On the force and torque on systems of rigid bodies: A remark on an integral formula due to Howe

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M. S. Howe [J. Fluid Mech. **206**, 131 (1989)] presented integral formulas for the force and torque on a rigid body that permit the identification of the separate influences of added mass, normal stresses induced by free vorticity, and viscous skin friction. Here a simple extension of Howe's formulas is done for systems of several rigid bodies. The goal is to separate these effects on each rigid body. These new formulas may help in understanding the hydrodynamic interaction between bodies. As an example, the interaction forces on a pair of bubbles rising at high Reynolds number side by side are computed. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2730481]

I. INTRODUCTION

The problem of finding the force on a rigid body moving inside a fluid is central in many branches of engineering, so there is an enormous literature dedicated to the subject. In principle, if the motion of the fluid is known, then the computation of forces reduces to the integration of the fluid stresses on the body surface. The point is that, except for bodies moving at very low speeds, there is no realistic problem for which the fluid velocity can be determined. Thus, many force and torque integral expressions alternative to integrating stresses on the body surface were proposed. These alternative expressions may be more suitable for experimental measurements, for numerical evaluation, or just for theoretical insight. The reader can find a partial list of the literature on the subject in Refs. 1–4.

This paper is concerned with a particular form of force and torque expressions proposed by Howe.[1] These are based on formulas used in the theory of aerodynamic sound previously obtained by the same author.[5–7] Expressions of the same type were obtained independently by Quartapelle and Napolitano[8] and Chang.[9] The advantage of the force formulas proposed by these authors, especially those in Refs. 1 and 9, is that they allow a partial identification of the separated influence of added mass, skin friction, and normal stresses induced by vorticity (see also Ref. 10 about this separation of effects).

Here we generalize the force and torque expressions given in Howe[1] for a single moving body, to the case of a system of many moving bodies, in such a way that the above-mentioned separation of effects can be, at least partially, performed independently on each body. Howe himself indicated implicitly this generalization in Ref. 5; it is also natural in the context of the work of Quartapelle and Napolitano[8] and Chang.[9] The particular case of the interaction between a plane wall and an airfoil fixed to it was explicitly treated by Howe[11] using the same idea as that used here. So, our main contribution in this paper is to pursue the details of this natural generalization of the work by Howe to any system of many moving bodies and to show that it may be useful in the study of the problem of hydrodynamic interaction between bodies.

Our work is divided as follows. In Sec. II, we present a generalization of Howe's force expression to the case of many bodies. The way we obtain this generalization is the same one that Chang[9] used to get his formulas, which is a little different from the procedure of Howe's.[1] Section II has a subsection about the particular case of no-slip boundary conditions.

In Sec. III, we generalize the torque formula of Howe to the case of many bodies. The procedure is very similar to that used for the force formula. In Sec. IV, we compute the drag and the lift on a pair of bubbles rising side by side with the same constant velocity and at high Reynolds number. For this, we were guided by the detailed numerical work of Legendre *et al.*[12] The idea was to apply the force formula obtained in Sec. II to a system in which the velocity field could be estimated analytically. The problem and its difficulties are discussed in Sec. IV.

II. FORCE FORMULAS

The hypotheses and notation used in this paper are as follows. Consider a system of N rigid bodies whose boundaries are smooth (i.e., continuously differentiable) surfaces. In each body, labeled $\alpha \in \{1,2,\ldots,N\}$, fix a reference point and an orthogonal reference frame K_α centered on it. Let $K = \{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ denote a reference frame fixed in space. The configuration of the system is determined by N position vectors $\mathbf{R}_\alpha$ and N orthogonal transformations T_α, where $\mathbf{R}_\alpha$ is the position of the reference point of solid α with respect to K, and $T_\alpha: K_\alpha \to K$ describes the orientation of body α with respect to K. (T_α is the attitude matrix of body α.) The velocity and angular velocity of body α with respect to K will be denoted by $\dot{\mathbf{R}}_\alpha$ and $\mathbf{\Omega}_\alpha$, respectively. The set of points of body α will be denoted by B_α, and its boundary by ∂B_α. The

