4 The Wave Equation

In this chapter, we study the wave equation

$$u_{tt} - c^2 \Delta u = 0 \tag{82}$$

and some variations, such as adding a forcing to the right-hand side. We have the following motivations for this study:

- The wave equation appears in a number of important applications, such as sound waves, electromagnetic waves, surface and internal waves in the ocean, and vibrating strings and membranes. It is one of the fundamental equations of theoretical physics.
- It allows us to extend naturally some of the concepts of the first two chapters (characteristics, domains of dependence and influence) to more than one spatial dimension.
- The equation is *linear*. Because of this, the principle of *superposition* applies (the sum of solutions is a solution; the effects of various causes add up.) In the context of the wave equation, this principle has given rise to elegant and insightful classical methodologies.

4.1 Physical origin

The wave equation arises in many physical scenarios. In the context of sound waves, it follows from the linearization of the isentropic gas dynamics equations (24, 25), extended to three–dimensional flows:

$$\begin{aligned} \rho_t + \nabla \cdot (\rho\, \vec{u}) &= 0 \qquad (83)\\ (\rho\, \vec{u})_t + \nabla \cdot (\rho\, \vec{u} \otimes \vec{u}) + \nabla P(\rho) &= 0\,, \qquad (84) \end{aligned}$$

where $\vec{u}$ represents the three-dimensional velocity field. The symbol "$\otimes$" above is to be interpreted so that the the ith component of the equation (84) for momentum conservation reads

$$(\rho\, u_i)_t + \sum_{j=1}^{3} \frac{\partial}{\partial x_j} (\rho\, u_i\, u_j) + \frac{\partial}{\partial x_i} P(\rho) = 0,$$

stating that the ith component of the momentum in an element of volume is forced by its flux across the boundaries of the element, represented by the divergence term, and the derivative of the pressure along the direction x_i.

For small perturbations around a quiescent state with density ρ_0, we propose an expansion

$$\rho = \rho_0 + \epsilon \rho_1\,, \quad \vec{u} = \epsilon \vec{u}_1\,, \quad \epsilon \ll 1$$

and obtain, neglecting all terms quadratic in ϵ, the system

$$\begin{aligned} \rho_t + \nabla \cdot (\rho_0\, \vec{u}) &= 0\\ (\rho_0\, \vec{u})_t + P'(\rho_0)\, \nabla \rho &= 0\,, \end{aligned}$$

where we have dropped the subindices 1 from ρ and $\vec{u}$. Cross–differentiating to eliminate $\vec{u}$, we obtain the wave equation

$$\rho_{tt} - c^2 \Delta \rho = 0\,,$$

where $c^2 = P'(\rho_0)$ is the partial derivative of the pressure with respect to the density at constant entropy, and Δ is the three-dimensional Laplacian

$$\Delta = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Similarly, the extension of the shallow water equations (22, 23) to two spatial dimensions, modeling lakes and oceans, followed by linearization about a quiescent state with $h = h_0$, yields

$$h_{tt} - c^2 \Delta h = 0\,,$$

where $c^2 = g\,h_0$; and Δ is the two-dimensional Laplacian.

For either a channel or for gas flow in a pipe, the one dimensional equations (22, 23) and (24, 25) yield, after linearization, the one-dimensional wave equation

$$u_{tt} - c^2\, u_{xx} = 0\,, \tag{85}$$

where u stands for either h or ρ, and c^2 for $g\,h_0$ or $P'(\rho_0)$ respectively.

Equation (85) arises also in transversal wave propagation along tense strings, with u representing the string's vertical displacement (assumed small), and c^2 given by T/ρ, where T is the tensional force along the spring, and ρ its mass per unit length. Finally, the three-dimensional wave equation

$$u_{tt} - c^2 \Delta u = 0 \tag{86}$$

appears also ubiquitously in electromagnetics. Here c is the speed of light, and u represents either the electric and magnetic fields or their corresponding potentials. This instance of (86) is closely related to the origin of Special Relativity. In this context, all of Maxwell's equations in the vacuum can be written rather succinctly as

$$\Box\, A = 0\,,$$

where $\Box = \partial_{tt} - c^2 \Delta$ is the wave operator or "D'Alambertian", and A is the four-vector potential.

