7 Laplace and Poisson equations

In this section, we study Poisson's equation

$$\Delta u = f(x). \tag{152}$$

When $f = 0$, the equation becomes Laplace's:

$$\Delta u = 0. \tag{153}$$

More often than not, the equations will apply in an open domain Ω of R^n, with suitable boundary conditions on $\delta\Omega$. These boundary conditions are typically the same that we have discussed for the heat equation: Dirichlet, Neumann or mixed (Newton's), though without any reference to time.

7.1 Motivation

These are the first equations that we study where time plays no role. Even though one might be tempted to think that "time" is just a suggestive name placed on an otherwise undistinguishable independent variable, models of time-evolution, such as the wave and heat equation, display phenomena very different from those where time is either absent or plays only a parametric role. We will observe some of this different phenomenology in the examples of this chapter, and build it later into a general classification of partial differential equations using characteristics.

Regarding physical instances of the equations, it is clear that they will show up whenever an evolution modeled by the heat equation reaches a steady state. All phenomena modeled by forced wave equations also include a Poisson component, corresponding to their time-independent solutions. Unlike the heat equation though, that dissipates the energy in all unsteady modes, the wave equation will typically "radiate" these out of the domain. Also, we saw in homework 5 that a *reduced wave equation*, very similar in form and spirit to Laplace and Poisson's, shows up in the study of monochromatic waves.

We noticed before that the Laplacian is the variational derivative of the L^2 norm of the gradient. Hence Laplace and Poisson's equations appear in the description, for instance, of surfaces with minimal area, such as soup bubbles. This was in fact one of Richard Courant's main areas of research. We will study this variational view of the equations as we move along.

On a more purely mathematical line, Laplace's equation appears prominently in the theory of complex variables and in connection with issues of analyticity and smoothness. Part of our study here will elucidate how a *regularity gain* associated with the equations comes along.

7.2 The one-dimensional case

In one dimension, both Laplace and Poisson's equations are ODEs, not PDEs. Yet it would be a mistake to skip them just out of formal purism: in their simplicity, these one-dimensional cousins reveal most clearly much of what their higher-dimensional relatives are about. Laplace's equation, for starters, becomes the humble ODE

$$u''(x) = 0,$$

with general solution

$$u = ax + b.$$

Then the only solution bounded on the whole real line is a constant, and the solution satisfying the Dirichlet boundary conditions

$$u(x_l) = u_l, \quad u(x_r) = u_r$$

is the straight segment

$$u(x) = u_l + (u_r - u_l)\frac{x - x_l}{x_r - x_l}. \tag{154}$$

This makes sense from the variational viewpoint: we know that $u(x)$ minimizes the norm of $u'(x)$:

$$I(u) = \int_{x_l}^{x_r} \left(u'(x)\right)^2 \, dx.$$

Problem: Show by induction on the cardinality of the partition of (x_l, x_r) into sub-segments, that the straight segment (154) minimizes I over all continuous piecewise linear functions satisfying the boundary conditions.

More generally, for small smooth *variations* $\eta(x)$ with $\eta(x_l) = \eta(x_r) = 0$,

$$I(u + \eta) - I(u) \approx 2\int_{x_l}^{x_r} u'(x)\eta'(x) \, dx = -2\int_{x_l}^{x_r} u''(x)\eta(x) \, dx.$$

Therefore, if $u''(x)$ is continuous and $u''(x) \neq 0$ at any point $x = x_0$, we can always find an $\eta(x)$ with support concentrated near x_0, in such a way that $I(u + \eta) < I(u)$. It follows that, if there is a minimum for $I(u)$ among those functions with continuous second derivatives, it needs to satisfy Laplace's equation $u''(x) = 0$.