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bodies move in a prescribed way, without collision, through an incompressible liquid of density ρ and dynamic viscosity μ. The region outside the bodies and occupied by the fluid will be denoted by Fl. The region Fl can be either bounded or unbounded. If Fl is bounded, then its "external" boundary will be denoted by ∂B_N. In this case, ∂B_N can be understood as the boundary of a distinguished body B_N which contains the liquid and all other bodies in its interior. If ∂B_N is at rest, then $\dot{\mathbf{R}}_N$ and $\mathbf{\Omega}_N$ are null for all time. The case of partially bounded domains can also be treated but it will not be explicitly considered here. The velocity field of the fluid will be denoted by $\mathbf{v}$ and its vorticity field by $\omega=\nabla\times\mathbf{v}$. If Fl is unbounded, it will be assumed that the velocity and vorticity fields decay sufficiently fast at infinity, such that the several surface integrals at infinity that appear below can be neglected. It suffices that $\mathbf{v}=\mathcal{O}(1/|\mathbf{x}|^2)$ and $\omega=\mathcal{O}(1/|\mathbf{x}|^3)$. Notice that these surface integrals at infinity may also be zero under milder hypotheses as in the case in which the velocity and vorticity decay varies with the direction with slower decay within narrow wake regions. However, we will not try to present the most general hypotheses under which our results hold. Decay estimates for the velocity and vorticity can be found in several places such as, for instance, the appendix of Ref. 1 or lemma 2.1 of Ref. 3 The pressure p is assumed bounded at infinity.

In Howe's formulas, a prominent role is played by certain velocity potentials. Here, for given body α and direction $\mathbf{e}_i$, the analogue potential is provided by the solution of the following problem:

$$\begin{aligned}
&\Delta\phi_{\alpha i}(\mathbf{x},t)=0 \quad \text{for } \mathbf{x}\in \mathrm{Fl},\\
&\nabla\phi_{\alpha i}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t)=\mathbf{e}_i\cdot\mathbf{n}(\mathbf{x},t) \quad \text{for } \mathbf{x}\in\partial B_\alpha,\\
&\nabla\phi_{\alpha i}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t)=0 \quad \text{for } \mathbf{x}\in\partial B_\beta,\ \beta\neq\alpha,\\
&|\nabla\phi_{\alpha i}(\mathbf{x},t)|=\mathcal{O}(1/|\mathbf{x}|^3)\to 0 \quad \text{as } |\mathbf{x}|\to\infty,
\end{aligned} \tag{1}$$

where the last condition applies to the case in which Fl is unbounded. The irrotational velocity field $\nabla\phi_{\alpha i}$ is that of an ideal fluid initially at rest acted upon by the motion of body B_α with velocity $\mathbf{e}_i$ while all other bodies remain at rest. Throughout this paper, $\mathbf{n}(\mathbf{x},t)$ denotes the unit normal vector at boundary points $\mathbf{x}\in\partial B_\alpha$ at time t, which is directed toward the inside of B_α. Equation (1) depends on t through the position and orientation of bodies B_β, $\beta=1,\ldots,N$, which change as the bodies move. So, $\phi_{\alpha i}$ also depends on t. In the usual case of a single body, it is enough to solve Eq. (1) at $t=0$ for $i=1,2,3$. The solutions to problems (1) for all other positions and orientations can be obtained through convenient translations and rotations of those at $t=0$. In the case of many bodies, this is not true anymore. Nevertheless, if the pairwise distance between bodies is sufficiently large, useful approximated solutions for problem (1) in terms of one-body solutions can still be obtained (see, for instance, Refs. 13–16). This will be done for the problem treated in Sec. IV.

The following derivation of force expressions is based on that of Howe,[1] although we start differently writing the i component of the force, $F_{\alpha i}$, which acts on body B_α as the integral of the stress tensor over ∂B_α,

$$F_{\alpha i}=\int_{\partial B_\alpha} p\mathbf{e}_i\cdot\mathbf{n}-\mu\int_{\partial B_\alpha}(\partial_{x_j}v_i+\partial_{x_i}v_j)n_j, \tag{2}$$

where the sum convention over repeated indices is assumed. Using the identity

$$0=\int_{\partial B_\alpha}\nabla\times(\mathbf{v}\times\mathbf{e}_i)\cdot\mathbf{n}=\int_{\partial B_\alpha}(\partial_{x_i}v_j)n_j,$$

the viscous part of (2) can be rewritten as

$$\begin{aligned}
-\mu\int_{\partial B_\alpha}(\partial_{x_j}v_i+\partial_{x_i}v_j)n_j&=-\mu\int_{\partial B_\alpha}(\partial_{x_j}v_i-\partial_{x_i}v_j)n_j\\
&=-\mu\int_{\partial B_\alpha}\mathbf{e}_i\times\omega\cdot\mathbf{n}.
\end{aligned} \tag{3}$$

