Another application

Recall Lemma: Given $x \in \mathbb{R}^n$, $\exists\, \mathcal{H}^{(n)}(v)$ s.t. $\mathcal{H}^{(n)}(v)\, x = \alpha e_1$

Again, given $A \in \mathbb{R}^{n \times n}$

$$A = \left(x_1 \,\middle|\, A_1 \in \mathbb{R}^{n \times (n-1)} \right) ; \quad \text{Let } \mathcal{H}_1 x_1 = \alpha_1 e_1$$

$$\mathcal{H}_1^{(n)} A = \left(\begin{array}{c|c} \alpha_1 & \\ 0 & \mathcal{H}_1^{(n)} A_1 \\ \vdots & \\ 0 & \end{array} \right) \qquad \mathcal{H}_1^{(n)} A_1 \in \mathbb{R}^{n \times (n-1)}$$

$$= \left(\begin{array}{c|cc} \alpha_1 & \multicolumn{2}{c}{y^T} \\ \hline 0 & & \\ \vdots & x_2 & A_2 \\ 0 & & \end{array} \right) \qquad y \in \mathbb{R}^{n-1},\ x_2 \in \mathbb{R}^{n-1},\ A_2 \in \mathbb{R}^{(n-1) \times (n-2)}$$

Let $\mathcal{H}_2^{(n-1)}(x_2) = \alpha_2 e_1 \in \mathbb{R}^{n-1}$

$$\mathcal{H}_2^{(n)} = \left(\begin{array}{c|c} 1 & 0^T \\ \hline 0 & \mathcal{H}_2^{(n-1)} \end{array} \right)$$

$$\mathcal{H}_2^{(n)} \mathcal{H}_1^{(n)} = \left(\begin{array}{c|c} 1 & 0^T \\ \hline 0 & \mathcal{H}_2^{(n-1)} \end{array} \right) \left(\begin{array}{c|c|c} \alpha_1 & \multicolumn{2}{c}{y_1^T} \\ \hline 0 & & \\ \vdots & x_2 & A_2 \\ 0 & & \end{array} \right)$$

$$= \left(\begin{array}{c|c|c} \alpha_1 & \multicolumn{2}{c}{y_1^T} \\ \hline 0 & & \\ \vdots & \underbrace{\mathcal{H}_2^{(n-1)} x_2}_{\in \mathbb{R}^{n-1}} & \underbrace{\mathcal{H}_2^{n-1} A_2}_{\in \mathbb{R}^{(n-1) \times (n-2)}} \\ 0 & & \end{array} \right) = \left(\begin{array}{c|c|c|c} \alpha_1 & \multicolumn{3}{c}{y_1^T} \\ \hline 0 & \alpha_2 & \multicolumn{2}{c}{y_2^T} \\ \cline{2-4} 0 & 0 & & \\ \vdots & \vdots & x_3 & A_3 \\ 0 & 0 & & \end{array} \right)$$

Given $x_3 \in \mathbb{R}^{n-2}$, let

$$\mathcal{H}_3^{(n-2)} x_3 = \alpha_3 \hat{e}_1 \in \mathbb{R}^{n-2}$$

$$\mathcal{H}_3^{(n)} \cdot A = \left(\begin{array}{cc|c} 1 & 0 & 0^T \\ 0 & 1 & 0^T \\ \hline 0 & 0 & \mathcal{H}_3^{(n-2)} \end{array} \right) \left(\begin{array}{c|c|c|c} \alpha_1 & \multicolumn{3}{c}{y_1^T} \\ \hline 0 & \alpha_2 & \multicolumn{2}{c}{y_2^T} \\ \cline{2-4} \vdots & 0 & & \\ 0 & 0 & x_3 & A_3 \end{array} \right) = \left(\begin{array}{c|c|c|c|c} \alpha_1 & \multicolumn{4}{c}{y_1^T} \\ \hline 0 & \alpha_2 & \multicolumn{3}{c}{y_2^T} \\ \cline{2-5} \vdots & 0 & \alpha_3 & \multicolumn{2}{c}{y_3^T} \\ \cline{3-5} 0 & 0 & 0 & x_4 & A_4 \end{array} \right)$$

That is,

$$R = H_{n-1} \cdots H_2 H_1 A = Q^T A$$

$$Q^T Q = H_{n-1} \cdots H_1 H_1 \cdots H_{n-1} = I$$

Q is $\perp$ & $\boxed{A = QR}$ i.e. H.T.s can be used to form a QR decomposition.

