mathbf{F} = \int_S dA\, \mathbf{s} = 6\pi\mu a \mathbf{U}$$

## The Jeffrey equation for ellipsoidal particles

Consider an axisymmetric ellipsoid of length $l$ and diameter $d$ rotating in a linear flow $\mathbf{u} = \mathbf{U} + \mathbf{A}\mathbf{x}$ so that $\nabla\mathbf{u} = \mathbf{A} = \mathbf{W} + \mathbf{E}$. Let the unit vector $\mathbf{p}(t)$ point in the direction of the major axix, $\mathbf{X}_c(t)$ be the ellipsoid center, and assume that no force or torque acts upon the ellipsoid. Then (Jeffrey, 1922)

$$\begin{aligned}
\dot{\mathbf{X}}_c &= \mathbf{U} + \mathbf{A}\mathbf{X}_c \\
\dot{\mathbf{p}} &= \mathbf{W}\mathbf{p} + \frac{\lambda^2 - 1}{\lambda^2 + 1}(\mathbf{I} - \mathbf{p}\mathbf{p})\mathbf{E}\mathbf{p} \\
&= (\mathbf{I} - \mathbf{p}\mathbf{p})\left[\mathbf{W} + \frac{\lambda^2 - 1}{\lambda^2 + 1}\mathbf{E}\right]\mathbf{p}
\end{aligned}$$

with $\lambda = l/d$.

- *Sphere:* $\lambda = 1 \Rightarrow \dot{\mathbf{p}} = \mathbf{W}\mathbf{p} = \frac{1}{2}\boldsymbol{\omega} \times \mathbf{p}$. Rotation of the director about the vorticity vector. The strain flow contributes nothing to the rotation of the sphere.

  **Sidenote**: Consider the tensor $\mathbf{Q} = \mathbf{p}\mathbf{p}^T$. Then

  $$\begin{aligned}\dot{\mathbf{Q}} = \dot{\mathbf{p}}\mathbf{p}^T + \mathbf{p}\dot{\mathbf{p}}^T &= \mathbf{W}\mathbf{Q} - \mathbf{Q}\mathbf{W}, \text{ or}\\ \dot{\mathbf{Q}} + \mathbf{Q}\mathbf{W} - \mathbf{W}\mathbf{Q} &= 0\end{aligned}$$

  $\overset{\triangle}{\mathbf{Q}} = \dot{\mathbf{Q}} + \mathbf{Q}\mathbf{W} - \mathbf{W}\mathbf{Q}$ is called the Jaumann or co-rotational derivative.
- *Slender rod:* $\lambda = \infty \Rightarrow \dot{\mathbf{p}} = (\mathbf{I} - \mathbf{p}\mathbf{p})\nabla\mathbf{u}\,\mathbf{p}$. Rotated by the flow, but constrained from stretching.
- *Plate:* $\lambda = 0 \Rightarrow \dot{\mathbf{p}} = (\mathbf{I} - \mathbf{p}\mathbf{p})[\mathbf{W} - \mathbf{E}]\mathbf{p} = -(\mathbf{I} - \mathbf{p}\mathbf{p})\nabla\mathbf{u}^T\mathbf{p}$

## Fundamental Solutions of the Stokes Equations

Because the Stokes equations are constant coefficient linear PDEs, solutions to them can be represented in terms of Green's functions. There are several important fundamental solutions for the Stokes equations, such as the Stokeslet, Rotlet, and Stresslet.

*Formal Construction of the Stokeslet:* Find a solution to the equation

$$\nabla \cdot \tilde{\boldsymbol{\sigma}} = -\nabla q + \mu\Delta\mathbf{v} = -\hat{\mathbf{e}}\,\delta(\mathbf{x}) \text{ and } \nabla \cdot \mathbf{v} = 0$$

where $\hat{\mathbf{e}}$ is an arbitrary unit vector, and $\delta$ is the 3-d $\delta$-function. Recall that the 3d free-space Green's function for the Laplacian is

$$G = \frac{-1}{4\pi}\frac{1}{|\mathbf{x}|}$$

i.e., $\Delta G = \delta(x)$. Taking a divergence gives $\Delta q = \hat{\mathbf{e}} \cdot \nabla\delta = \hat{\mathbf{e}} \cdot \nabla\Delta G = \Delta(\hat{\mathbf{e}} \cdot \nabla G)$ and so we choose

$$q = \hat{\mathbf{e}} \cdot \nabla G = \frac{1}{4\pi}\frac{\mathbf{x}}{|\mathbf{x}|^3} \cdot \hat{\mathbf{e}} = \frac{1}{4\pi}\frac{\hat{\mathbf{x}}}{|\mathbf{x}|^2} \cdot \hat{\mathbf{e}} = P_k \hat{e}_k$$

Hence, the fundamental solution for the pressure is

$$P_k = \frac{1}{4\pi}\frac{\hat{x}_k}{|\mathbf{x}|^2}$$

We then have

$$\nabla q = \frac{1}{4\pi}\frac{1}{|\mathbf{x}|^3}[\mathbf{I} - 3\hat{\mathbf{x}}\hat{\mathbf{x}}]\hat{\mathbf{e}}$$

$$\Rightarrow \mu\Delta\mathbf{v} = -\hat{\mathbf{e}}\delta + \nabla(\hat{\mathbf{e}} \cdot \nabla G)$$

Now we construct two functions $B_1,\ B_2$ that satisfy

$$\Delta B_1 = \delta \ \ \& \ \ \Delta B_2 = G$$

and let $\mathbf{v} = \frac{1}{\mu} - \hat{\mathbf{e}}B_1 + \nabla(\hat{\mathbf{e}} \cdot \nabla B_2)$. Now, plucking out only the radially symmetric particular solutions for $B_{1,2}$ gives:

$$B_1 = G \ \text{ and } \ B_2 = \frac{-1}{8\pi}|\mathbf{x}|$$

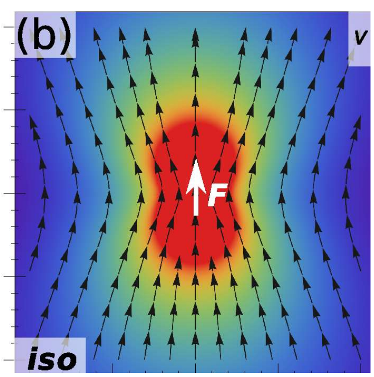
where further calculation gives

$$\mathbf{v} = \frac{1}{8\pi\mu}\left[\frac{\mathbf{I} + \hat{\mathbf{x}}\hat{\mathbf{x}}}{|\mathbf{x}|}\right]\hat{\mathbf{e}} = \mathbf{S}(\mathbf{x})\hat{\mathbf{e}}$$

The rank-two tensor

$$\mathbf{S} = \frac{1}{8\pi\mu}\left[\frac{\mathbf{I} + \hat{\mathbf{x}}\hat{\mathbf{x}}}{|\mathbf{x}|}\right] \ \text{ or } \ S_{ik} = \frac{1}{8\pi\mu}\left[\frac{\delta_{ik} + \hat{x}_i\hat{x}_k}{|\mathbf{x}|}\right]$$

is called the *Stokeslet* or the *Oseen tensor*. It has a long-range $R^{-1}$ decay and is a negative definite matrix. It can be used to construct other relevant fundamental solutions.




The Stokeslet flow field

We define the *Stresslet* as the rank-three tensor $T_{ijk}$ satisfying $\tilde{\sigma}_{ij} = T_{ijk}\hat{e}_k$, or

$$T_{ijk} = -P_k\delta_{ij} + \mu\left(\frac{\partial S_{ik}}{\partial x_j} + \frac{\partial S_{jk}}{\partial x_i}\right)$$
$$= -\frac{3}{4\pi}\frac{\hat{x}_i\hat{x}_j\hat{x}_k}{|\mathbf{x}|^2}$$

**Sidenote**: Let's calculate the response to a anti-aligned force dipole. We write the Stokeslet as satisfying a linear functional equation where $\delta(\mathbf{x})\mathbf{e}$ is the force applied **to the fluid** (this is important for interpretation!):

$$\mathsf{L}[\mathbf{u} = \mathbf{S}(\mathbf{x})\mathbf{e}] + \delta(\mathbf{x})\mathbf{e} = \mathbf{0} \Leftrightarrow \mathsf{L}[u_i = -S_{ij}(\mathbf{x})e_j] = -\delta(\mathbf{x})e_i$$

and form

$$\mathsf{L}\left[u_i = \frac{S_{ij}(\mathbf{x}+h\mathbf{e}) - S_{ij}(\mathbf{x}-h\mathbf{e})}{2h}e_j\right] + \frac{\delta(\mathbf{x}+h\mathbf{e}) - \delta(\mathbf{x}-h\mathbf{e})}{2h}e_i = 0$$

and let $h \to 0$. This yields

$$\mathsf{L}\left[u_i = e_k\frac{\partial S_{ij}}{\partial x_k}(\mathbf{x})e_j\right] = -\frac{\partial\delta}{\partial x_j}(\mathbf{x})e_je_i = -\frac{\partial}{\partial x_j}[\delta(\mathbf{x})e_ie_j] = -\nabla\cdot(\delta(\mathbf{x})\mathbf{e}\mathbf{e}^T)$$