To simplify notation, we shall rescale space and time so that the speed c equals one. Whenever reference is needed to the linearized shallow water or gas dynamics equations, we shall also choose our units so that g, h_0, ρ_0 and $P'(\rho_0)$ all equal one.

4.2 The one–dimensional wave equation

In one dimension, the wave equation (85) becomes, after normalization,

$$u_{tt} - u_{xx} = 0\,. \tag{87}$$

If we think of this equation as arising from the linearized shallow water system

$$h_t + u_x = 0 \tag{88}$$

$$u_t + h_x = 0\,, \tag{89}$$

with initial data

$$h(x,0) = h_0(x)$$
$$u(x,0) = u_0(x)\,,$$

then we have already discussed how to find the solution in terms of Riemann invariants. Adding and substracting (88) and (89), we obtain their *characteristic form*

$$(u+h)_t + (u+h)_x = 0$$
$$(u-h)_t - (u-h)_x = 0\,.$$

Then $R^{\pm} = u \pm h$, $\lambda^{\pm} = \pm 1$ (information travels to the right and left at speed 1), and the solution is

$$u(x,t) + h(x,t) = u_0(x-t) + h_0(x-t)$$
$$u(x,t) - h(x,t) = u_0(x+t) - h_0(x+t)\,,$$

so

$$\begin{aligned} h(x,t) &= \frac{h_0(x-t)+h_0(x+t)}{2} + \frac{u_0(x-t)-u_0(x+t)}{2} \\ u(x,t) &= \frac{u_0(x-t)+u_0(x+t)}{2} + \frac{h_0(x-t)-h_0(x+t)}{2}\,. \end{aligned} \tag{90}$$

On the other hand, if equation (87) models a vibrating string, more natural initial data are

$$u(x,0) = f(x) \tag{91}$$

$$u_t(x,0) = g(x)\,, \tag{92}$$

i.e. the initial vertical displacement and velocity of the string. The solution (90) can be translated easily to this case by realizing that, in the context of linearized shallow waters,

$$u_t(x,0) = -h_x(x,0)\,,$$

so

$$u(x,t) = \frac{f(x-t)+f(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds\,. \tag{93}$$

This is the full solution to the initial value problem for the wave equation in one spatial dimension. Another, more customary derivation, writes the general solution to (87) as a sum of left and right traveling waves:

$$u(x,t) = F(x-t) + G(x+t)\,, \tag{94}$$

and then finds (93) by fitting F and G to satisfy the initial data (92).

4.3 A semi-infinite string

We shall now solve the wave equation (87) with boundary condition

$$u(0,t) = 0$$

and initial data given by (92), but only for positive values of x. Our motivation is two-fold. On the one hand, we would like to learn about the effects of boundary conditions; in this respect, the semi-infinite string with a fixed end is the simplest problem with a boundary. On the other, we shall find in subsection 4.4 that, by a mathematical twist, the solution to this problem allows us to write the solution to the initial value problem for the wave equations in the unbounded three-dimensional space in a rather simple way.

The wave equation preserves the oddity of solutions. To see this, note that changing x into $-x$ leaves equation (92) unchanged, as does turning u into $-u$. Hence, if $u(x,t)$ is a solution, so is $-u(-x,t)$. If the initial data are odd ($f(x) = -f(-x)$, $g(x) = -g(-x)$), then both $u(x,t)$ and $-u(-x,t)$ will solve the same initial value problem, which has a unique solution. So $u(x,t)$ and $-u(-x,t)$ must be equal, and the solution remains odd for all times.