Let us now move on to the one-dimensional Poisson problem:

$$u''(x) = f(x), \quad u(x_l) = u_l, \quad u(x_r) = u_r.$$

By sustracting the solution above to Laplace's equation, we can limit ourselves to the consideration of homogeneous boundary conditions:

$$u(x_l) = u(x_r) = 0.$$

We could solve this problem by integrating f twice, and fit the constants of integration to the boundary conditions. However, it is more illuminating to introduce the Green's functions $G(x, y)$, satisfying

$$G_{xx} = \delta(x - y), \quad G(x_l, y) = G(x_r, y) = 0. \tag{155}$$

Then $u(x)$ is the result of the superposition

$$u(x) = \int_{x_l}^{x_r} f(y)\ G(x,y)\ dy.$$

As for G, it solves Laplace's equation and one boundary condition on each side of $x = y$, so

$$G(x,y) = \begin{cases} a\,(x - x_l) & \text{for } x < y \\ b\,(x - x_r) & \text{for } x > y \end{cases}.$$

Continuity of G at $x = y$ implies that

$$a\,(y - x_l) = b\,(y - x_r)\,,$$

and the jump in G_x: $G_x(y^+, y) - G_x(y^-, y) = 1$, brought about by the δ-function on the right-hand side, implies that

$$b - a = 1.$$

It follows that

$$G(x,y) = \begin{cases} \frac{(x_r - y)(x_l - x)}{x_r - x_l} & \text{for } x < y \\ \frac{(y - x_l)(x - x_r)}{x_r - x_l} & \text{for } x > y \end{cases}, \tag{156}$$

a function displayed in figure (12). Notice the somewhat unexpected symmetry:

$$G(x,y) = G(y,x). \tag{157}$$

In words, the effect of a force at point y over the solution u at point x is the same as that of a force at point x over the solution at point y. In the engineering literature, this is known as *Maxwell's reciprocity principle*.

We do not need the exact formula (156) to derive this principle; we could have derived it directly from the problem (155) defining the Green's functions. For this, we need a particular instance of the divergence theorem: for any two functions $u(x)$ and $v(x)$,

$$\int_{x_l}^{x_r} (u\ v'' - v\ u'')\ dx = (u\ v' - v\ u')\Big|_{x_l}^{x_r}, \tag{158}$$

which follows from simple integration by parts. Applied to

$$u(x) = G(x,y), \quad v(x) = G(x,z),$$

this yields

$$G(z,y) - G(y,z) = 0,$$

the principle we wanted to prove. The identity (158) is nothing but the proof that the 1d-Laplacian operator with homogeneous boundary conditions is self-adjoint under the canonical inner product. Hence Maxwell's reciprocity principle

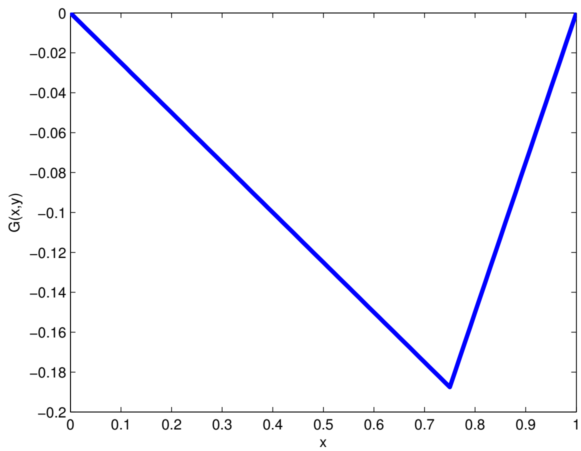

Figure 12: Green's function for the unit segment; here y=0.75.

(157) applies to far more general scenarios than 1d Poisson: if L is a self-adjoint operator, i.e.

$$\int_{\Omega} u \ L(v) \ dx = \int_{\Omega} v \ L(u) \ dx,$$

and

$$L[G(x,y)] = \delta(x-y),$$

then $G(x,y) = G(y,x)$. The space where the functions u, v and G above live is typically -but not necessarily- defined by homogeneous conditions on the boundary $\delta\Omega$.