Givens Rotations

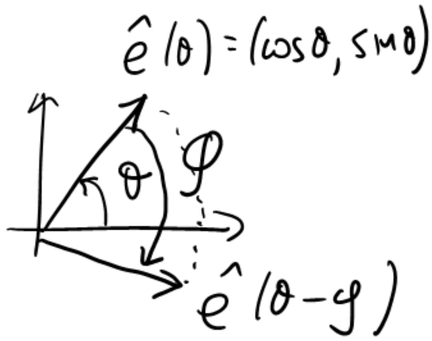
Rotation matrices, like reflections, are $\perp$ matrices. $AA^T = A^T A = I$

2D rotation matrix

$$R(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$$

$\varphi = 0 \cdot R = I$

$$R(\varphi)\hat{e}(\theta) = \begin{pmatrix} \cos\theta\cos\theta + \sin\theta\sin\varphi \\ \sin\theta\cos\varphi - \cos\theta\sin\varphi \end{pmatrix}$$



$$= \begin{pmatrix} \cos(\theta-\varphi) \\ \sin(\theta-\varphi) \end{pmatrix} = \hat{e}(\theta-\varphi)$$

<u>Properties</u> $R^T(\varphi) = R(\varphi)^{-1} = R(-\varphi)$

$$R(\varphi)R(-\varphi) = I$$

Plane rotation in $\mathbb{R}^n$

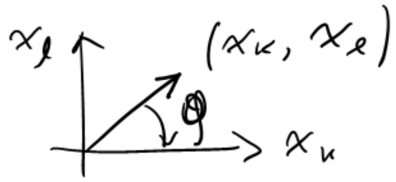
$$
R^{kl} = \begin{array}{c} \\ \\ k \\ \\ l \\ \\ \\ \end{array}
\begin{pmatrix}
1 & & & & & & \\
& \ddots & & & & & \\
& & 1 & & & & \\
& & & c & \cdots & s & \\
& & & \vdots & 1 \ddots 1 & \vdots & \\
& & & -s & \cdots & c & \\
& & & & & & \ddots \\
& & & & & & & 1
\end{pmatrix}
= \begin{pmatrix}
I & 0 & 0 \\
0 & \begin{smallmatrix} c & & s \\ & 1 \ddots 1_l & \\ -s & & \end{smallmatrix} & 0 \\
0 & 0 & I
\end{pmatrix}
$$

$$\uparrow_k \qquad \uparrow_l$$

$$
= I + (c-1)e_k e_k^T + (c-1) e_l e_l^T + s(e_k e_l^T - e_l e_k^T)
$$

$$
R^{kl}x = \begin{pmatrix} x_1 \\ \vdots \\ x_{k-1} \\ cx_k + sx_l \\ x_{k+1} \\ \vdots \\ x_{l-1} \\ -sx_k + cx_l \\ x_{l+1} \\ \vdots \\ x_n \end{pmatrix};
\quad \|R^{kl}x\|_2^2 = \sum_{j\neq k,l} x_j^2 + (cx_k + sx_l)^2 + (-sx_k + cx_l)^2 = \sum x_j^2 = \|x\|_2^2
$$

Now, given the two coordinates x_k, x_l



Find the φ that rotates

$$\begin{pmatrix} x_k \\ x_l \end{pmatrix} \to \begin{pmatrix} r \\ 0 \end{pmatrix} \qquad r = (x_k^2 + x_l^2)^{1/2}$$

$$\cos\varphi = x_k / r$$

$$\sin\varphi = x_l / r$$

$$
\Rightarrow R^{kl}(\varphi_{kl})x = \begin{pmatrix} x_1 \\ \vdots \\ x_{k-1} \\ r \\ x_k \\ \vdots \\ x_{l-1} \\ 0 \\ x_{l+1} \\ \vdots \end{pmatrix}
$$

Givens rotations can also be used to reduce A to a triangular matrix