Hence, to solve the semi-infinite string problem, we extend it to an odd solution on the whole real line, since odd solutions automatically satisfy $u(0,t) = -u(0,t) = 0$. Then the solution (93) applies, and we have

$$u(x,t) = \begin{cases} \frac{f(x+t)+f(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds & \text{for } x > t \\ \frac{f(x+t)-f(t-x)}{2} + \frac{1}{2}\int_{t-x}^{t+x} g(s)\,ds & \text{for } x < t. \end{cases} \tag{95}$$

To the right of the line $x = t$, the solution does not see the boundary at $x = 0$, and has the same form as for the doubly infinite string. On the left of $x = t$, on the other hand, we do observe the effects of wave reflection at $x = 0$: f changes sign, and the reflected g cancels part of the incoming one.

4.4 The method of spherical means

In order to solve the initial value for the wave equation in dimensions higher than one, we shall use the method of spherical means, due to Hadamard. The intuitive grounds for this methodology lies in our conception of waves as being made of a superposition of spherically symmetric fronts emanating from point sources. Huygens was a pioneer of this view of waves, which he used to show that the laws of optics suggested that light was a wave phenomenon.

A building block for the method is the exact solution to the wave equation for spherically symmetric waves in n-dimensional space: [2]

$$u_{tt} - \left(u_{rr} + \frac{n-1}{r}u_r\right) = 0\,. \tag{96}$$

---

[2]For a geometrical derivation of the form that the Laplacian adopts under spherical symmetry, first write it in divergence form: $\Delta u = \nabla\cdot(\nabla u)$, and then apply the divergence theorem to a volume elemement V bounded by two surfaces of constant radius and a cone with vertex at the origin: $\int_V \Delta u\, dV = \int_{\delta V} u_n\, dS$, or $\Delta u\, r^{n-1}\, dS\, dr = (r^{n-1}u_r)_r\, dS\, dr$, where dS is the area element of a ball of unitary radius. It follows that $\Delta u = u_{rr} + \frac{n-1}{r}u_r$.

When n is odd, equation (96) can be reduced to the one-dimensional wave equation (87). We perform here this reduction for $n = 3$, the most ubiquitous case in applications. In this case, it is enough to introduce a variable $v = r\,u$. Clearly $v_{tt} = r\,u_{tt}$, and $v_{rr} = r\,u_{rr} + 2\,u_r$. Then (96) is just

$$v_{tt} - v_{rr} = 0\,,$$

the one-dimensional wave equation for v. Notice that v is defined only for positive values of r, and that $v(0,t)$ vanishes. Then v solves the semi-infinite string problem of the prior subsection. Its solution (95), in terms of $u = v/r$, reads

$$u(r,t) = \begin{cases} \frac{(r+t)\,u(r+t,0)+(r-t)\,u(r-t,0)}{2\,r} + \frac{1}{2\,r}\int_{r-t}^{r+t} s\,u_t(s,0)\,ds & \text{for } r > t \\ \frac{(r+t)\,u(r+t,0)-(t-r)\,u(t-r,0)}{2\,r} + \frac{1}{2\,r}\int_{t-r}^{t+r} s\,u_t(s,0)\,ds & \text{for } r < t. \end{cases} \tag{97}$$

With the exact solution to the spherically symmetric case in hand, we could write the solution to the three-dimensional wave equation with general initial data as a superposition of infinitely many spherically symmetric waves, each arising from an individual point at time zero. In a language to be introduced later, in the context of the heat equation, this would be the *Green's function* approach to constructing a solution. The method of spherical means, that we describe now, can be though of as a short-cut for the construction of Green's functions, or rather as a twist of the idea: instead of looking for the solution generated by each local piece of the initial data, we switch the problem around, and figure out which points at time zero contribute to the solution for each value of (x,t).