Hence, I can interpret the consequent velocity $\mathbf{u}$ as being in response to an extra-stress $\boldsymbol{\sigma}^e = \delta(\mathbf{x})\mathbf{e}\mathbf{e}^T$.

$$\frac{\partial S_{ij}}{\partial x_k}(\mathbf{x})e_je_k = \frac{1}{8\pi\mu}\frac{\partial}{\partial x_k}\left[\frac{\delta_{ij}}{|\mathbf{x}|} + \frac{x_ix_j}{|\mathbf{x}|^3}\right]e_je_k = \frac{1}{8\pi\mu}\left[-\frac{\delta_{ij}\hat{x}_k}{|\mathbf{x}|^2} + \frac{\delta_{ik}\hat{x}_j + \hat{x}_i\delta_{jk}}{|\mathbf{x}|^2} - \frac{3\hat{x}_i\hat{x}_j\hat{x}_k}{|\mathbf{x}|^2}\right]e_je_k$$
$$= \frac{1}{8\pi\mu}\left[-\frac{\hat{x}_ke_k}{|\mathbf{x}|^2}e_i + \frac{(e_j\hat{x}_j)e_i + \hat{x}_i}{|\mathbf{x}|^2} - \frac{3(e_j\hat{x}_j)(e_k\hat{x}_k)}{|\mathbf{x}|^2}\hat{x}_i\right]$$

or, expressing the consequent velocity $u_i$:

$$\mathbf{u} = \frac{1}{8\pi\mu}\frac{1-3(\mathbf{e}\cdot\hat{\mathbf{x}})^2}{|\mathbf{x}|^2}\hat{\mathbf{x}}$$

This is a contractile dipolar flow streaming **inwards** along the $\mathbf{e}$-axis, as is consistent with the form of the extra-stress and of the forcing.

**End sidenote**

# The Lorentz Identity

A fundamental identity satisfied by any two solutions $(\boldsymbol{\sigma}, \mathbf{u})$ and $(\tilde{\boldsymbol{\sigma}}, \mathbf{v})$ of the Stokes equation is the Lorentz identity:

$$\nabla\cdot\left(\boldsymbol{\sigma}\,\mathbf{v} - \tilde{\boldsymbol{\sigma}}\mathbf{u}\right) = 0 \text{ or } \frac{\partial}{\partial x_k}(\sigma_{ki}v_i - \tilde{\sigma}_{ki}u_i) = 0$$

Using symmetry of the stress tensor, we can write:

$$\nabla\cdot(\boldsymbol{\sigma}\mathbf{v} - \tilde{\boldsymbol{\sigma}}\mathbf{u}) = \boldsymbol{\sigma} : \nabla\mathbf{v} - \tilde{\boldsymbol{\sigma}} : \nabla\mathbf{u}$$
$$= [-p\mathbf{I} + 2\mu\mathbf{E}_u]:[\mathbf{E}_v + \mathbf{W}_v] - [-q\mathbf{I} + 2\mu\mathbf{E}_v]:[\mathbf{E}_u + \mathbf{W}_u] = 0$$

Integrating this identity in the exterior of a moving body with boundary $\Gamma(t)$ and exterior normal $\mathbf{n}$, and using the divergence theory yields the integral form

$$\int_{\Gamma(t)} dS\,\mathbf{n}\,\boldsymbol{\sigma}\,\mathbf{v} = \int_{\Gamma(t)} dS\,\mathbf{n}\,\tilde{\boldsymbol{\sigma}}\,\mathbf{u} \Leftrightarrow \int_{\Gamma(t)} dS\,\mathbf{s}\cdot\mathbf{v} = \int_{\Gamma(t)} dS\,\tilde{\mathbf{s}}\cdot\mathbf{u}$$

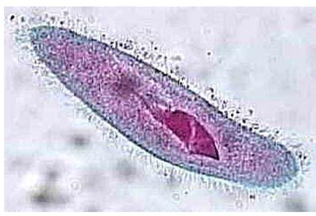
The Lorentz identity is extremely useful:

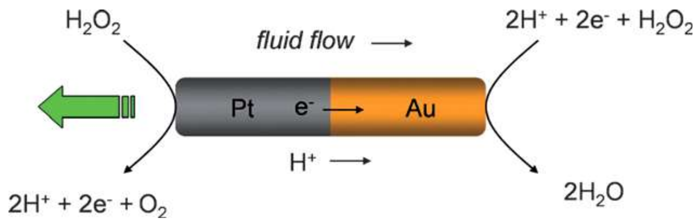
## (1) The speed of a Blake squirmer (Stone & Samuel *PRL* '96)

Consider a freely-moving spherical body upon which a surface velocity $\mathbf{u}_b(\mathbf{y},t)$ (body frame) is prescribed so that $\mathbf{u}^\Gamma = \mathbf{U}(t) + \mathbf{u}_b(\mathbf{y},t)$ (in the lab frame) for $\mathbf{y} \in \Gamma$. Its total force is zero. Let's say the body is a sphere of radius $a$. Then, use Stokes solution $(\tilde{\mathbf{s}}, \tilde{\mathbf{v}})$ for a sphere to derive:

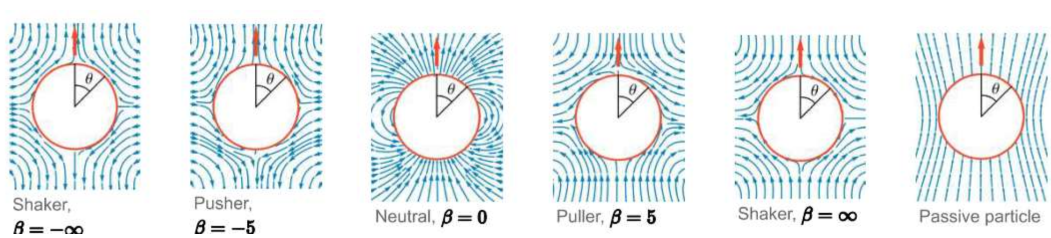
$$\mathbf{U} = -\frac{1}{4\pi a^2}\int_\Gamma dS_y \mathbf{u}_b(\mathbf{y},t)$$

Blake squirmer models (1971): the surface velocity $\mathbf{u}_b$ is tangential and axisymmetric, with $\mathbf{u}_b(\phi) = u_\phi \hat{\boldsymbol{\phi}}$, with $\phi$ the polar angle and $u_\phi = B_1 \sin\phi + B_2 \sin 2\phi$. $\beta = |B_1|/B_2$ and $\mathbf{U} = \alpha B_1 \hat{\mathbf{z}}$ (independent of $B_2$). This is originally a model for paramecia and now of Janus particles, and gives a trivial example of Stokes reversibility.

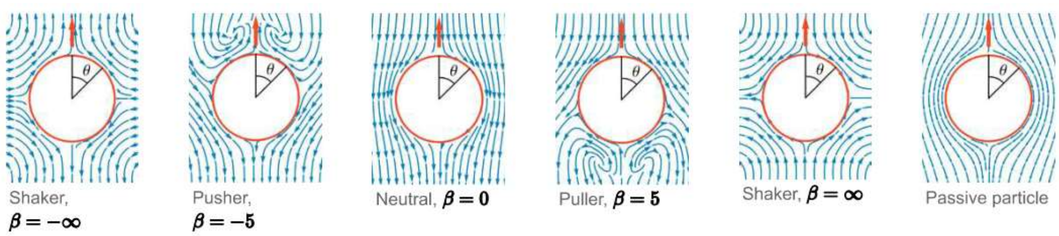





Blake squirmers in the lab frame:




in the swimmer frame (tangential flows):




## The Classical Boundary Integral Formulation

The Stokeslet and Stresslet can be used to construct a boundary integral representation for solutions of the Stokes equations, which we very roughly outline (see Pozrikidis for a more detailed derivation). Consider an infinite fluid domain $\Omega$ containing a closed body $B$ with surface $\Gamma$ and outer normal $\hat{\mathbf{n}}$, and with a surface stress distribution $\boldsymbol{\zeta}$ (acting upon the fluid) and surface velocity $\mathbf{u}_\Gamma$. Let $(\boldsymbol{\sigma}, \mathbf{u})$ be the Stokes solution satisfying $\boldsymbol{\sigma}|_\Gamma \hat{\mathbf{n}} = -\boldsymbol{\zeta}$ and $\mathbf{u}|_\Gamma = \mathbf{u}^\Gamma$.