To handle the pressure term, we multiply the Navier-Stokes equation

$$\rho\partial_t\mathbf{v}+\nabla\left(p+\frac{\rho}{2}|\mathbf{v}|^2\right)=\rho\mathbf{v}\times\omega-\mu\nabla\times\omega \tag{4}$$

by $\nabla\phi_{\alpha i}$, and integrate over the region Fl occupied by the fluid,

$$\begin{aligned}
&\rho\int_{\mathrm{Fl}}\nabla\phi_{\alpha i}\cdot\partial_t\mathbf{v}+\int_{\mathrm{Fl}}\nabla\phi_{\alpha i}\cdot\nabla\left(p+\frac{\rho}{2}|\mathbf{v}|^2\right)\\
&\quad=\rho\int_{\mathrm{Fl}}\nabla\phi_{\alpha i}\cdot\mathbf{v}\times\omega-\mu\int_{\mathrm{Fl}}\nabla\phi_{\alpha i}\cdot\nabla\times\omega.
\end{aligned} \tag{5}$$

Each term in this sum will be analyzed separately. First,

$$\int_{\mathrm{Fl}}\nabla\phi_{\alpha i}\cdot\nabla p=\int_{\mathrm{Fl}}\mathrm{div}(p\nabla\phi_{\alpha i})=\int_{\partial B_\alpha}p\mathbf{e}_i\cdot n,$$

where we have used the facts that $\nabla\phi_{\alpha i}\cdot\mathbf{n}$ equals $\mathbf{e}_i\cdot\mathbf{n}$ if $\mathbf{n}\in\partial B_\alpha$ and vanishes if $\mathbf{n}\in\partial B_\beta$, with $\beta\neq\alpha$, and that the surface integral at infinity of $p\nabla\phi_{\alpha i}\cdot\mathbf{n}$ is zero. (For simplicity, we will always refer to the case of an unbounded fluid domain.) Notice that the expression on the right-hand side of the above equation is exactly the pressure force term that appears in Eq. (2). In a similar way, we get

$$\begin{aligned}
\frac{\rho}{2}\int_{\mathrm{Fl}}\nabla\phi_{\alpha i}\cdot\nabla|\mathbf{v}|^2&=\frac{\rho}{2}\int_{\mathrm{Fl}}\mathrm{div}(|\mathbf{v}|^2\nabla\phi_{\alpha i})\\
&=\frac{\rho}{2}\int_{\partial B_\alpha}|\mathbf{v}|^2\mathbf{e}_i\cdot n
\end{aligned}$$

and

$$-\mu\int_{\mathrm{Fl}}\nabla\phi_{\alpha i}\cdot\nabla\times\omega=\mu\int_{\mathrm{Fl}}\mathrm{div}(\nabla\phi_{\alpha i}\times\omega)$$
$$=\mu\sum_{\beta=1}^{N}\int_{\partial B_\beta}\nabla\phi_{\alpha i}\times\omega\cdot\mathbf{n}.$$

Finally,

$$\int_{\mathrm{Fl}}\nabla\phi_{\alpha i}\cdot\partial_t\mathbf{v}=\int_{\mathrm{Fl}}\partial_t(\nabla\phi_{\alpha i}\cdot\mathbf{v})-\int_{\mathrm{Fl}}\mathbf{v}\cdot\partial_t\nabla\phi_{\alpha i}$$
$$=\partial_t\int_{\mathrm{Fl}}\nabla\phi_{\alpha i}\cdot\mathbf{v}-\int_{\mathrm{Fl}}\mathbf{v}\cdot\nabla(\mathbf{v}\cdot\nabla\phi_{\alpha i})$$
$$-\int_{\mathrm{Fl}}\mathbf{v}\cdot\nabla\partial_t\phi_{\alpha i}$$
$$=\partial_t\int_{\mathrm{Fl}}\mathrm{div}(\phi_{\alpha i}\mathbf{v})-\int_{\mathrm{Fl}}\mathrm{div}[(\mathbf{v}\cdot\nabla\phi_{\alpha i})\mathbf{v}]$$
$$-\int_{\mathrm{Fl}}\mathrm{div}(\mathbf{v}\partial_t\phi_{\alpha i})$$
$$=\partial_t\left\{\sum_{\beta=1}^{N}\int_{\partial B_\beta}\phi_{\alpha i}\mathbf{v}\cdot\mathbf{n}\right\}$$
$$-\sum_{\beta=1}^{N}\int_{\partial B_\beta}(\partial_t\phi_{\alpha i}+\mathbf{v}\cdot\nabla\phi_{\alpha i})\mathbf{v}\cdot\mathbf{n},$$

