Introduction to PDEs 2018, third assignment, due Monday October 1st

1) Prove equations (63, 64, 65) in the notes: that the "conservation of waves" equation

$$k_t + \nabla_x \omega = 0,$$

where $k = \nabla_x \theta$, $\omega = -\theta_t$ and $\omega(x,t) = \Omega(k(x,t),x,t)$, implies the characteristic equations:

$$\dot{x} = C_g = \nabla_k \Omega \quad \Rightarrow \quad \dot{k} = -\nabla_x \Omega.$$

2) In the Geometrical Optics approximation, the light-fronts $\Psi(x,y)$ –fronts along which the phase is constant– satisfy the Eikonal equation

$$\Psi_x{}^2 + \Psi_y{}^2 = \frac{1}{c^2}$$

where c is the speed of light. Consider a flat Earth at $y = 0$, and model the variation of c with height with the simple exponential for c^{-2}

$$\frac{1}{c^2} = 1 + e^{-y}$$

(That is, the speed of light increases with height, as the air becomes more rarefied.)

Compute the trajectory of all rays of light –i.e. the characteristics of the Eikonal– leaving the point $(x,y) = (0,0)$ with angles between 0 and $\pi/4$ with the horizontal, until they touch ground again, and plot a few of these rays. Does this solution shed light on why we often see distant mountains as if they were floating above the horizon? What happens to those rays of light which leave ground with an angle larger than $\pi/4$?

Bonus question: Propose and solve a similar model that explains why, while driving on very sunny days, one sees ghost water puddles on the hot pavement (Hint: it is the sky that you are seeing.)