Now let $(\tilde{\boldsymbol{\sigma}}, \mathbf{v})$ be the Stresslet/Stokeslet pair translated to place their singularities at a point $\mathbf{y}$ in the fluid domain. Then, integrate the Lorentz identity over the punctured fluid domain $\Omega / D_\varepsilon(\mathbf{y})$ (with normal into the domain) where $D_\varepsilon(\mathbf{y})$ is the $\varepsilon$-ball about $\mathbf{y}$, hence excluding the singular point from the domain. The divergence theorem then gives

$$0 = \int_{\Gamma+\{|\mathbf{x}'-\mathbf{y}|=\varepsilon\}} [v_i(\mathbf{x}')\sigma_{ik}(\mathbf{x}')n_k(\mathbf{x}') - u_i(\mathbf{x}')\tilde{\sigma}_{ik}(\mathbf{x}')n_k(\mathbf{x}')]dA_{x'}$$

(1) On $|\mathbf{x}-\mathbf{y}| = \varepsilon$: Note that on the boundary of $D_\varepsilon(\mathbf{y})$, $\widehat{\mathbf{x}'-\mathbf{y}} = \mathbf{n}$,

$$\int_{|\mathbf{x}'-\mathbf{y}|=\varepsilon} v_i(\mathbf{x}')\sigma_{ik}(\mathbf{x}')n_k(\mathbf{x}')dA_{x'} = \int_{|\mathbf{x}'-\mathbf{y}|=\varepsilon} S_{ij}(\mathbf{x}'-\mathbf{y})\sigma_{ik}(\mathbf{x}')n_k(\mathbf{x}')dA_{x'}$$
$$= \frac{1}{8\pi\mu}\int_{|\mathbf{x}'-\mathbf{y}|=\varepsilon} \frac{\delta_{ij}+n_in_j}{\varepsilon}\sigma_{ik}(\mathbf{x}')n_k(\mathbf{x}')dA_{x'} \to 0 \text{ as } \varepsilon \to 0$$

since the area element scales as $\varepsilon^2$. Now the second term is given by:

$$-\int_{\Gamma+\{|\mathbf{x}-\mathbf{y}|=\varepsilon\}} u_i(\mathbf{x}')\tilde{\sigma}_{ik}(\mathbf{x}')n_k(\mathbf{x}')dA_{x'} = \frac{3}{4\pi}\int_{|\mathbf{x}-\mathbf{y}|=\varepsilon} u_i\frac{n_in_jn_k}{\varepsilon^2}n_kdA_{x'} = \frac{3}{4\pi}\varepsilon^{-2}\int_{|\mathbf{x}-\mathbf{y}|=\varepsilon} u_in_in_jdA_{x'}$$
$$= \frac{3}{4\pi}\varepsilon^{-2}\varepsilon^2\int_0^{2\pi}d\theta\int_0^{\pi}d\phi\sin\phi(u_i(\mathbf{y})+O(\varepsilon))n_i(\theta,\phi)n_j(\theta,\phi)$$
$$= \frac{3}{4\pi}\int_0^{2\pi}d\theta\int_0^{\pi}d\phi\sin\phi\,[\mathbf{n}\,\mathbf{n}]\mathbf{u}(\mathbf{y}) + O(\varepsilon)$$
$$\to \frac{3}{4\pi}\int_0^{2\pi}d\theta\int_0^{\pi}d\phi\sin\phi\begin{bmatrix}\cos^2\theta\sin^2\phi & 0 & 0\\ 0 & \sin^2\theta\sin^2\phi & 0\\ 0 & 0 & \cos^2\phi\end{bmatrix}\mathbf{u}(\mathbf{y})$$
$$= u_j(\mathbf{y})$$

This yields

$$0 = u_j(\mathbf{y}) + \int_\Gamma [S_{ij}(\mathbf{x}'-\mathbf{y})\sigma_{ik}(\mathbf{x}')n_k(\mathbf{x}') - u_i^\Gamma(\mathbf{x}')T_{ijk}(\mathbf{x}'-\mathbf{y})n_k(\mathbf{x}')]dA_{x'}$$
$$= u_j(\mathbf{y}) - \int_\Gamma [S_{ij}(\mathbf{x}'-\mathbf{y})\zeta_i(\mathbf{x}') - u_i^\Gamma(\mathbf{x}')T_{ijk}(\mathbf{x}'-\mathbf{y})n_k(\mathbf{x}')]dA_{x'}$$

One can thus show that:

$$u_j(\mathbf{y}) = \int_\Gamma [S_{ji}(\mathbf{x}'-\mathbf{y})\zeta_i(\mathbf{x}') + u_i^\Gamma(\mathbf{x})T_{ijk}(\mathbf{x}'-\mathbf{y})n_k(\mathbf{x})]dA_x$$

or in nicer notation

$$\boxed{\mathbf{u}(\mathbf{y}) = \int_\Gamma [\mathbf{S}(\mathbf{x}'-\mathbf{y})\boldsymbol{\zeta}(\mathbf{x}') + \mathbf{u}^\Gamma(\mathbf{x}')\mathbf{T}(\mathbf{x}'-\mathbf{y})\hat{\mathbf{n}}(\mathbf{x}')]dA_x'} \checkmark$$

Note that **S** is even wrt its argument, while **T** is odd. So, in a more standard convolution form we would have

$$\mathbf{u}(\mathbf{y}) = \int_\Gamma [\mathbf{S}(\mathbf{x}'-\mathbf{y})\boldsymbol{\zeta}(\mathbf{x}') - \mathbf{u}^\Gamma(\mathbf{x}')\mathbf{T}(\mathbf{y}-\mathbf{x}')\hat{\mathbf{n}}(\mathbf{x}')]dA_{x'}$$

Hence, we have expressed the velocity at every point in the fluid as a function of the surface stress and velocity. Of course the surface velocity and the fluid velocity are related by the no-slip condition, and so it remains to take the limit $\mathbf{y} \to \mathbf{x} \in \Gamma$. The hard one is the stresslet, so let's do that one first. The dominant part of the limit to the surface should arise from this integral:

$$I = \int_{|\mathbf{x}-\mathbf{x}_0|\le\varepsilon} u_i(\mathbf{x})T_{ijk}(\mathbf{x}-\mathbf{y})n_k(\mathbf{x})dA_x$$

where $\mathbf{x}_0$ is the closest point to $\mathbf{y}$. Let's replace the $\varepsilon$ –patch with a flat disk, and assume that $\mathbf{y}-\mathbf{x} = r\,\mathbf{n}_0 + \rho\,\mathbf{R}_0\,\mathbf{e}(\theta)$ where $\mathbf{R}_0$ is a rotation matrix ($\mathbf{R}_0\hat{\mathbf{z}} = \mathbf{n}_0$) and $\mathbf{e}(\theta) = (\cos\theta, \sin\theta, 0)$. That is, this is a little $(\rho,\theta)$ coordinate system on the patch. Then

$$\widehat{\mathbf{y}-\mathbf{x}} = \frac{r\,\mathbf{n}_0 + \rho\,\mathbf{R}_0\,\mathbf{e}}{(r^2+\rho^2)^{1/2}}$$

:

$$\begin{aligned}
I &= \frac{3}{4\pi}\int_0^{2\pi} d\theta \int_0^{\varepsilon} d\rho\,\rho\,\frac{\mathbf{u}\cdot(r\mathbf{n}_0+\rho\mathbf{R}_0\mathbf{e})\;\mathbf{n}_0\cdot(r\mathbf{n}_0+\rho\mathbf{R}_0\mathbf{e})}{(r^2+\rho^2)^{5/2}}(r\mathbf{n}_0+\rho\mathbf{R}_0\mathbf{e})\\
&= \frac{3}{4\pi}r\int_0^{2\pi} d\theta \int_0^{\varepsilon} d\rho\,\rho\,\frac{\mathbf{u}\cdot(r\mathbf{n}_0+\rho\mathbf{R}_0\mathbf{e})}{(r^2+\rho^2)^{5/2}}(r\mathbf{n}_0+\rho\mathbf{R}_0\mathbf{e})\\
&= \frac{3}{4\pi}r\int_0^{2\pi} d\theta \int_0^{\varepsilon} d\rho\,\rho\,\frac{r^2(\mathbf{u}\cdot\mathbf{n}_0)\mathbf{n}_0+\rho^2(\mathbf{u}\cdot\mathbf{R}_0\mathbf{e})\mathbf{R}_0\mathbf{e}}{(r^2+\rho^2)^{5/2}}
\end{aligned}$$

Ok, we need to calculate

$$\begin{aligned}
\int_0^{2\pi} d\theta(\mathbf{u}\cdot\mathbf{R}_0\mathbf{e})\mathbf{R}_0\mathbf{e} &= \mathbf{R}_0\left(\int_0^{2\pi} d\theta\,\mathbf{e}\mathbf{e}\right)\mathbf{R}_0^T\mathbf{u}\\
&= \pi\mathbf{R}_0(\mathbf{I}-\hat{\mathbf{z}}\hat{\mathbf{z}}^T)\mathbf{R}_0^T\mathbf{u}\\
&= \pi(\mathbf{u}-(\mathbf{u}\cdot\mathbf{n}_0)\mathbf{n}_0)
\end{aligned}$$