where we have again used the fact that the surface integrals at infinity vanish, due to the decay hypotheses. Collecting all the above terms, we obtain a force expression that is an analogue to Howe's equation (2.11) in Ref. 1,

$$F_{\alpha i}=-\rho\partial_t\left\{\sum_{\beta=1}^{N}\int_{\partial B_\beta}\phi_{\alpha i}\mathbf{v}\cdot\mathbf{n}\right\}$$
$$+\rho\sum_{\beta=1}^{N}\int_{\partial B_\beta}(\partial_t\phi_{\alpha i}+\mathbf{v}\cdot\nabla\phi_{\alpha i})\mathbf{v}\cdot n$$
$$-\frac{\rho}{2}\int_{\partial B_\alpha}|\mathbf{v}|^2\mathbf{e}_i\cdot n+\rho\int_{\mathrm{Fl}}\nabla\phi_{\alpha i}\cdot\mathbf{v}\times\omega$$
$$+\mu\sum_{\beta=1}^{N}\int_{\partial B_\beta}\nabla\phi_{\alpha i}\times\omega\cdot\mathbf{n}-\mu\int_{\partial B_\alpha}\mathbf{e}_i\times\omega\cdot\mathbf{n}.\qquad(6)$$

Remark: Since each B_β is a rigid body, the factor $\mathbf{v}\cdot\mathbf{n}$ that appears in all surface integrals above is given by

$$\mathbf{v}\cdot\mathbf{n}=[\dot{\mathbf{R}}_\beta+\mathbf{\Omega}_\beta\times(\mathbf{x}-\mathbf{R}_\beta)]\cdot\mathbf{n},$$

where $\mathbf{x}\in\partial B_\beta$. □

Remark: In the case of a single body, denoted just by α, Eq. (6) is not exactly equation (2.11) in Ref. 1 obtained by Howe. Indeed, Howe's expression is more compact, and it is written not in terms of the function $\phi_{\alpha i}$, but of $X_i=x_i-\mathbf{R}_{\alpha i}-\phi_{\alpha i}$. For a single body in $\mathbb{R}^3$, the following identity holds:

$$\int_{\partial B_\alpha}\frac{|\mathbf{v}|^2}{2}\mathbf{e}_i\cdot\mathbf{n}=\int_{\mathrm{Fl}}\mathrm{div}\left[\frac{|\mathbf{v}|^2}{2}\mathbf{e}_i\right]$$
$$=\int_{\mathrm{Fl}}\nabla\left[\frac{|\mathbf{v}|^2}{2}\right]\cdot\mathbf{e}_i$$
$$=\int_{\mathrm{Fl}}\mathbf{v}\times\omega\cdot\mathbf{e}_i+\int_{\mathrm{Fl}}(\mathbf{v}\cdot\nabla\mathbf{v})\cdot\mathbf{e}_i$$
$$=\int_{\mathrm{Fl}}\mathbf{v}\times\omega\cdot\nabla x_i+\int_{\partial B_\alpha}v_i\mathbf{v}\cdot\mathbf{n}.\qquad(7)$$

Using this identity in Eq. (6), one easily obtains Howe's expression,

$$F_{\alpha i}=-\rho\partial_t\int_{\partial B_\alpha}\phi_{\alpha i}\mathbf{v}\cdot\mathbf{n}-\rho\int_{\partial B_\alpha}(\partial_tX_i+\mathbf{v}\cdot\nabla X_i)\mathbf{v}\cdot\mathbf{n}$$
$$+\rho\int_{\mathrm{Fl}}\nabla X_i\cdot\omega\times\mathbf{v}+\mu\int_{\partial B_\alpha}\omega\times\nabla X_i\cdot\mathbf{n}.$$

If there is more than one body, however, it is not possible to eliminate the integrals $\int|\mathbf{v}|^2\mathbf{e}_i\cdot\mathbf{n}$ from the surfaces of all bodies. So, although Eq. (6) can be rewritten in other forms using functions like X_i, the new expressions do not look simpler than Eq. (6). □

