2 More on Conservation Laws

We apply a model entirely similar to the traffic flow model of section 1 to river flow. In this context, some of the model's limitations become apparent, which motivates us to switch from a single equation to systems of conservation laws. For general systems, we study characteristics, Riemann invariants, simple waves and shocks.

2.1 Flood waves: a kinematic model

The analysis developed in section 1 has applications far beyond traffic flow. Here we extend it to long waves in rivers, where the principle of car conservation is replaced by volume conservation, under the assumption of negligible evaporation, infiltration and rain. Since the volume between two cross–sections of the river at positions x_1 and x_2 is given by

$$V = \int_{x_1}^{x_2} S(x,t)\, dx \,,$$

where S is the cross–sectional area of the river up to the free surface of the water, volume conservation takes the form

$$S_t + Q_x = 0 \,, \tag{21}$$

entirely analogous to (2), with S replacing the car density ρ and $Q(x,t)$ representing the volume flow per unit time through the river's cross–section at position x and time t:

$$Q(x,t) = \int_{S(x,t)} u(x,y,z,t)\, dy\, dz.$$

Here u is the component of the fluid velocity normal to the cross-section. As for traffic flow, we have the kinematic constraint that

$$Q = S\,U \,,$$

where U is the fluid's mean velocity across the section:

$$U(x,t) = \frac{\int_{S(x,t)} u(x,y,z,t)\, dy\, dz}{S(x,t)}.$$

In analogy with traffic flow, we may close equation (21) invoking a relation between Q and S. Hydraulic engineers denote such a relation a *hydrological law.* This is customarily measured in various cross–sections of the world's main rivers: a vertical stick measures the water height h, a surrogate for the area S if the geometry of the river bed is known, while a variety of devices are used for measuring the water speed u at various points. Integrating these velocity measurements across S yields an estimate for Q.

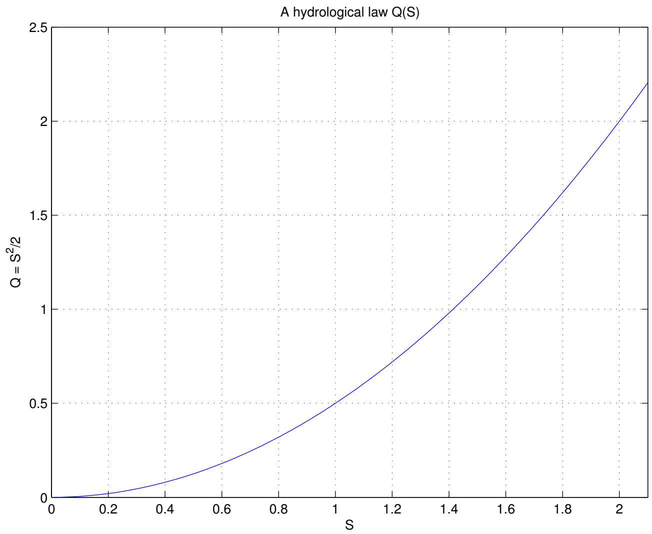

Figure 8: Example ($Q = S^2/2$) of a hydrological law for a river cross-section.

Hence we shall assume that a functional relation $Q = Q(S)$ exists. An important difference with the traffic model is that U is generally an increasing function of S, and so $Q(S)$ is convex instead of concave (see figure 8). The reason for this is that the mean speed U follows from a balance between two forces: gravity, which pushes the water downslope toward the sea, and lateral friction, which slows down the flow. Since the latter is proportional to the wetted perimeter of the river while the former is a body force, proportional to the area, and the area grows faster than the perimeter, the mean speed grows as the water level increases.