And so

$$\begin{aligned}
I &= -\frac{3}{4}r\int_0^{\varepsilon} d\rho\,\rho\,\frac{2r^2(\mathbf{u}\cdot\mathbf{n}_0)\mathbf{n}_0+\rho^2(\mathbf{u}-(\mathbf{u}\cdot\mathbf{n}_0)\mathbf{n}_0)}{(r^2+\rho^2)^{5/2}}\\
&= -\frac{3}{2}\int_0^{\varepsilon} d(\rho/r)\,\frac{(\rho/r)}{\left(1+(\rho/r)^2\right)^{5/2}}(\mathbf{u}\cdot\mathbf{n}_0)\mathbf{n}_0 - \frac{3}{4}\int_0^{\varepsilon} d(\rho/r)\frac{(\rho/r)^3}{\left(1+(\rho/r)^2\right)^{5/2}}(\mathbf{u}-(\mathbf{u}\cdot\mathbf{n}_0)\mathbf{n}_0)\\
&= -\frac{3}{2}\int_0^{\delta}\frac{x}{(1+x^2)^{5/2}}dx\,(\mathbf{u}\cdot\mathbf{n}_0)\mathbf{n}_0 - \frac{3}{4}\int_0^{\delta}\frac{x^3}{(1+x^2)^{5/2}}dx\,(\mathbf{u}-(\mathbf{u}\cdot\mathbf{n}_0)\mathbf{n}_0)\ ;\ \delta = \varepsilon/r\\
&= -\frac{1}{2}\frac{(\delta^2+1)^{\frac{3}{2}}-1}{(\delta^2+1)^{\frac{3}{2}}}(\mathbf{u}\cdot\mathbf{n}_0)\mathbf{n}_0 + \frac{1}{4}\frac{1}{(\delta^2+1)^{\frac{3}{2}}}\left(3\delta^2-2(\delta^2+1)^{\frac{3}{2}}+2\right)(\mathbf{u}-(\mathbf{u}\cdot\mathbf{n}_0)\mathbf{n}_0)
\end{aligned}$$

Now, we need to take $r\to 0$ for $\varepsilon$ fixed, that is, $\delta\to\infty$. This yields

$$I = -\frac{1}{2}(\mathbf{u}\cdot\mathbf{n}_0)\mathbf{n}_0 - \frac{1}{2}(\mathbf{u}-(\mathbf{u}\cdot\mathbf{n}_0)\mathbf{n}_0) = -\frac{1}{2}\mathbf{u}$$

And so, in this limit we have

$$\begin{aligned}
u_j(\mathbf{x}) &= \int_\Gamma S_{ji}(\mathbf{x}'-\mathbf{x})\zeta_i(\mathbf{x}')dA_{x'}\\
&\frac{1}{2}u_j(\mathbf{x}) + P\int_\Gamma u_i(\mathbf{x}')T_{ijk}(\mathbf{x}'-\mathbf{x})n_k(\mathbf{x}')dA_{x'}
\end{aligned}$$

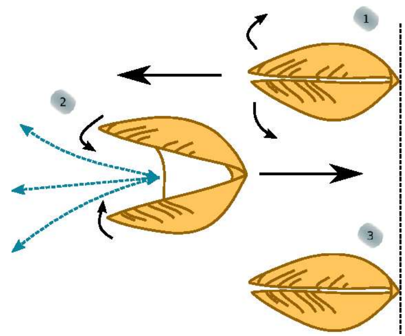
$$\boxed{\tfrac{1}{2}\mathbf{u}^\Gamma(\mathbf{x}) = \int_\Gamma \mathbf{S}(\mathbf{x}'-\mathbf{x})\boldsymbol{\zeta}(\mathbf{x}')dA'_x + P\int_\Gamma \mathbf{u}^\Gamma(\mathbf{x}')\mathbf{T}(\mathbf{x}'-\mathbf{x})\hat{\mathbf{n}}(\mathbf{x}')dA'_x}\ \checkmark$$

or, as an integral equation in convolution form

$$\boxed{\tfrac{1}{2}\mathbf{u}^\Gamma(\mathbf{x}) + P\int_\Gamma \mathbf{u}^\Gamma(\mathbf{x}')\mathbf{T}(\mathbf{x}-\mathbf{x}')\hat{\mathbf{n}}(\mathbf{x}')dS'_x = \int_\Gamma \mathbf{S}(\mathbf{x}-\mathbf{x}')\boldsymbol{\zeta}(\mathbf{x}')dS'_x}$$

or a Fredholm integral equation of the second kind for the surface velocity, or a first-kind equation for the surface stress. If we are given the surface stress, then this equation can in principal be solved for the surface velocity, and then used to give the fluid velocity everywhere in the fluid domain. This is one of the fundamental relations of the Stokes equations.

**Comment**: Establishing this linear integral relation between the surface velocity and the surface stress establishes Purcell's Scallop Theorem (1977), which states that if a freely moving body goes through a set of surface deformations (which may not lead to shape changes – this is a statement about the motion of material points of the surface) on the time interval $[0, T]$ and then exactly reverses that set of surface deformations on $[T, 2T]$, the body will end up just where it started (that is, in the same position and orientation). Further, the rate at which the deformation take place is not important. One could speed up time on the second part of the cycle, and displacement would still be zero. Establishing this theorem is intuitively obvious but technically subtle (see K. Ishimoto, SIAP 2012).




"Freely moving" means here that there are no external forces, and so in this regime we have the constraints:

$$\int_\Gamma dS_x \boldsymbol{\zeta}(\mathbf{x}) = \int_\Gamma dS_x (\mathbf{x} - \mathbf{X}_c) \times \boldsymbol{\zeta}(\mathbf{x}) = \mathbf{0}$$

which become linear equations for translational and rotational velocities. The scallop-theorem is broken by nonlinear fluid rheology, multiple bodies, fluid elasticity. Still, it establishes why one doesn't typically see time reversible motion of swimmers at low Reynolds number.

Now, for further illustration, consider a rigid body moving under an applied force **F** and torque **L**, that is

$$\int_\Gamma dS_x\, \boldsymbol{\zeta}(\mathbf{x})\hat{\mathbf{n}}(\mathbf{x}) = \mathbf{F} \text{ and } \int_\Gamma dS_x (\mathbf{x} - \mathbf{X}_c) \times \boldsymbol{\zeta}(\mathbf{x}) = \mathbf{L}$$

and being rigid means that for the surface velocity $\mathbf{u}^\Gamma = \mathbf{U} + (\mathbf{x} - \mathbf{X}_c(t)) \times \boldsymbol{\Omega}(t)$. Before inserting this into the integral equation we note two identities for $\mathbf{x} \in \Omega$ (the fluid domain):

$$P\int_\Gamma \mathbf{v}\mathbf{T}(\mathbf{x} - \mathbf{x}')\hat{\mathbf{n}}(\mathbf{x}')dS_{x'} = \mathbf{0} \text{ for any constant vector } \mathbf{v}.$$

$$P\int_\Gamma x'_l T_{ijk}(\mathbf{x} - \mathbf{x}')n_k(\mathbf{x}')dS_{x'} = 0$$

or

$$\boxed{\mathbf{u}(\mathbf{x}) = \int_\Gamma \mathbf{S}(\mathbf{x} - \mathbf{x}')\boldsymbol{\zeta}(\mathbf{x}')dS'_x}$$

which means that for rigid bodies, taking the limit $\mathbf{x} \to \Gamma$, we have:

$$\boxed{\mathbf{U} + (\mathbf{x} - \mathbf{X}_c(t)) \times \boldsymbol{\Omega}(t) = \int_\Gamma \mathbf{S}(\mathbf{x} - \mathbf{x}')\boldsymbol{\zeta}(\mathbf{x}')dS_x/}$$

which is an integral equation for $\boldsymbol{\zeta}$ in terms of the two unknowns $\mathbf{U}$ and $\boldsymbol{\Omega}$. The system is closed by the specification of the force and torque. The body is then evolved via

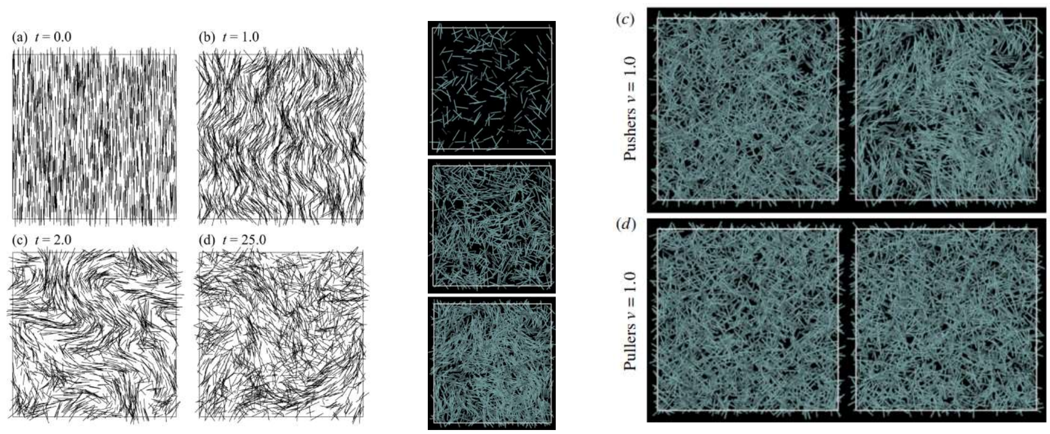
$$\dot{\mathbf{X}}_c = \mathbf{U} \text{ and } \dot{\boldsymbol{\omega}} = \boldsymbol{\Omega}$$

Note however that this is essentially a first-kind integral equation for the surface stress $\boldsymbol{\zeta}$, and is widely used but ill-conditioned.

### The formulation of Power & Miranda

This I am suppressing to a footnote:

## Rod-like swimmers




Three observations from 3D simulations of rod-like swimmers into the semi-dilute regime $\left(N\,l^3 \sim L^3\right)$.