For a function $u(x,t)$ satisfying the wave equation

$$\Box\, u = u_{tt} - \Delta u = 0 \tag{98}$$

with initial data

$$u(x,0) = f(x)\,, \quad u_t(x,0) = g(x)\,, \tag{99}$$

we introduce the spherical mean $U(x,t,r)$ as the average value of $u(s,t)$ on the surface of a sphere with center x and radius r:

$$U(x,t,r) = \int_{B(x,r)} u(s,t) dS_r = \int_{B(x,1)} u(x + r\,\psi, t) dS_1(\psi)\,, \tag{100}$$

where $B(x,r)$ is the surface of a ball or radius r centered at x, dS_r, the area element, is normalized so as to integrate to one, and ψ is a unit vector spanning all spherical angles. If we find an expression for U, the solution u can be found easily averaging it over a vanishingly small sphere:

$$u(x,t) = lim_{r\to 0} U(x,t,r)\,. \tag{101}$$

A priori, it is not clear why finding U, a function of one extra variable, should be any easier than computing $u(x,t)$. The "magical" reason is that, for fixed x, U solves a simpler PDE:

$$U_{tt} - \left(U_{rr} + \frac{n-1}{r} U_r \right) = 0\,, \tag{102}$$

i.e., the spherically symmetric wave equation (96). Since we know how to solve this equation exactly, at least for $n = 3$, writing the solution $u(x,t)$ from (101) becomes a straightforward exercise.

If $u(x,t)$ were spherically symmetric around x, then $U(x,t,r)$ and $u(x+r\,\psi,t)$ would agree, so U would have to satisfy (102). The easiest way to see that (102) holds for general data, is to average the wave equation (98) over a sphere of radius r around x:

$$\int_{B(x,1)} \left[u_{tt}(x + r\,\psi, t) - \Delta u(x + r\,\psi, t) \right] dS_1(\psi) = 0\,.$$

Clearly the first term yields $U(x,t,r)_{tt}$. To evaluate the second term, we can write the Laplacian in polar coordinates centered at x. Since all angular derivatives integrate to zero over the surface of the sphere, the second term reduces to

$$\begin{aligned} &\textstyle\int_{B(x,1)} \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} \right) u(x + r\,\psi, t) dS_1(\psi) = \\ &= \textstyle\left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} \right) \int_{B(x,1)} u(x + r\,\psi, t) dS_1(\psi) = \\ &= \textstyle\left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} \right) U(x,t,r)\,, \end{aligned}$$

thus proving (102).

We are now ready to write the exact solution $u(x,t)$ to the three-dimensional wave equation, using (97) and (101):

$$\begin{aligned} u(x,t) &= \textstyle\lim_{r\to 0} U(x,t,r) = \\ &= \textstyle\lim_{r\to 0} \left(\frac{(r+t)\,F(x,r+t) - (t-r)\,F(x,t-r)}{2\,r} + \frac{1}{2\,r} \int_{t-r}^{t+r} s\,G(x,s)\,ds \right) = \\ &= \textstyle\frac{\partial}{\partial t}(t\,F(x,t)) + tG(x,t)\,, \end{aligned} \tag{103}$$

where $F(x,r)$ and $G(x,r)$ are the spherical means of $f(x)$ and $g(x)$ respectively.

Notice that this solution has the property that the value of $u(x,t)$ depends only on the initial data lying *exactly* at a distance t from x. In other words, information propagates at speed one along sharp fronts; once a front has past though a point, its information is immediately forgotten. This property, the *strong version of Huygens' principle*, is valid not only for $n = 3$, but in all odd-dimensional spaces other than $n = 1$. It does not apply, however, to the wave equation in spaces of even dimensionality. There, even though information still propagates at speed one, it does not do it through sharp fronts, leaving instead a trace behind as it passes through a point. Hence, when a tsunami shakes the two-dimensional ocean, it leaves significant wave action behind its leading front. We shall see now that this is the case in two dimensions, using the *method of* *descent*. The same methodology applies to all even dimensions $n = 2d$, once we have the general solution to the initial value problem in the odd-dimensional space $n = 2d + 1$.