Thinking in terms of adjoints brings to mind the issue of solvability. Consider the 1d Poisson equation with homogeneous Neumann boundary data:

$$u''(x) = f(x), \quad u'(x_l) = u'(x_r) = 0.$$

Does it have a unique solution? The Fredholm alternative would ask us to look at the homogeneous adjoint problem

$$v''(x) = 0, \quad v'(x_l) = v'(x_r) = 0.$$

Since this has the nontrivial solution

$$v = 1,$$

the original problem will have solutions only if $f(x)$ is orthogonal to $v = 1$:

$$\int_{x_l}^{x_r} f(x) \ dx = 0.$$

Moreover, the solution $u(x)$, if it exists, is not unique, since we can add to it any multiple of v (i.e., any constant.) [4]

The linear algebra behind the Fredholm alternative is clear: if

$$L(u) = f \quad \text{and} \quad L^*(v) = 0,$$

then

$$(v, f) = (v, L(u)) = (L^*(v), u) = 0.$$

The physics is also clear, in terms for instance of the heat equation: a problem with insulating boundaries can only have a steady solution is there is no net generation of heat within the domain, i.e. if $\int f \, dx = 0$.

Questions: Are there similar constraints on existence of solutions and lack of uniqueness for 1-d Poisson with homogeneous Dirichlet data? And for non-homogeneous Neumann data?

7.3 Multidimensions

Most of what we have said about 1-d Poisson extends immediately to the multidimensional scenario, except for the readily availability of exact solutions. Let us summarize the results here. If u solves the Poisson problem with Dirichlet boundary data

$$\Delta u = f(x) \text{ in } \Omega, \quad u(x) = u_b(x) \text{ on } \delta\Omega,$$

we can think of u as a superposition of two solutions,

$$u = u_1 + u_2,$$

where u_1 satisfies Laplace's equation

$$\Delta u_1 = 0 \text{ in } \Omega, \quad u_1(x) = u_b(x) \text{ on } \delta\Omega,$$

and u_2 satisfies Poisson's with homogeneous boundary data,

$$\Delta u_2 = f(x) \text{ in } \Omega, \quad u_2(x) = 0 \text{ on } \delta\Omega.$$

To find u_2, we introduce the Green's functions $G(x, y)$, satisfying

$$\Delta_x G(x, y) = \delta(x - y) \text{ for } x \in \Omega, \quad G(x, y) = 0 \text{ on } \delta\Omega.$$

In terms of these,

$$u_2(x) = \int_\Omega G(x, y) \, f(y) \, dy.$$

From Green's identity (an instance of the divergence theorem)

$$\int_\Omega (u\Delta v - v\Delta u) \, dx = \int_{\delta\Omega} (u\nabla v - v\nabla u) \cdot \vec{n} \, dx, \tag{159}$$

[4] We are using for this last statement the fact that the Laplacian is self-adjoint; otherwise, the free solution to add would be a solution to the homogeneous version of the original problem, not its adjoint.

we infer Maxwell's reciprocity principle (157), by setting $u = G(x, y)$ and $v = G(x, z)$. It so happens that we can use the same Green's functions to solve Laplace's equation with non-homogeneous boundary data. To this end, we can invoke (159) again, but this time setting $u = u_1$ and $v = G(x, y)$. We obtain

$$u_1(y) = \int_{\delta\Omega} u_b(x) \nabla_x G(x, y) \cdot \vec{n} \, dx.$$

Exchanging x and y for notational uniformity, and invoking Maxwell's reciprocity principle, we obtain

$$u_1(x) = \int_{\delta\Omega} u_b(y) \nabla_y G(x, y) \cdot \vec{n} \, dy. \tag{160}$$

Then the effect of a boundary value at the point y on the solution to Laplace's equation at point x is given by the derivative of the Green's function normal to the boundary at y.

Question: What's the intuition behind this fact? *Hint:* Work out what it means in 1d.

As for the solvability of the Poisson problem with Neumann boundary data, the reasoning and results are identical to those in one dimension.