- Initially aligned suspensions of swimmers (Pushers or Pullers) become disaligned (SS PRL 2007)
- For Pusher suspensions there seems to be a critical concentration at which the uniform isotropic state becomes unstable (SS JSRI 2014)
- Puller suspensions remain in a resolutely uniform and isotropic state (SS JRSI 2014).

## Slender-Body Theory

See Tornberg & Shelley (J. Comp. Phys. 196, 8-40 (2004); TS2004) for discussion and references (most especially Keller & Rubinow ( JFM 1976), Johnson (JFM 1980), and Gotz (PhD thesis 2000)) on nonlocal slender-body theory.

**The leading-order Local SBT**: Consider a slender rod of length $l$, with center-line $\mathbf{X}(s,t)$, and radius $r(s) = 2\varepsilon\sqrt{s^2 - l^2/4}$ (ellipsoidal ends) so that $r(s=0) = \varepsilon l$. Take $\varepsilon = a/l << 1$. It moves in a background flow $\mathbf{u}(\mathbf{x},t)$ and has center-line velocity $\mathbf{V}(s)$. Let $f(s) = r(s)\int_0^{2\pi} d\theta\, \boldsymbol{\xi}(r(s),\theta)$ be the force per unit length exerted upon the fluid by the fiber. Then

$$\eta[\mathbf{V}(s) - \mathbf{u}(\mathbf{X}(s))] = (\mathbf{I} + \mathbf{X}_s\mathbf{X}_s)\mathbf{f}(s) + \left[\text{nonlocal HOTs}\right]$$

where $\eta = 8\pi\mu/|c|$, with $c = \ln(e\varepsilon^2)$. This is a local balance between applied and drag forces. In many cases the arclength $s$ is a material parameter.

### (1) The dynamics of a small rigid rod moving freely in a background flow

For a rigid rod we can write

$$\mathbf{X}(s,t) = \mathbf{X}_c(t) + s\mathbf{p}(t) \Rightarrow$$
$$\mathbf{X}_t = \mathbf{V} = \dot{\mathbf{X}}_c + s\dot{\mathbf{p}}$$
$$\mathbf{X}_s = \mathbf{p}$$

Assume that the length of the rod is very small relative to the length-scale of the flow, i.e., $\mathbf{u}(\mathbf{x}) \approx \mathbf{u}(\mathbf{X}_c) + \nabla\mathbf{u}(\mathbf{X}_c)(\mathbf{x} - \mathbf{X}_c)$

$$\eta[\dot{\mathbf{X}}_c - \mathbf{u}(\mathbf{X}_c) + s(\dot{\mathbf{p}} - \nabla\mathbf{u}(\mathbf{X}_c)\mathbf{p})] = (\mathbf{I} + \mathbf{p}\mathbf{p}^T)\mathbf{f}$$
$$\text{with } \int_{-L/2}^{L/2} ds\, \mathbf{f}(s) = \mathbf{0} \text{ and}$$
$$\int_{-L/2}^{L/2} ds\, (\mathbf{X}(s,t) - \mathbf{X}_c(t)) \times \mathbf{f}(s) = \mathbf{p} \times \int_{-L/2}^{L/2} ds\, s\mathbf{f}(s) = \mathbf{0}$$

we have

$$\mathbf{f} = \eta\left(\mathbf{I} - \frac{1}{2}\mathbf{p}\mathbf{p}^T\right)\left[\dot{\mathbf{X}}_c - \mathbf{u}(\mathbf{X}_c) + s\left(\dot{\mathbf{p}} - \nabla\mathbf{u}(\mathbf{X}_c)\,\mathbf{p}\right)\right]$$

and from the force-free condition and oddness of **f** gives

$$\dot{\mathbf{X}}_c(t) = \mathbf{u}(\mathbf{X}_c) \text{ and hence } \mathbf{f} = \eta s\left(\mathbf{I} - \frac{1}{2}\mathbf{p}\mathbf{p}^T\right)\left(\dot{\mathbf{p}} - \nabla\mathbf{u}(\mathbf{X}_c)\,\mathbf{p}\right)$$

Zero torque gives:

$$\mathbf{p} \times \left(\mathbf{I} - \frac{1}{2}\mathbf{p}\mathbf{p}^T\right)\left(\dot{\mathbf{p}} - \nabla\mathbf{u}(\mathbf{X}_c)\,\mathbf{p}\right) = \mathbf{0} \Leftrightarrow$$
$$\mathbf{p} \times \left(\dot{\mathbf{p}} - \nabla\mathbf{u}(\mathbf{X}_c)\,\mathbf{p}\right) = \mathbf{0}$$

Now we use the identity $\mathbf{p} \times \mathbf{p} \times \mathbf{g} = -(\mathbf{I} - \mathbf{p}\mathbf{p})\mathbf{g}$ and that $\mathbf{p} \cdot \dot{\mathbf{p}} = 0$ to get

$$\dot{\mathbf{p}} = (\mathbf{I} - \mathbf{p}\mathbf{p}^T)\nabla\mathbf{u}(\mathbf{X}_c)\,\mathbf{p}$$

i.e. **Jeffery's equation for rods**. Finally, we calculate the force itself:

$$\mathbf{f} = -s\eta\left(\mathbf{I} - \frac{1}{2}\mathbf{p}\mathbf{p}^T\right)(\mathbf{p}\mathbf{p}^T : \nabla\mathbf{u})\mathbf{p} = -s\frac{\eta}{2}(\mathbf{p}\mathbf{p}^T : \nabla\mathbf{u})\mathbf{p}$$

Note that the force is entirely in the **p** direction and is linear in $s$. This force is the so-called constraint force that maintains the rigidity of the rod against the flow. If we write $\mathbf{f} = T_s\mathbf{p}$ then the tension $T = -\frac{\eta}{4}(\mathbf{p}\mathbf{p}^T : \nabla\mathbf{u})(s^2 - 1/4)$, which will be negative for a compressive flow along the rod. This constraint force generates a stress in the surrounding fluid. The single particle "extra-stress" density is given by **Batchelor's formula**

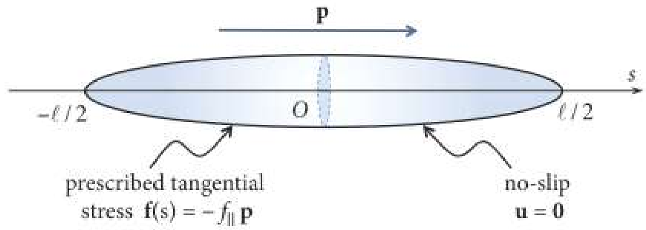
$$\boldsymbol{\Sigma}_p = -\int_{-l/2}^{+l/2} ds\, \mathbf{f}(s)\mathbf{X}(s)^T = -\int_{-l/2}^{+l/2} ds\, s\mathbf{f}(s)\mathbf{p}(t)^T = \frac{l^3}{24}\eta(\mathbf{p}\mathbf{p}^T : \nabla\mathbf{u})\mathbf{p}\mathbf{p}^T$$

This term is generally relaxational, contributing an anisotropic viscosity term.

## Applied to Simple Swimmers

(1) Here I am putting into a footnote an extended discussion of the application of SBT to undulatory swimming. This includes an argument that small amplitude traveling waves, say $O(\delta)$, give rise to $O(\delta^2)$ translation speeds, establishing the result of Taylor:

# 2. A swimming rod in a linear background flow.




Again, consider a slender rod with center-line position $\mathbf{X}(s,t) = \mathbf{X}_c(t) + s\mathbf{p}(t)$ with $-l/2 \leq s \leq l/2$ where we pose a propulsive surface stress for negative $s$ and a no-slip condition and consequent drag for positive $s$. Slender body theory:

$$\eta[\mathbf{V} - \mathbf{u}(\mathbf{X})] = (\mathbf{I} + \mathbf{ss})\mathbf{f}$$

where $\eta = 8\pi\mu/|c| > 0$ (following TS2004; $c = \ln \varepsilon^2 e < 0$) and $\mathbf{f}$ is the force/length acting on the fluid by the filament, and we interpret $\mathbf{V}$ as the averaged surface velocity on the fiber. What is the swimming speed $V_s$ of this particle, and what is its dynamics?