4.5 The method of descent

The method of descent, also due to Hadamard, consists simply of thinking of any solution to the wave equation in even ($n = 2d$) dimensions, as a solution in one more dimension that happens not to depend on one of the space variables. In two dimensions, in particular, we can write

$$u(x, y, t) = \tilde{u}(x, y, z, t)\,,$$

where $\tilde{u}$ is a solution to the three–dimensional wave equation with initial data that do not depend on z:

$$\tilde{u}(x, y, z, 0) = \tilde{f}(x, y, z) = f(x, y)\,, \quad \tilde{u}_t(x, y, z, 0) = \tilde{g}(x, y, z) = g(x, y)\,.$$

For $\tilde{u}$ we have the exact formula (103), so the same applies to u. However, by definition, the corresponding $\tilde{F}(x, r)$ and $\tilde{G}(x, r)$ are the spherical means over three–dimensional balls of functions $\tilde{f}(x)$ and $\tilde{g}(x)$ that do not depend on z. Then we have

$$\tilde{F}(x, r) = \int_{B(x,r)} \tilde{f}(s)\, dS_r = \int_{S(x,r)} f(s)\, J\, dA\,,$$

where B is the surface of a three–dimensional sphere, S is the surface of a two–dimensional circle, and J is the Jacobian

$$J = \frac{r}{|s - x|}$$

that projects one area element onto the other. For our purposes, it is enough to notice that now the formula for u involves integrals over the *interior* of circles of radius t, not just their circumference. Hence the strong version of Huygens principle does not apply in two dimensions: the solution to the wave equation at point x and time t depends on all the initial data within a circle of radius t around x, not just on their values and derivatives on the circumference $|y - x| = t$.

4.6 Duhamel's principle

Having solved the initial value problem for the wave equation, now we switch to the *forced* wave equation

$$\Box\, u = f(x, t)\,, \tag{104}$$

where f is a prescribed function of space and time. Depending on the problem, f may represent an external force acting on the system, or a source or sink of mass, such as rain and infiltration for river flow. In Maxwell's equations in relativistic notation, where u is the four-vector potential A, f is the four current-density vector j.

Notice that, due to linearity, it is enough to consider the simplest initial data

$$u(x,0) = u_t(x,0) = 0\,, \tag{105}$$

since problems with more general data can be though of as the superposition of the forced wave equation (104) with the trivial data (105), and the unforced wave equation (98) with initial data (99), which we already know how to solve.

The idea behind Duhamel's principle is to think of the forcing $f(x,t)$ as the superposition of infinitely many, spatially dependent "pushes" acting at individual times t. Each of this pushes, or *impulses*, in the language of Newtonian mechanics, has the effect of making u_t jump by $f\,dt$. Hence the solution to (104) can be reduced to the superposition of very many initial value problems for the unforced wave equation (98), each with $u(x,t_0) = 0$ and $u_t(x,t_0) = f(x,t_0)$. Let us denote by $U(x,t,s)$ the solution for $t \geq s$ of the initial value problem

$$\Box U = 0\,, \quad U(x,s,s) = 0 \quad U_t(x,s,s) = f(x,s)\,. \tag{106}$$

Our claim is that $u(x,t)$ can be written as a linear superposition of the U's:

$$u(x,t) = \int_0^t U(x,t,s)\,ds\,. \tag{107}$$

To prove (107), it is enough to take the D' Alambertian of u:

$$\Box\, u = \Box \int_0^t U(x,t,s)\,ds = \int_0^t \Box\, U(x,t,s)\,ds + \frac{d}{dt} U(x,t,t) + U_t(x,t,t) = f(x,t)\,.$$

Notice that (107) implies a *causality* principle for the forced wave equation: the solution $u(x,t)$ depends on the forces acting from time zero up to t, not on those yet to act in the future.

4.7 More to come

There is much more to learn about the wave equation. We will postpone further treatment until we learn more about the Laplacian and about Fourier synthesis.