7.4 Fundamental solutions

In the subsection above, we wrote most results in terms of the Green's functions $G(x, y)$, but did not even attempt to find these. For general domains Ω, there is no closed formula for G. For some very symmetric domains though, the situation is not so hopeless. Inspired by our previous work on the heat equation, we start by solving the following equation in all of R^n:

$$\Delta F = \delta(x).$$

Because of the invariance of the equation under rotations, the answer can only depend on r, so

$$\left(r^{n-1} F'(r)\right)' = 0 \text{ for } r > 0,$$

yielding

$$F = -\frac{1}{2\pi} \log(r) \text{ for } n = 2,$$

and

$$F = \frac{c_n}{r^{n-2}} \text{ for } n > 2,$$

the *fundamental solutions* to Laplace's equation, where the constants c_n are chosen so that ΔF integrates to one over a sphere centered at the origin; in particular, $c_3 = \frac{1}{4\pi}$. With these, we can already solve Poisson's equation on all of R^n,

$$\Delta u = f(x),$$

using as Green's function $G(x, y) = F(x - y)$.

Consider next the half-space $x_1 > 0$. To satisfy the boundary condition $G(x, y) = 0$ on $x_1 = 0$, it is enough to add to the source at y an equivalent sink at its *image* y^*, where $y_1^* = -y_1$ and $y_j^* = y_j$ for $j \neq 1$:

$$G(x, y) = F(x - y) - F(x - y^*).$$

This is the *method of images,* which can also be applied to compute the Green's function for the unit n-dimensional ball. To this end, we notice –or remember, if we have studied some analytic geometry before ...– that, for any point y inside the ball, there is another point outside,

$$y^* = \frac{y}{\|y\|^2},$$

the image of y, such that, for all points x on the surface of the ball,

$$\frac{\|x - y^*\|}{\|x - y\|} = \frac{1}{\|y\|},$$

a value independent of x (It is enough to see that this holds in 2d, since y, y^*, x and the center of the sphere all lie on the same plane.) Then

$$G(x, y) = \frac{1}{2\pi} \log \left(\|y\| \frac{x - y^*}{x - y} \right) \text{ for } n = 2$$

and

$$G(x, y) = c_n \left[\|x - y\|^{2-n} - (\|y\| \, \|x - y^*\|)^{2-n} \right] \text{ for } n > 2$$

satisfy

$$\Delta_x G = \delta(x - y) \text{ for } \|x\| < 1, \quad G(x, y) = 0 \text{ for } \|x\| = 1,$$

and so they are the required Green's functions. The resulting formula (160) for the solution to Laplace's equation within a ball goes by the name of Poisson's.

7.5 Laplace's equation mean-value properties

When we studied the heat equation, we noticed that the Laplacian operator Δ could be thought of as modeling diffusion, a process by which a field is rendered more uniform through the local mixing brought about by a fluctuating field. Then Laplace's equation, that models the end product of this diffusion, should have solutions that are as well-mixed as can be. In particular, the solution at each point within the domain should agree with the local average of the solution nearby. This is the physical content of the mean-value property of Laplace's equation. As a result, we should not expect to find any irregularity in the solution within the domain; not even smooth maxima or minima. Thus the mean-value property can be used as a building block for a theory of *regularity*.

In one dimension, the solutions to Laplace's equation are linear. Now, linear functions can be characterized precisely by their mean-value property: the value of the solution at each point is the average of the values at any two equidistant points, one to the right and the other to the left. It is also the average of the solution over any segment centered at the point. This suggests that not only a mean-value property should hold for all solutions to Laplace's equation, but also that the property alone should suffice to characterize these solutions.

For more motivation, consider the standard finite-difference approximation to the Laplacian in two-dimensions:

$$\Delta_d u_i^j = \frac{1}{\Delta x^2}\left(u_{i+1}^{j+1} + u_{i+1}^{j-1} + u_{i-1}^{j+1} + u_{i-1}^{j-1} - 4u_i^j\right).$$

Then $\Delta_d u_i^j = 0$ states that the value of u at each point is the average of its values over the four neighboring points on the lattice [5].