First consider the case with no background flow, so consider is the following (lab frame) system:

$$-l/2 \leq s \leq 0: \ \eta[V_s + u_\parallel(s)]\mathbf{p} = (\mathbf{I} + \mathbf{pp})\mathbf{f}_1$$

$$\text{where } \mathbf{f}_1 = -f_\parallel(s)\mathbf{p} \text{ with } f_\parallel > 0 \text{ and}$$

$$0 \leq s \leq l/2: \ \eta V_s \mathbf{p} = (\mathbf{I} + \mathbf{pp})\mathbf{f}_2$$

That is, $u_\parallel(s)$ is the surface slip, $\mathbf{f}_1$ is the propulsive force per unit length, and $\mathbf{f}_2$ (to be determined) is the drag force per unit length. This system is completed by the requirement of zero total force. Note that $\mathbf{f}_2$ must be constant and in the $\mathbf{p}$ direction. Given $f_\parallel$ we have the three equations

$$\eta[V_s + u_\parallel] = -2f_\parallel$$

$$\eta V_s = 2f_2$$

$$-\int_{-l/2}^{0} ds\, f_\parallel + \frac{l}{2} f_2 = 0$$

with the solution:

$$f_2 = \frac{1}{l/2}\int_{-l/2}^{0} ds\, f_\parallel$$

$$V_s = \frac{2}{\eta}\frac{1}{l/2}\int_{-l/2}^{0} ds\, f_\parallel > 0$$

$$u_\parallel = -V_s - \frac{2}{\eta} f_\parallel = -\frac{4}{\eta l}\int_{-l/2}^{0} ds\, f_\parallel - \frac{2}{\eta} f_\parallel < 0$$

**Example**: Take $f_\parallel$ a constant.

$$f_2 = f_\parallel \text{ (i.e., equal and opposite to the propulsive thrust)}, V_s = \frac{2}{\eta} f_\parallel, u_\parallel = -\frac{4}{\eta} f_\parallel$$

Or, using that $f_\parallel = 2\pi a g_\parallel$ with $g_\parallel$ the surface stress:

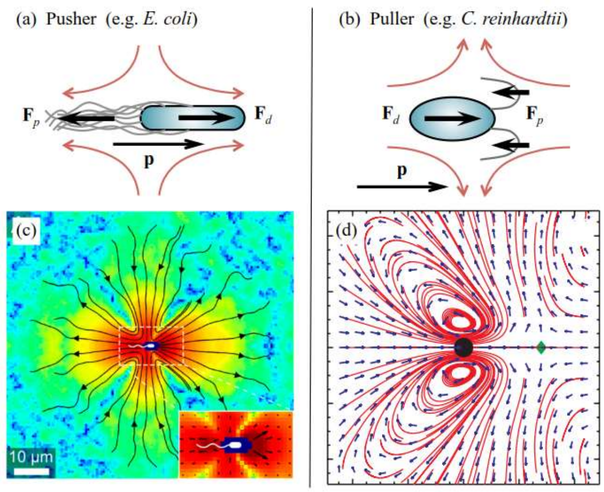
$$V_s = \frac{2}{8\pi\mu/|\ln \varepsilon^2 e|} 2\pi a g_\parallel = \frac{\varepsilon |\ln \varepsilon^2 e|}{2}\frac{l g_\parallel}{\mu} = \kappa_2(\varepsilon)\frac{l g_\parallel}{\mu}$$

That is, $\kappa_2$ is a geometric constant.

Let's calculate the "active" extra-stress contributions. From Batchelor's formula, its density has the form:

$$\mathbf{\Sigma}_a = -\int_{-l/2}^{+l/2} ds\, \mathbf{f}(s)\mathbf{x}^T(s) = -\left[\int_{-l/2}^{0} ds(-f_\parallel \mathbf{p})(\bar{\mathbf{X}}(t) + s\mathbf{p}(t))^T + \int_0^{l/2} ds(f_2\mathbf{p})(\bar{\mathbf{X}}(t) + s\mathbf{p}(t))^T\right]$$
$$= -\left[\int_{-l/2}^{0} ds\, sf_\parallel\, \mathbf{pp}^T - \int_0^{l/2} ds\, sf_\parallel\, \mathbf{pp}^T\right] = \frac{-1}{2}\left[-s^2|_{-l/2}^{0} + s^2|_0^{l/2}\right] f_\parallel \mathbf{pp}^T = \frac{-l^2}{4} f_\parallel \mathbf{pp}^T$$
$$= -\frac{l^2}{4}(2\pi a g_\parallel)\mathbf{pp}^T = -\frac{\pi\varepsilon}{2} l^3 g_\parallel \mathbf{pp}^T = -\kappa_1(\varepsilon) l^3 g_\parallel \mathbf{pp}^T$$

Note that $\kappa_{1,2}$ are solely geometric constants.




These are flows generated by force-dipoles. The one we just calculated is for a "Pusher" particle, like a bacterium. If we take the propulsive stress $g_\parallel$ to be negative, then the direction of motion reverses with propulsion leading drag, and this is called a "Puller". In either case we have

$$\mathbf{\Sigma}_a = \sigma_0\, \mathbf{pp}^T$$

where $\sigma_0$ (the "stresslet", units of force times length) is negative for Pushers, and positive for Pullers. We refer to this as an "active stress" as it arises only through swimming. This is different from what has been recently studied by Brady and others, the so-called "swim stress", which is collisional.

Including a linear background flow just turns out to be a combination of the simple rigid rod case for its rotation and constraint force and stress, and the active swimmer's velocity and active stress. The result is

$$\dot{\mathbf{X}}_c = V_s\mathbf{p} + \mathbf{u}(\mathbf{X}_c)$$
$$\dot{\mathbf{p}} = (\mathbf{I} - \mathbf{pp}^T)\nabla\mathbf{u}(\mathbf{X}_c)\mathbf{p}$$
$$\mathbf{\Sigma}_p = \sigma_0\mathbf{pp}^T + \frac{l^3}{24}\eta(\mathbf{pp}^T : \nabla\mathbf{u})\mathbf{pp}^T$$

Note that if $\nabla\mathbf{u}$ is being produced by the active stresses of the swimmers themselves, then the first "active" term is dominant in the dilute regime.

Here I put into a footnote a calculation of swimming velocity and extra stress for a more general placement of the active stress upon the body swimming in a background flow: . It also includes a calculation of the total power of a swimmer.

# b. The extra stress calculation

Consider having $N$ total swimmers in a volume $L^3$. From Batchelor's formula:

$$\langle \boldsymbol{\sigma}_a \rangle = -\frac{1}{l_b^3} \sum_m^M \int_{B_m} dS\, \mathbf{g}\mathbf{X}^T = -\frac{1}{l_b^3} \sum_m^M \int_{\Gamma_m} ds\, \mathbf{f}\mathbf{x}^T$$

$$= -\frac{N}{L^3} \frac{M/l_b^3}{N/L^3} \sigma_0 \frac{1}{M} \sum_m^M \mathbf{p}_m \mathbf{p}_m = -n\, \sigma_0\, C\, \frac{1}{M} \sum_m^M \mathbf{p}_m \mathbf{p}_m$$

where $C = (M/l_b^3)/(N/L^3)$ is the local concentration.

## c. Scaling

The normalization from SS2008 is

$$\frac{1}{L^3} \int_\Omega dV_x \int_S dS_p\, \Psi = n = \frac{N}{L^3}$$

Now rescale as $x \to l_c x$, $u \to Uu$, and $\Psi \to n\Psi$. Normalization becomes

$$\frac{1}{(L/l_c)^3} \int_\Omega dV_x \int_S dS_p\, \Psi = 1$$

where $\Psi = 1/4\pi$ if $\Psi$ is a constant. Fluxes become

$$\dot{\mathbf{x}} = \mathbf{p} + \mathbf{u} - \frac{D_p}{l_c U} \nabla_x \ln \Psi$$

$$\dot{\mathbf{p}} = (\mathbf{I} - \mathbf{pp}) \nabla_x \mathbf{u}\, \mathbf{p} - \frac{d_p l_c}{U} \nabla_p \ln \Psi$$

and momentum balance:

$$-\Delta \mathbf{u} + \boldsymbol{\nabla} q = \frac{l_c^2}{\mu U} \frac{1}{l_c} n \sigma_0 \nabla_x \cdot \int dS_p \Psi \mathbf{pp}$$

$$= l_c \frac{n \kappa_1 l^3 g_\parallel}{\mu \kappa_2 l g_\parallel / \mu} \nabla_x \cdot \int dS_p \Psi \mathbf{pp}$$

$$= l_c \frac{N l^3}{L^3} l^{-1}\, \frac{\kappa_1}{\kappa_2} \nabla_x \cdot \int dS_p \Psi \mathbf{pp}$$

$$= l_c (\nu l^{-1}) \frac{\kappa_1}{\kappa_2} \nabla_x \cdot \int dS_p \Psi \mathbf{pp}$$

Here $\nu = \frac{N l^3}{L^3}$ is the effective volume concentration. So, choose $l_c = l/\nu$ and $\alpha = \kappa_1/\kappa_2$ so that

$$-\Delta \mathbf{u} + \nabla q = \alpha\, \nabla_x \cdot \int dS_p \Psi \mathbf{pp}$$

and the fluxes are then

$$\dot{\mathbf{x}} = \mathbf{p} + \mathbf{u} - \left( \frac{D_p \nu}{l U} \right) \nabla_x \ln \Psi$$

$$\dot{\mathbf{p}} = (\mathbf{I} - \mathbf{pp}) \nabla_x \mathbf{u}\, \mathbf{p} - \left( \frac{d_p l}{\nu U} \right) \nabla_p \ln \Psi$$

Here I suppress a remark about low $\nu$ scaling of the diffusions:

At any rate, the adimensional system is

$$\Psi_t + \nabla \cdot (\dot{\mathbf{x}}\Psi) + \nabla_p \cdot (\dot{\mathbf{p}}\Psi) = 0 \text{ or}$$
$$\Psi_t + \nabla \cdot ((\mathbf{p} + \mathbf{u})\Psi) + \nabla_p \cdot \left((\mathbf{I} - \mathbf{p}\mathbf{p}^T)\nabla_x \mathbf{u}\,\mathbf{p}\,\Psi\right) = D\Delta\Psi + d\,\nabla_p^2\Psi$$
$$-\Delta\mathbf{u} + \nabla q = \alpha\,\nabla_x \cdot \mathbf{D} \;\&\; \nabla \cdot \mathbf{u} = 0 \text{ where}$$
$$\mathbf{D} = \int dS_p \Psi\,\mathbf{p}\mathbf{p}^T$$

These equations are, for all intents and purposes, identical to the Doi model for rigid fiber suspension, and is generally in the class of Doi-Onsager models. Their differing characteristics lie in the propulsive part of the positional flux $\dot{\mathbf{x}}$, and in the active stress. For the active stress, $\alpha > 0$ corresponds to Pullers, and $\alpha < 0$ corresponds to Pushers.