What dynamical consequences does this upward concavity bring? The characteristic lines are given by

$$\frac{dx}{dt} = Q'(S).$$

Along characteristics the water height remains constant. Notice that, unlike the situation for traffic flow, here all characteristics have positive slope; i.e., they move downstream. Moreover, $\frac{dx}{dt} \geq U$, so information travels faster than the water particles themselves. Hence fine suspensions and floating objects are caught by these long waves from behind. Finally, when shocks form, it is regions with higher values of S that catch up with ones with lower water levels, so shock waves arise during *floods*. Such discontinuous and violent flood waves are often observed in mountain rivers, sometimes with tragic consequences for nearby campers, who are taken by surprise by these nearly vertical walls of water coming with no notice, due to rain or thaw far upstream. The end of a flood, on the other hand, is much more gradual, as it is brought up by a smooth

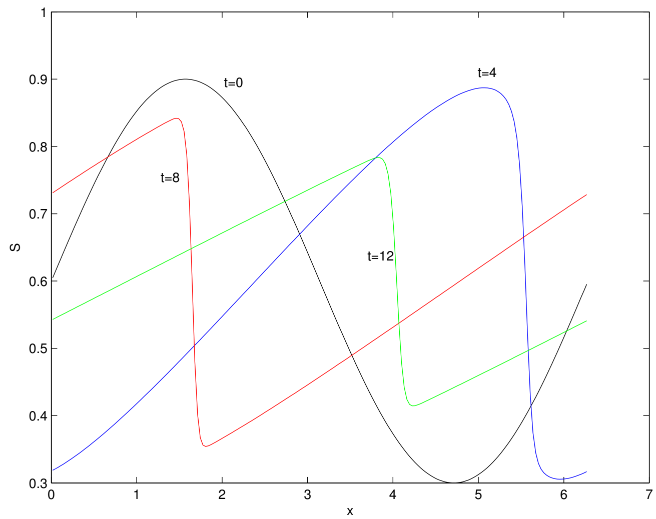

Figure 9: Discontinuous flood wave produced by the inviscid Burgers equation.

rarefaction wave.

A numerical run with $Q = S^2/2$, initial data $S = 0.3(2+\sin(x))$ and periodic boundary conditions is shown in figure 9. One can see a sharp discontinuity developing from smooth initial data, and then slowly decaying as characteristics carrying extreme data end at the shock. Equation (21) with this particular choice of $Q(S)$ is called the *inviscid Burgers* or *Hopf* equation.

2.2 Insufficiency of the kinematic model

The kinematic model developed above is quite useful for the fast computation of the main features of large flood waves. Hydraulic engineers have in fact made much use of models of this kind, which in the hydraulic literature go under the name of Muskingham. However, there is abundant evidence that such simple models cannot capture many important ingredients of real river flows:

- For very long waves, the water height $h(S)$ and the volume flux Q are definitely related. However, they are not quite functions of each other. Figure 10 sketches typical measurements, which show a *hysteresis* phenomenon: Q does not depend only on S, but also on its history. Typically, for the same value of the wetted area S, the flux Q is larger at the beginning of a flood, when S is growing, than at the end, when it is decreasing. The reason is that, as the flood starts, the water level is higher upstream than downstream, and so gravity acts strongly. Later on, the downstream area is also flooded, while upstream is returning to more regular conditions. This yields a smaller surface slope, and hence reduces the resulting

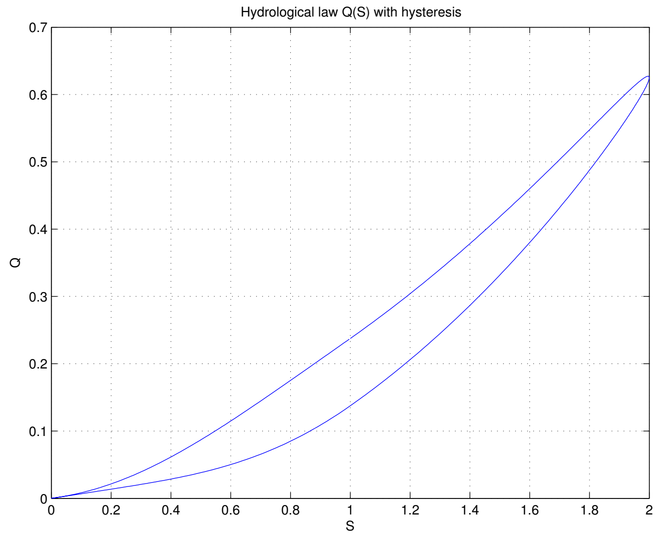

Figure 10: Hydrological law with hysteresis.

water speed.