Remark (Ideal flows): Howe[1] makes an interesting analysis of forces in inviscid irrotational and rotational flows for the case of a single body. The analysis becomes much more complicated in the presence of more bodies. Let us briefly discuss the case of irrotational flows. In this case, only the first two lines of Eq. (6) are non-null. For a single body in $\mathbb{R}^3$, denoted by B_α, it is possible to show that the second line is also null. Indeed, in this case $v=\nabla\varphi$ and Eq. (7) becomes

$$\int_{\partial B_\alpha}\frac{|\nabla\varphi|^2}{2}\mathbf{e}_i\cdot\mathbf{n}=\int_{\partial B_\alpha}(\partial_{x_i}\varphi)\mathbf{v}\cdot\mathbf{n}.$$

Then the second line of (6) becomes

$$\rho\int_{\partial B_\alpha}(\partial_t\phi_{\alpha i}+\nabla\varphi\cdot\nabla\phi_{\alpha i})\mathbf{v}\cdot n-\rho\int_{\partial B_\alpha}(\partial_{x_i}\varphi)\mathbf{v}\cdot\mathbf{n}=0.\qquad(8)$$

This last identity is proven in the appendix of Ref. 5. So, in the case of one body. the only remaining term in (6) is the one in the first line. It corresponds to the well-known added-mass forces of potential flows (see Ref. 1, Sec. 2.2). With more than one body, the second line of (6) does not cancel out anymore. In Sec. IV, for instance, the potential flow of a pair of spheres moving parallel to each other with uniform velocity will be analyzed. For this system, the term in the first line of (6) vanishes, but the one in the second line does not, yielding instead an attractive force between the two spheres. For arbitrary motion and body geometries, even in the context of irrotational flows, the first two lines in Eq. (6) can give rise to very complex interaction forces (see, for instance, Ref. 15 for a treatment of this question using Lagrangian formalism). It is worth mentioning that the most complete analysis of the forces and torques on a deformable single body moving unsteadily in a weakly nonuniform, nonstationary, irrotational flow field was given by Galper and Miloh.[17] □

A. No-slip boundary conditions

Some simplifications of expression (6) occur when a no-slip boundary condition is imposed. In this case, the velocity at the surface of each body is given by

$$\mathbf{v} = \dot{\mathbf{R}}_\alpha + \mathbf{\Omega}_\alpha \times (\mathbf{x} - \mathbf{R}_\alpha), \quad \text{for } \mathbf{x} \in \partial B_\alpha. \tag{9}$$

This implies that

$$-\frac{\rho}{2}\int_{\partial B_\alpha} |\mathbf{v}|^2 \mathbf{e}_i \cdot n = \rho Vol(B_\alpha)(\mathbf{\Omega}_\alpha \times \dot{\mathbf{R}}_\alpha)\cdot \mathbf{e}_i + \rho \mathbf{\Omega}_\alpha \times (\mathbf{\Omega}_\alpha \times \boldsymbol{\xi}_\alpha)\cdot \mathbf{e}_i, \tag{10}$$

where $\mathrm{Vol}(B_\alpha)$ is the volume of B_α and

$$\boldsymbol{\xi}_\alpha = \int_{B_\alpha} (\mathbf{x} - \mathbf{R}_\alpha) \tag{11}$$

is the un-normalized center of volume of B_α, measured from its reference point. In particular, if $\mathbf{R}_\alpha$ coincides with the center of volume of B_α, then ξ_α is null. Moreover, if $\phi_{\alpha i}(\mathbf{x},t)$ is known, then the other terms in lines one and two of Eq. (6) are also known.

Remark (Skin friction): The no-slip boundary condition hypothesis implies that the stresses on the surface of each body can be written as $p\mathbf{n}+\mu\mathbf{n}\times(\omega-2\mathbf{\Omega}_\alpha)$. So, the last term in Eq. (6) is an integral of tangential viscous stresses on body B_α, representing the so called skin friction. The other term in the last line of Eq. (6) arises from the integral of the pressure on the surface of B_α. Howe gave this interpretation in Ref. 18 (Sec. 2.3) and Ref. 19 (Sec. 4.4.3) and illustrated the difference computing these terms for the Stokes drag on a sphere. The distinction between both terms is more evident here than in the case of a single body, since the integral of the first term in the last line of Eq. (6) is over the surface of all bodies, not only B_α, and this is not consistent with the concept of skin friction. This point is pursued further in Ref. 10. □