Other effects can and have been included, such as alignment forces (Maier-Saupe theory) leading to liquid crystallinity (leading to active nematics), advection of and interaction with other fields as is relevant to chemotaxis, different stresses and fluxes relevant to microtubule/motor systems, etc.

Let's consider these equations posed either in all of space, or with periodic boundary conditions. Here are some of their properties:

1. The state of uniform isotropy is given by $\Psi \equiv \Psi_0 = 1/4\pi$ is an exact solution. If $\Psi \equiv \Psi_0$ then $\mathbf{D} = \mathbf{I} \Rightarrow \mathbf{u} \equiv \mathbf{0}$.
2. From HS2010:

> When diffusional processes are absent, the system Eqs. (1)–(5) has several interesting aspects: (i) the swimmer concentration $\nu$ appears only through the normalized system size $L = L_p/l_c$; (ii) if $t$ and $\mathbf{x}$ are rescaled as $(\mathbf{x},t) \to (\mathbf{x},t)/|\alpha|$, then up to its sign, $\alpha$ can be scaled out of the dynamics and so only the cases $\alpha = \pm 1$ need be considered; and (iii) if $\alpha < 0$, then the dynamics for $\alpha > 0$ is gotten by simply reversing time, orientation, and velocity, i.e., $(t,\mathbf{p},\mathbf{v}) \to -(t,\mathbf{p},\mathbf{v})$. This reflects the reversibility of the single microswimmer. Rotational or translational diffusion destroys this symmetry.

   The $\alpha \to -\alpha \Leftrightarrow (t,\mathbf{p},\mathbf{v}) \to -(t,\mathbf{p},\mathbf{v})$ property means that for any stability analysis below where $d = D = 0$, any eigenvalue will obey $\alpha \to -\alpha \Rightarrow \sigma \to -\sigma$.

3. Conformational Entropy:

$$E = \int dV_x \int dS_p \frac{\Psi}{\Psi_0} \ln \frac{\Psi}{\Psi_0}$$

   The entropy $E$ satisfies $E \geq 0$ and is zero iff $\Psi \equiv \Psi_0$. Entropies are natural energy-like quantities for these types of conservation equations. Using just the Fokker-Planck equation, a direct calculation yields

$$4\pi\dot{E} = 3\int dV_x \mathbf{E} : \mathbf{D} - \int dV_x \int dS_p \left[D|\nabla \ln\Psi|^2 + d|\nabla_p \ln\Psi|^2\right]$$

   where the $\mathbf{E} : \mathbf{D}$ term comes from the Jeffrey's term. Now, contracting the momentum equation against $\mathbf{u}$ and integrating yields

$$2\int dV_x \mathbf{E} : \mathbf{E} = -\alpha \int dV_x \mathbf{E} : \mathbf{D}$$

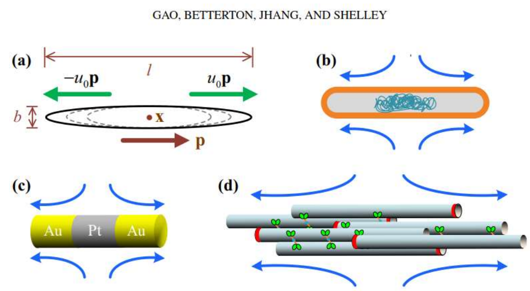
   This is the rate of viscous dissipation being balanced by the rate-of-work (power) being done by the swimmers. Note that for Pullers, if this expression is nontrivial, will be aligned with negative directions of strain, while Pushers will be aligned with positive directions of strain.

That is, for the latter aggregates will create an aligned extensional flow structure.

$$4\pi\dot{E} = -\frac{6}{\alpha}\int dV_x \mathbf{E} : \mathbf{E} - \int dV_x \int dS_p \left[ D|\nabla \ln \Psi|^2 + d|\nabla_p \ln \Psi|^2 \right]$$

Thus, we have for suspensions of Pullers the conformational entropy will always decrease to isotropy. However, velocity fluctuations drive up entropy for Pushers (and thus departures from istropy).

**Important Point**: This conclusion does not depend upon swimming, only upon having an active stress with negative coefficient. Indeed, all of the instabilities discussed below occur in simplified form for this case.




4. Moments:

This is a 5 + 1 dimensional theory. Integrating 1, **p**, **pp**, etc against the FP equation gives evolution for the **p** moments of the Ψ distribution. It does not in general close (called the moment-closure problem). Define $\langle f \rangle = \int dS_p f(\mathbf{p})\,\Psi(\mathbf{x},\mathbf{p},t)$.

$$\begin{aligned}
D_t c &= -V_s \nabla\cdot(c\mathbf{n}) + D\nabla^2 c,\\
D_t(c\mathbf{n}) &= -V_s[\nabla\cdot(c\mathbf{Q}) + (1/3)\nabla c] + D\nabla^2(c\mathbf{n})\\
&\quad + (c\mathbf{I}\mathbf{n} - \langle\mathbf{ppp}\rangle) : (\beta\mathbf{E} + \mathbf{W}) - 2dc\mathbf{n},\\
D_t(c\mathbf{Q}) &= -V_s[\nabla\cdot\langle\mathbf{ppp}\rangle - (\mathbf{I}/3)\nabla\cdot(c\mathbf{n})] + D\nabla^2(c\mathbf{Q})\\
&\quad + \beta c[\mathbf{E}\cdot(\mathbf{Q}+\mathbf{I}/3) + (\mathbf{Q}+\mathbf{I}/3)\cdot\mathbf{E}]\\
&\quad + c[\mathbf{W}\cdot\mathbf{Q} - \mathbf{Q}\cdot\mathbf{W}] - 2\beta\langle\mathbf{pppp}\rangle : \mathbf{E} - 6dc\mathbf{Q},
\end{aligned}$$

$$\phi(\mathbf{x},t) = \langle 1 \rangle = \int dS_p \Psi(\mathbf{x},\mathbf{p},t), \text{ the concentration}$$

$$\mathbf{n}(\mathbf{x},t) = \phi^{-1}\langle \mathbf{p} \rangle = \phi^{-1}\int dS_p \mathbf{p}\Psi(\mathbf{x},\mathbf{p},t), \text{ the polarity}$$

$$\mathbf{Q}(\mathbf{x},t) = \phi^{-1}\langle \mathbf{pp} \rangle = \phi^{-1}\int dS_p(\mathbf{pp} - \mathbf{I}/3)\Psi(\mathbf{x},\mathbf{p},t) = \phi^{-1}\mathbf{D} - \mathbf{I}/3, \text{ the tensor order parameter}$$

Let's put the swimming velocity $V_s$ back in just to keep track of the origin of terms:

$$\frac{D\phi}{Dt} + V_s\nabla\cdot(\mathbf{n}\phi) = D\Delta\phi \text{ or}$$

$$\phi_t + (\mathbf{u} + V_s\mathbf{n})\cdot\nabla\phi = -V_s\phi\nabla\cdot\mathbf{n} + D\Delta\phi$$

that is, fluctuations in swimmer concentration are driven by divergence in the swimmer polarity field.

$$\frac{D}{Dt}(\phi\mathbf{n}) = V_s\nabla\cdot\mathbf{D} + (\phi\mathbf{In} - \langle\mathbf{ppp}\rangle) : \nabla\mathbf{u} + D\Delta(\phi\mathbf{n}) - 2d\phi\mathbf{n}$$

Note that zeroeth-moment depended upon first-moment, and that first-moment depended upon second- and third-moments. And so it goes. Closure methods – such as Bingham – are very useful and much studied.