- All shocks of the kinematic model correspond to flood waves traveling downstream. As mentioned above, these shocks are observed in mountain rivers. However, shocks (*bores* in hydraulic terminology) are also observed traveling upstream, carrying the information from high tides in the ocean; two famous examples are the *Mascaret* wave of the Seine and the 9m high bores of the Qiantang river in China. Big standing backward shocks, denoted *hydraulic jumps*, also occur downstream of a dam's spillway (and, in a much smaller scale and with circular shape, also on dishes under a faucet.) This suggests that there is a second characteristic family, giving the problem a certain amount of left–right symmetry.

In the following subsection, we construct a more complex model that includes this second characteristic family and the associated phenomena. For simplicity though, we do not include in this model the frictional and bottom slope effects that gave rise to the hydrological law $Q(S)$. Hence our kinematic model will not be derivable as a limit of the more complex one. Rather, the two models should be thought of as standing side by side, representing different aspects of the rich phenomenology of river flow.

2.3 Shallow waters and gas dynamics

For simplicity, we consider a rectangular channel with constant width b. Then the wetted area S is given by

$$S = bh\,,$$

where h represents the water height, and the water flux by

$$Q = bhu\,.$$

(We switch to lower case for the mean speed u for notational homogeneity.) The volume conservation equation (21) becomes

$$h_t + (hu)_x = 0\,.$$

Instead of closing this equation with an hydrological law, as before, we take now a more fundamental approach, whereby we write the equation for conservation of momentum:

$$(hu)_t + \left(hu^2 + P\right)_x = 0\,.$$

Here ρbhu is the momentum density (mass times velocity per unit length of the river); we have factored out of the equation the constant water density ρ and width b. Multiplying the momentum density by the speed u (assumed here uniform at each cross-section) yields the momentum flux hu^2, similar to the volume flux $Q = hu$ in the previous equation. The variable P represents the area integral of the pressure forces acting between the fluid parcels at the left and right of a river's cross–section. Unfortunately, in writing down this new equation, we also brought in the new unknown P. In order to close the system, we invoke now the *hydrostatic approximation*, valid for fluids at rest, whereby the pressure at each point is given by the fluid weight per unit area above it. Under this hypothesis, it follows that the integral of the pressure is given by $P = g\,h^2/2$, where g is the gravity constant, and the equations for mass and momentum conservation constitute a closed system, the **one–dimensional shallow water equations:**

$$h_t + (h\,u)_x = 0 \tag{22}$$

$$(h\,u)_t + \left(h\,u^2 + g\,h^2/2\right)_x = 0\,. \tag{23}$$

This is an example of a **system of conservation laws**. Another prototypical example is given by one–dimensional isentropic gas dynamics:

$$\rho_t + (\rho\,u)_x = 0 \tag{24}$$

$$(\rho\,u)_t + \left(\rho\,u^2 + P\right)_x = 0\,. \tag{25}$$

Here ρ is the gas density, u its speed, and $P = P(\rho)$ its pressure at constant entropy. As for the kinematic model for rivers, this model for gas dynamics can be shown to fail in the presence of big shocks, for which the isentropic assumption does not hold. When this is the case, the next natural step is to introduce the temperature T, and use the full equation of state $P(\rho, T)$. Since T is a new unknown, a new equation is required. This is provided by the principle of *energy conservation*

$$\left(e + \rho\, u^2/2\right)_t + \left(e\, u + \rho\, u^3/2 + P\, u\right)_x = 0\,, \tag{26}$$

where $e = e(T)$ is the internal ("thermal") energy of the gas, $\rho\, u^2/2$ the kinetic energy density, and $P\, u$ the work per unit time performed by the pressure.