The mean-value properties in all of their generality are the following: if $u(x)$ is a solution to Laplace's equation in $\Omega \in R^n$, then, for any interior point x and any ball $B_r(x)$ lying entirely within the domain Ω, the value of u at x is the average of its values over both the surface $\delta B_r(x)$ of the ball and its interior $B_r(x)$. To prove this, we introduce the average

$$\bar{u}(x,r) = \int_{\delta B_1(x)} u\left(x + r\vec{n}(s)\right)\, ds,$$

where ds is normalized so as to integrate to one over $\delta B_1(x)$. Then

$$\frac{d}{dr}\bar{u}(x,r) = \int_{\delta B_1(x)} \frac{\delta u}{\delta n}\left(x + r\vec{n}(s)\right)\, ds = \int_{B_1(x)} \Delta u \, dx = 0.$$

Since clearly $\lim_{r\to 0} \bar{u}(x,r) = u(x)$, it follows that

$$u(x) = \bar{u}(x,r) \qquad (161)$$

[5] The same applies to any number of dimensions. Yet the statement is true only for equilateral grids; if the grid cells were to be rectangular, the averaging procedure would have to weight the various neighbors by their inverse square distance to the point.

for all $r > 0$. The fact that $u(x)$ is also the average over all of $B_r(x)$ follows from integrating u over $B_r(x)$ along concentric spherical surfaces, and applying the mean value property (161) to each of these.

To prove the converse of these properties, that a function $u(x)$ with continuous second derivatives that satisfies (161) is necessarily *harmonic* –i.e., it satisfies Laplace's equation–, it is enough to notice that, if $\Delta u \neq 0$ at some point x, then one can always find a small enough radius r such that Δu does not change sign within $B_r(x)$. It follows from the divergence theorem then that $\frac{d}{ds}\bar{u}(x, s) \neq 0$ for $s \leq r$, so $\bar{u}(x, s)$ cannot possibly equal $u(x)$, which does not depend on s.

Tu summarize the results of this subsection, the following two statements are equivalent: 1) $u(x)$ satisfies Laplace's equation in a domain Ω; and 2) $u(x)$ has continuous second derivatives and satisfies the mean-value property (161) within Ω.

7.6 Some consequences of the mean-value properties

The mean-value properties above allow us to make quite strong statements about the nature of harmonic functions. We survey here some of the most important ones:

The maximum principle: A function $u(x)$ harmonic in a connected domain Ω cannot achieve its maximum or minimum value in the interior of Ω, unless u is a constant: $\max_\Omega u = \max_{\delta\Omega} u$.
Proof: The average $\bar{u}(x, r)$ cannot be bigger than the maximum value of u on $\delta B_r(x)$, nor smaller than its minimum. Then, if u achieves its maximum value u_{max} on $x \in \Omega$, u must equal u_{max} in all balls $B_r(x)$ that fit within Ω. Since Ω is connected, we can cover it with overlapping balls, so u must equal u_{max} everywhere, i.e. it must be a constant.

Uniqueness: The Dirichlet problem for Poisson's equation,

$$\Delta u = f(x) \text{ in } \Omega, \quad u(x) = u_b(x) \text{ on } \delta\Omega,$$

has at most one solution.
Proof: The difference $w = u_1 - u_2$ between two solutions, u_1 and u_2, satisfies Laplace's equation with homogeneous boundary data,

$$\Delta w = 0 \text{ in } \Omega, \quad w(x) = 0 \text{ on } \delta\Omega.$$

Since w is bounded above and below from its values on $\delta\Omega$, we conclude that w is identically zero, and so $u_1 = u_2$.

When we studied the one-dimensional version of Laplace's equation, we saw that a harmonic function defined on the whole real line is linear, and so it cannot be bounded unless it is a constant. This is a particular instance of the following more general theorem:

Liouville's theorem: A harmonic function $u(x)$ defined over all of R^n that is bounded either above or below is necessarily a constant.
Proof: Denote the bound by D. If a function is harmonic, so is its first derivative. Then, for any $y \in R^n$ and $r > 0$,

$$|\nabla u(y)| = \left| \frac{1}{c_n r^n} \int_{B_r(y)} \nabla u \ dx \right| = \left| \frac{1}{c_n r^n} \int_{\delta B_r(y)} u \ \vec{n}(s) \ ds \right| \leq \left| \frac{n \ D}{r} \right|.$$

Leting $r \to \infty$, we obtain that $|\nabla u(y)| = 0$. It follows that u is a constant.