**5**. Orientational instability:

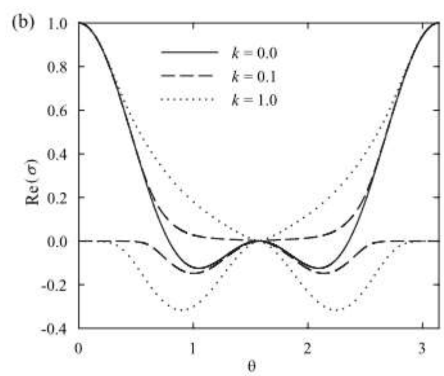
Let's study the stability of oriented suspensions. At this stage we'll need to drop diffusive terms in order to get a base state. So, set $D = d = 0$. We'll seek "sharply aligned" solutions of the form $\Psi(\mathbf{x},\mathbf{p},t) = \phi(\mathbf{x},t)\delta(\mathbf{p} - \mathbf{n}(\mathbf{x},t))$ where $\phi$ is then the density and $\mathbf{n}$ is the direction of swimming. Inserting above gives

$$\phi_t + \nabla \cdot [(\mathbf{u} + \mathbf{n})\phi] = 0$$

$$\mathbf{n}_t + (\mathbf{u} + \mathbf{n}) \cdot \nabla\mathbf{n} = (\mathbf{I} - \mathbf{nn})\nabla\mathbf{u}\,\mathbf{n}$$

$$-\Delta\mathbf{u} + \nabla q = \alpha\, \nabla_x \cdot (\phi\mathbf{nn}) \;\;\&\;\; \nabla \cdot \mathbf{u} = 0$$

This has a solution $\phi \equiv \bar{\phi} = \frac{1}{4\pi}$, $\mathbf{u} \equiv \mathbf{0}$, and $\mathbf{n} \equiv \hat{\mathbf{z}}$ (to make a choice). One can linearize this system around this state and seek plane wave solutions, i.e., write $\phi = \frac{1}{4\pi} + \varepsilon A(t)\exp(i\mathbf{k} \cdot \mathbf{x})$, for $\varepsilon << 1$. This leads to a complex-coefficient quadratic eigenvalue problem for a growth rate $\sigma(\phi,k)$ where $\mathbf{k} = k\left(\sin\phi\, \hat{\mathbf{e}}(\theta) + \cos\phi\, \hat{\mathbf{z}}\right)$. One can show that for every $(k,\phi)$ pair there is an eigenvalue with positive real part (unstable) and one with negative real part (stable). The maximal growth occurs at $\phi = 0$ (wave-vector aligned with the direction of swimmer alignment) given $k > 0$, and is maximized as $k \to 0$. One branch of the $k = 0$ case was first discovered by Simha & Ramaswamy (PRL '02) (the other branch is zero in that limit).

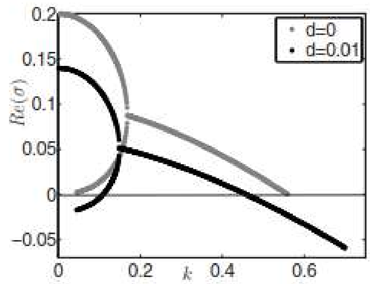



From Saintillan & Shelley, 2008a,b

For some reason, $\theta$ is used here instead of $\phi$ for the angle of the wave-vector relative to the $\hat{\mathbf{z}}$ direction.

**6**. Stability of nearly isotropic suspensions & transition to "turbulence": (Saintillan & Shelley, Phys. Fluids '08, Hohenegger & Shelley, PRE '10). Seek solutions of the form

$\Psi = \frac{1}{4\pi}\left(1 + \varepsilon\tilde{\Psi}(\mathbf{p},t)e^{i\mathbf{k}\cdot\mathbf{x}}\right)$ (note that this is not necessarily described as an eigenvalue problem).




From Hohenegger & Sh, 2010

No instability for $\alpha > 0$. For $\alpha < 0$ there is a long-wave instability occurring at the system's fundamental scale $k = k^*$ which can be recast as

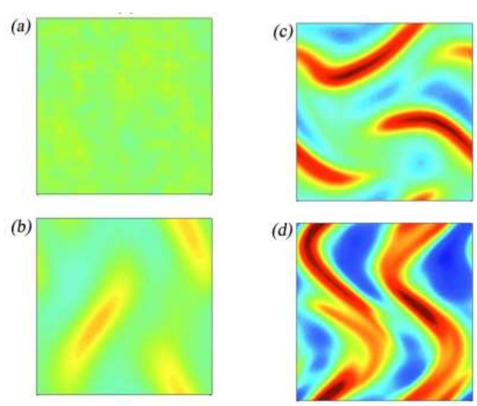
$$A = |\alpha|\nu\frac{L}{l} > A^*$$

This can be recast as a more general condition on the ratio of total active stress to viscous stress:

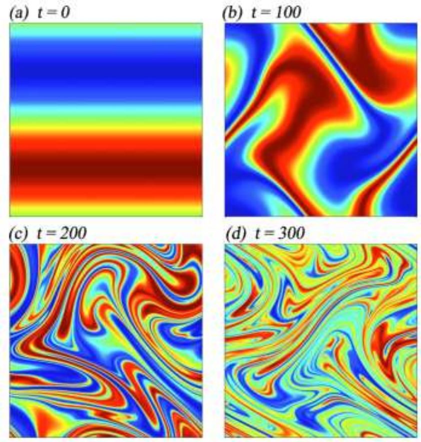
$$A \sim \frac{N(\sigma_0/L^3)}{\frac{U}{L}\mu}$$

The linear structure is quite complex, and it can be shown that one needs only examine the dynamics on the $m = 1$ azimuthal mode on the $p$-sphere. The instability is *orientational*. Concentration fluctuations decay! Only nonlinearity yields concentration fluctuations. The maximal growth rate always occurs at the system scale, and having a mode of maximal growth requires some other effect, such as a source of external drag (another medium, a wall, etc), or perhaps inertia, or ...?

From Saintillan & Shelley 2008b, the development of flow instability and concentration fluctuations:




Mixing by the unstable flow:




Mixing in an active suspension

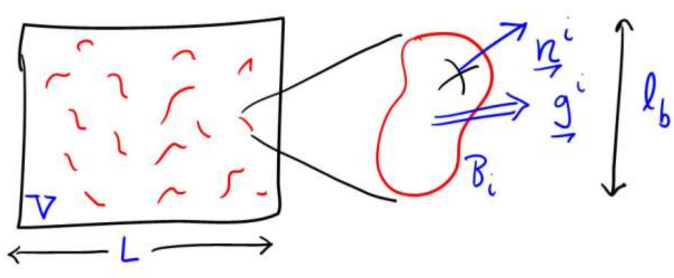
Some reviews which include extensions to other problems:

- Theory of active suspensions, Saintillan & Shelley in Complex Fluids in Biological Systems (2015)
- Active Suspensions and Their Nonlinear Models, Saintillan & Shelley, in Comptes Rendes Physique (2013)
- The dynamics of microtubule/motor-protein assemblies in biology and physics, M. Shelley, in Annual Reviews of Fluid Mechanics (2016)

Here the active stress is communicated through the fluid. Other active materials problems are different with active stresses communicated through the microstructure itself (see Foster, Fuerthauer, Shelley, Needleman, eLife 2015).

## Deriving the Batchelor formula

Consider a system volume **V** of volume $L^3$ and containing $N$ particles of length-scale $l$. Assume that **V** can be parcellated into many averaging subvolumes of length-scale $l_a$. We will make some separation of scale arguments when we need them, such as $l << l_a << L$. (this isn't used in the discussion below) ($l_b$ in the figure below should be $l$).




We would like to calculate the total average stress in a volume containing a Newtonian liquid in which are immersed many small bodies, where each body exerts a stress $\mathbf{g}^i$ on the surrounding fluid. Center the volume on point $\mathbf{x}$ and write $\Omega[\mathbf{x}]$. Let the fluid subdomain be $\tilde{\Omega}_f$ and the particle subdomain be $\Omega_p = \cup_n B_n$. We assume that both fluid and particle is described by the zero divergence stress tensors $\boldsymbol{\sigma}^f$ and $\boldsymbol{\sigma}^p$, respectively, and require that $\mathbf{g} = \boldsymbol{\sigma}^p|_{\partial B_i}\mathbf{n} = -\boldsymbol{\sigma}^f|_{\partial B_i}\mathbf{n}$. The bodies have outward normals while $\partial\Omega$ has inward.

Here I suppress into a footnote a derivation of Batchelor's formula for force-free particles (it needs cleaning up). The main result I leave below.

In summary then, we have derived the Kirkwood-Batchelor formula:

$$\nabla \cdot \bar{\boldsymbol{\sigma}} = \mathbf{0},\ \nabla \cdot \bar{\mathbf{u}} = 0 \text{ with}$$

$$\bar{\boldsymbol{\sigma}} = \frac{V_f}{V}\left[-\bar{p}\mathbf{I} + \mu(\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T)\right] + \frac{1}{V_f}\left[\int_{\partial\Omega_p} dS_y\, \mathbf{g}\mathbf{y}^T - \mu\int_{\partial\Omega_p} (\mathbf{u}\mathbf{n}^T + \mathbf{n}\mathbf{u}^T)\right] \text{ or}$$

$$-\nabla\bar{p} + \Delta\bar{\mathbf{u}} = -\nabla \cdot \boldsymbol{\sigma}^e \quad \text{and} \quad \nabla \cdot \bar{\mathbf{u}} = 0$$

$$\text{with } \boldsymbol{\sigma}^e = -\frac{1}{V_f}\left[\int_{\partial\Omega_p} dS_y\, \mathbf{g}\mathbf{y}^T - \mu\int_{\partial\Omega_p} (\mathbf{u}\mathbf{n}^T + \mathbf{n}\mathbf{u}^T)\right]$$

# a. Two spheres connected by a spring in a linear background flow.

I suppress as a footnote a little calculation using Batchelor's formula to calculate the extra-stress from two spheres connected by a spring. It leads to the Oldroyd-B single-particle stress .