Note: a common feature of all of our model derivations so far is that we start with one equation in two unknowns (typically some kind of mass conservation), and, as we add new equations, they bring in new unknowns. At some point, we decide to call it a day and introduce a *closure assumption*, valid under some local equilibrium approximation. For traffic and the kinematic flood models, we did this already at the level of the mass equation, introducing a flux law $Q(\rho)$ or $Q(S)$, valid near equilibrium, when the solution is nearly uniform in space and time. For the shallow water and the isentropic gas dynamical models, we added instead a momentum equation and closed the system by introducing a hydrostatic or *polytropic* law $P(S)$ or $P(\rho)$, again valid only for small perturbations of a uniform state. For real gases, we went one step further and closed the equations at the level of the energy, introducing the *equation of state* $P = P(\rho, T)$, valid under the assumption of thermodynamical equilibrium. For very dilute gases or very violent shocks, this hypothesis may fail as well, and we may need to move toward a more complex system, such as the Boltzman equations or, in the case of water waves, to the fully two–dimensional Euler equations. An area of research that has received much recent attention, *multiscale simulation*, involves replacing such equilibrium closures by more sophisticated statistical or numerical representations of the fluxes.

2.4 General systems of conservation laws

The traffic flow equation (2), the kinematic model for river flows (21), the shallow water system (22, 23), and the gas dynamic equations (24, 25, 26), are all systems of conservation laws. In one spatial dimension, these take the general form

$$U_t + F(U)_x = 0\,, \tag{27}$$

where $U = U(x, t)$ is the n-dimensional vector field of densities of the corresponding conserved quantities $\int U\, dx$, and $F(U)$ is a vector function of U with n components, the corresponding *fluxes*. The associated, more fundamental integral form is

$$\int_{\delta\Omega} (U\, dx\; -\; F\, dt) \;=\; 0 \tag{28}$$

for any domain Ω in the (x, t)-plane, a straightforward generalization of (10).

In areas where the solution is smooth, equation (27) can be rewritten in the form

$$U_t + A(U)\, U_x = 0\,, \tag{29}$$

where A, an $n \times n$ matrix, is the Jacobian of F:

$$A_{i,j} = \frac{\partial F_i}{\partial U_j}\,.$$

The system (29) is called *hyperbolic* when all the eigenvalues of the matrix A are real and the corresponding eigenvectors span R^n. All the systems that we have written down so far belong to this category under reasonable conditions (such as that the water height h be positive.)

Hyperbolic systems can be written in a particularly revealing form. To this end, consider a complete set of left eigenvectors of A, l_j, with corresponding eigenvalues λ_j:

$$l_j^t A = \lambda_j l_j^t\,,$$

where a superindex t denotes the transpose. Multiplying the system (29) on the left by l_j, we obtain its *characteristic form*

$$l_j^t\, U_t + \lambda_j\, l_j^t\, U_x = 0\,, \tag{30}$$

a scalar equation reminiscent of (5). The next subsection is devoted to study to what extent this analogy carries through to determining the information flow along characteristics, and how much it tells us about the solution $U(x,t)$.

2.5 Riemann invariants: the linear case

If the flux vector F is a linear function of U, then A is a constant matrix, and its eigenvalues and eigenvectors are also constant. In this case, the characteristic equations (30) can be rewritten in the simpler form

$$R_{j\,t} + \lambda_j\, R_{j\,x} = 0\,, \tag{31}$$

where

$$R_j = \sum_i l_j^i\, U_i$$

is the system's jth *Riemann invariant.* Introducing the jth characteristic family of curves X_j through the ODEs

$$\frac{dX_j}{dt} = \lambda_j\,, \tag{32}$$

we notice that R_j does not vary along this characteristic family, hence the "invariant" in its name.

With the Riemann invariants in hand, we can solve the initial value problem consisting of the system (29) with initial data

$$U(x,0) = U_0(x)\,.$$

To find the solution at the point (x,t), we just trace the n characteristics with slopes λ_j back to $t = 0$. At these points, we compute the corresponding Riemann invariants from U_0. Since these are conserved along characteristics, now we know all n Riemann invariants at (x,t). But this is equivalent to knowing the solution $U(x,t)$, since the matrix l^i_j transforming U into R is invertible, due to the independence of the eigenvectors l_j.

