Introduction to PDEs 2018, seventh assignment, due Monday November 5th

Consider the following stochastic process in the interval $x \in [0, 1]$:

1. Divide the interval into n equal segments of size $\Delta x = \frac{1}{n}$, and introduce the points $x_i = i * \Delta x$, for $i = 0, 1, \ldots n$. Similarly, introduce the times $t_j = j * \Delta t$, with $\Delta t = \frac{1}{n^2}$.

2. Assign to each x_i a number of particles $n_i^0 = [n * f(x_i)]$, where the brackets stand for rounding off to the nearest integer, and $f(x) = \sin\left(\frac{\pi}{2}x\right)$. Call N the total number of particles assigned, i.e. $N = \sum_i n_i^0$.

3. Each particle k starts at position X_k^0 given by its assignment above, and then, for each j, it has $X_k^{j+1} = X_k^j - \Delta x$ with probability $\frac{1}{2} + \frac{1}{n}$ and $X_k^{j+1} = X_k^j + \Delta x$ with probability $\frac{1}{2} - \frac{1}{n}$. If particle k arrives at $X_k^{j+1} = -\Delta x$, it is eliminated. If it arrives at $X_k^{j+1} = 1 + \Delta x$ instead, it is reassigned to $X_k^{j+1} = 1$.

4. Count the number of particles n_i^j at position x_i at time t_j.

1) As n grows, n_i^j approaches $p(x_i, t_j)$, where $p(x, t)$ is a smooth function. Which equation, with which boundary conditions and initial data does p satisfy?

2) Simulate the process for $n = 50$ and $n = 100$ and plot your results for $t = 0$, $t = 0.1$ and $t = 0.3$, as well as the trajectories up to $t = 0.3$ (or up to their disappearance) of 5 particles for each of the two values of n. Make sure that at least one of the 10 particles chosen makes it to time 0.3!

A little Matlab hint: the instruction

$$r = rand(1, m) > pr;$$

generates m independent random numbers r that are 1 with probability $1 - pr$ and 0 with probability pr.

3) If you eliminate the *drift* terms $\pm\frac{1}{n}$ from the stochastic process, then the model that you derived in part **1)** reduces to one with a very simple exact solution. Compare it to the simulation of part **2)** (repeated without drift) by superimposing the corresponding plots.