Thus the main difference between a linear system of conservation laws and a scalar one is that in the former various pieces of information –the Riemann invariants– travel at different speeds, given by the eigenvalues λ_j of A.

2.6 Riemann invariants for shallow waters

How much of the simple characteristic structure of linear systems described above translates to the general nonlinear systems (29)? Before addressing this question in all of its generality, let us work out an important example, the shallow water equations (22, 23). In smooth areas, these can be rewritten in the simpler form

$$\begin{aligned} h_t + u\,h_x + h\,u_x &= 0 \\ u_t + u\,u_x + g\,h_x &= 0\,, \end{aligned}$$

or, equivalently, in the matrix form (29), with

$$U = \begin{pmatrix} h \\ u \end{pmatrix}$$

and

$$A = \begin{pmatrix} u & h \\ g & u \end{pmatrix}.$$

The eigenvalues of A are

$$\lambda_\pm = u \pm \sqrt{g\,h}\,, \tag{33}$$

and we shall choose to write the corresponding left eigenvectors in the form

$$l^t_\pm = \left(\pm\sqrt{\frac{g}{h}}, 1\right).$$

With this choice, the characteristic form (30) can be integrated and rewritten as

$$R_{\pm t} + \lambda_\pm R_{\pm x} = 0\,,$$

where the Riemann invariants $R_\pm$ are given by

$$R_\pm = u \pm 2\,\sqrt{g\,h}\,. \tag{34}$$

So Riemann invariants, constant along characteristics, exist for the nonlinear shallow water system as well. Notice though that, unlike the linear case, now the equations for the two Riemann invariants are weakly coupled, through the eigenvalues $\lambda_\pm$, which depend on both Riemann invariants:

$$\lambda_\pm = \frac{R_+ + R_-}{2} \pm \frac{R_+ - R_-}{4}\,. \tag{35}$$

Hence the characteristic curves are no longer necessarily straight lines. Consider, for concreteness, a characteristic of the positive family,

$$\frac{dX}{dt} = \lambda_+(R_+, R_-),$$

along which R_+ is a constant. For this to be a straight line, λ_+ cannot vary, so R_- has to be constant as well. But R_- is constant along its own characteristic curves, which are transversal to the positive characteristics. Then the positive characteristics are straight lines only in areas where R_- is uniform. This will be one of the characterizations of the *simple waves* described below.

2.7 Riemann invariants for general systems

The existence of Riemann invariants for linear systems and for shallow waters followed from the integrability of the differential form

$$l_j^t dU = \sum_{i=1}^{n} l_j^i dU_i.$$

If there exists an integrating factor $\mu_j(U)$ such that

$$\mu_j(U) \sum_{i=1}^{n} l_j^i dU_i = dR_j(U),$$

then the characteristic form (30) gives rise to the equations (31), stating that the Riemann invariants R_j are indeed invariant along their corresponding characteristics.

Integrating factors always exist for two-dimensional differential forms; hence if $n = 2$, as in shallow waters, Riemann invariants exist, simplifying enormously the solution of the equations. On the other hand, there are typically no Riemann invariants in nonlinear systems with more than two variables. For instance, there's only one, the entropy s, for non-isentropic gas dynamics equation (24, 25, 26), which have $n = 3$. The corresponding characteristic speed is given by the velocity u, and the characteristic equation

$$s_t + u s_x = 0 \tag{36}$$

simply states that the entropy of a fluid is carried by the flow. An entirely similar equation arises if we add ink with density s to the shallow water equations (22, 23). However, in this case the other two variables h and u evolve independently of s, and hence preserve their Riemann invariants. So shallow waters with ink represents one of the exceptional cases of nonlinear systems of conservation laws with more than two dependent variables that have a complete set of independent Riemann invariants.

2.8 Simple waves

Simple waves are fully nonlinear solutions to general systems of conservation laws whose computation reduces to the solution of systems of ODEs. They appear naturally in the solution to Riemann problems, in the study of waves moving into uniform states and, more generally, in the long-time behavior of initial value problems with initial data with compact support.

A simple wave is a solution $U(x,t)$ to (29) of the form

$$U(x,t) = U(\theta(x,t)) \,, \tag{37}$$

where $\theta(x,t)$ is a scalar. Hence the name "simple": instead of general vector functions, we are dealing with solutions where all components depend on the same scalar function; we can say that we are studying a *single mode* of the system. Replacing this ansatz into (29), we obtain

$$\left(A(U) + \frac{\theta_t}{\theta_x} I \right) U'(\theta) = 0. \tag{38}$$

This equation can be read as stating that

$$\lambda = -\frac{\theta_t}{\theta_x}$$

is an eigenvalue of $A(U)$, with corresponding eigenvector $U'(\theta)$. This implies that one can build simple waves through the following procedure:

- Build two functions, $U(\theta)$, $\lambda(\theta)$, one vectorial and the other scalar, by following these steps:
  - Pick an initial value $U(0)$, as well as a characteristic family j, with $j \in (1, \ldots, n)$.
  - Solve the system of ordinary differential equations

    $$U'(\theta) = V(\theta),$$

    where V is the jth eigenvector of $A(U(\theta))$ (suitably normalized):

    $$(A(U(\theta)) - \lambda(\theta) I) \, V(\theta) = 0$$

    and $\lambda(\theta)$ is the jth eigenvalue of $A(U(\theta))$.
- Assign arbitrary initial values to $\theta(x,t)$:

  $$\theta(x,0) = \theta_0(x).$$

- Solve, using characteristics, the scalar, nonlinear, first order equation

  $$\theta_t + \lambda(\theta)\theta_x = 0. \tag{39}$$

- Set
$$U(x,t) = U(\theta(x,t)).$$

So we first build a *phase space* representation of the simple wave, $U(\theta)$, $\lambda(\theta)$, and then develop it into *physical space* (x,t) by solving an initial value problem for $\theta(X,T)$, with arbitrary initial data.

When the system has Riemann invariants, an equivalent characterization of simple waves is that they are solutions where all but one of the n Riemann invariants are uniform in space and time.

Problem: Prove that this is so. (Hint: remember that, for any matrix A, the left and right eigenvectors corresponding to different eigenvalues are orthogonal to each other.)

2.9 Shock formation and jump conditions

We saw before that typical solutions of single nonlinear conservation laws broke after a finite time, developing an infinite slope. This followed from the nontrivial dependence of the characteristic speed $Q'(\rho)$ on ρ, which made characteristics carrying different values of ρ move at different speeds, and hence intersect if the faster characteristic starts behind the slower one. If $c = Q'(\rho)$ had been a constant, we would have had the linear advection equation

$$\rho_t + c\,\rho_x = 0\,, \tag{40}$$

with solution

$$\rho(x,t) = \rho_0(x - ct) \tag{41}$$

valid for all times. Can we make similar statements about systems of conservation laws?

General solutions to systems of conservation laws are more complex, with the various modes –the characteristic families– interacting nonlinearly; we can no longer just follow the solution along a characteristic disregarding what's going on nearby. Even when we have Riemann invariants (an exceptional situation for $n > 2$), constant along characteristics, the characteristic speeds are not known, since they depend on the other Riemann invariants as well [1]. Yet we have found in the subsection above a huge class of single-mode solutions, the simple waves. Will these break nonlinearly? From the scalar evolution equation (39), it is enough to check that the characteristic speed $\lambda(\theta)$ genuinely depends on θ; i.e. that it is not a constant in disguise. To this end, we can compute its derivative

[1] This leads to an elegant procedure to solve the equations: instead of thinking of the Riemann invariants as functions of space and time, think of the latter as functions of the former, i.e., exchange dependent and independent variables. You can find this *hodograph plane* procedure in the book of Courant and Friedrichs, **Supersonic flow and shock waves, 1948.** Unfortunately, we do not have the time to cover this material in class, or else we would never leave the subject of conservation laws!

$\frac{d\lambda}{d\theta}$. The way that λ depends on θ is convoluted: λ is an eigenvalue of A, which depends on U, which depends on θ. Then $\lambda = \lambda(U(\theta))$, and

$$\frac{d\lambda}{d\theta} = U'(\theta) \cdot \nabla_u \lambda = (V \cdot \nabla)\lambda, \tag{42}$$

the derivative of the eigenvalue λ of $A(U)$ along the direction of the corresponding right eigenvector. Characteristic families for which (42) is never zero are called *genuinely nonlinear*, those for which (42) is identically zero are called *linearly degenerate*, and those in between where (42) is zero here and there have not, as far as I know, been given a name. You would expect these nameless families to constitute the general rule, with the other ones being rather exceptional. Yet classical systems, such as shallow waters and gas dynamics, only have the named cases. In shallow waters, all characteristics are genuinely nonlinear; in gas dynamics, the two characteristics associated with *sound* waves (more on this later) are genuinely nonlinear, while the one associated with the entropy is linearly degenerate. This latter fact can be inferred from (36): the characteristic speed u, as a function of the state variables (u, T, s), depends only on u itself, so its partial derivative with respect to the entropy s is zero. Scalar conservation laws are genuinely nonlinear when the flux function $Q(\rho)$ is either convex or concave, and linearly degenerate when $Q(\rho)$ is linear. Nonlinear flux functions that are neither convex nor concave, which appear for instance in rivers with flood plains, give rise to interesting, unique phenomena, such as the arrival of floods in two separate shock waves moving at different speeds.

With these definitions in hand, we can now make a precise statement about breaking waves in systems of conservation laws: simple waves associated with genuinely nonlinear characteristic families typically break –i.e., develop an infinite slope– after a finite time. Once a solution breaks, we can, as in the scalar case, introduce discontinuities, or shock waves. Then again we need to consider weak solutions, satisfying the integral principles (28) that gave rise to the system of conservation laws, and again we derive the jump conditions for $c = \frac{dx}{dt}$, the speed of the shock:

$$c\left(U^+ - U^-\right) = F(U^+) - F(U^-)\,. \tag{43}$$

However, this is now a system of n equations, one for each component of U and F. Then it can no longer be thought of as an equation for the speed c in terms of the states U^+ and U^- on both sides of the shock, but rather as a set of constraints on these two states. Given, for instance, a state U^+ on the right of the shock, (43) yields a one-parameter family of compatible states U^- on the left, parameterized by the shock speed c. Notice that, when U^+ and U^- are close to each other, the speed c converges to an eigenvalue of the matrix $A(U)$, and the jump in U to the corresponding eigenvector. Then we can anticipate that there should exist n families of shock waves, one associated with each family of eigenvalues of A.

2.10 The Riemann problem

As for the scalar conservation law, the basic building block for systems is the Riemann problem, with initial conditions:

$$U(x,0) = \begin{cases} U^- & \text{for } x < 0 \\ U^+ & \text{for } x > 0 \end{cases} . \tag{44}$$

As in the scalar case, the invariance of the PDE and the initial conditions under stretching of space and time implies that the solution adopts the form

$$U(x,t) = U(\xi = x/t)\,,$$

yielding

$$(A(U) - \xi I)\; U'(\xi) = 0\,.$$

Because $-\frac{\xi_t}{\xi_x} = \xi$, this is precisely the equation that defines simple waves, with ξ playing the role of θ. Then these are simple waves where $\theta = \frac{x}{t}$, i.e., the equivalent of the scalar rarefaction fan. Other solutions are areas with $U(\xi)$ constant, and shock waves separating two such areas.

The remaining question is whether all these pieces are enough to solve the Riemann problem, matching U^- and U^+ at the two ends. To see the plausibility of a positive answer, at least when U^- and U^+ are close to each other, notice that we have n families of waves (that are shocks or rarefactions, depending on whether the corresponding characteristics at the two ends converge or diverge). These should be enough to go from U^- to U^+ through $n-1$ intermediate constant states, with each consecutive pair separated by either a shock or a rarefaction. We do not have time in this class, however, to discuss this in any more detail.

Problem: Solve the Riemann problem when $F(U)$ is a linear function: $F = AU$.