Remark: It is worth pointing out another big difference between the case of a single body and that of more than one body. In the case of a single body in $\mathbb{R}^3$, it is possible to obtain the three functions ϕ_1, ϕ_2, and ϕ_3 for any $t\in\mathbb{R}$ (we will omit the index α to simplify the notation) since they are known at $t=0$. This is a consequence of the invariance of the Laplacian under translations and rotations. The construction is as follows. Let $T(t)$ be a rotation matrix that describes the orientation of the body and which is given in the following way: At time $t=0$, choose an arbitrary point $\mathbf{x}_0$ in the body distinct from its reference point. If $\mathbf{x}$ is the position at time t of the chosen point, then $T(t)$ is such that $\mathbf{x}=\mathbf{R}(t)+T(t)[\mathbf{x}_0-\mathbf{R}(0)]$. To simplify the notation, let us suppose that $\mathbf{R}(0)=0$, $T(0)=$identity, and let us denote the function $\mathbf{x}_0\to\mathbf{R}(t)+T(t)\mathbf{x}_0$ by $\mathbf{x}\to\psi_t(\mathbf{x}_0)$. Notice that ψ_t is the flow of the rigid body motion velocity field $\dot{\mathbf{R}}+\mathbf{\Omega}\times(\mathbf{x}-\mathbf{R})$. If $\tilde{\phi}_j(\mathbf{x})$, $j=1,2,3$, denote the functions $\phi_j(\mathbf{x},0)$, $j=1,2,3$, at $t=0$, then it can be shown that for any t,

$$\phi_i(\mathbf{x},t) = T_{ij}(t)\tilde{\phi}_j[\psi_t^{-1}(\mathbf{x})].$$

Using this, we conclude that on the surface of the body, where the velocity field is the rigid-body one due to the no-slip boundary condition, the following holds:

$$\partial_t\phi_i(\mathbf{x},t) + \mathbf{v}\cdot\nabla\phi_i(\mathbf{x},t) = \frac{d}{dt}\,\phi_i(\psi_t(\mathbf{x}),t)\big|_{t=0} = \frac{d}{dt}\,T_{ij}(t)\tilde{\phi}_j(\mathbf{x})\big|_{t=0}.$$

Finally, from this identity, we obtain

$$\int_{\partial B} (\partial_t\phi_i + \mathbf{v}\cdot\nabla\phi_i)\mathbf{v}\cdot n = \epsilon_{ijk}\Omega_j\int_{\partial B}\phi_k(\mathbf{x},t)\mathbf{v}\cdot n. \tag{12}$$

This last equation is essentially expression (2.15) of Howe,[1] which can be easily rewritten in terms of added-mass coefficients. For the case of more than one body, however, this very explicit construction does not work. Something similar can be done when approximated explicit functions $\phi_{\alpha i}$ are used, as in Refs. 15 and 16. Yet the analogue to Eq. (12) becomes much more complicated. □

III. TORQUE FORMULAS

The following derivation of torque expressions is again based on that of Howe.[1] Nevertheless, here we assume from the beginning that, if μ is different from zero, then no-slip boundary conditions hold. So, if $\mu\neq 0$, the force per unit area on the surface of body α is

$$p\mathbf{n} + \mu\mathbf{n}\times(\omega - 2\mathbf{\Omega}_\alpha), \tag{13}$$

while if $\mu=0$, this force reduces to $p\mathbf{n}$.

Remark: The reason for assuming no-slip boundary conditions from the beginning is that expression (13) is not true under more general boundary conditions if $\mu\neq 0$. Consider, for instance, the velocity field $\mathbf{v}=(-x_1,-x_2,2x_3)$ and the surface $\Sigma=\{x_3=0\}$. This surface is rigid, in the sense that $\mathbf{v}\cdot\mathbf{n}=\mathbf{v}\cdot\mathbf{e}_3=0$. The deviatoric stress tensor $d_{ij}=\mu(\partial_{x_i}v_j+\partial_{x_j}v_i)$ is not tangential at Σ. Indeed, the deviatoric force per unit area on Σ is $\mu\mathbf{e}_i(\partial_{x_i}v_j+\partial_{x_j}v_i)\delta_{j3}=4\mu\mathbf{e}_3$, which does not agree with (13), since in this case $\mathbf{\Omega}=0$ and $\omega=0$. Howe in Ref. 1 also starts from expression (13), but does not discuss this point. There are problems, like the bubble one in the next section, where the flow is essentially irrotational, viscosity is not null, and yet the boundary conditions are not the no-slip ones. For this sort of problem, it is not clear that the torque expressions of this section apply. □

The i component of the torque about the reference point $\mathbf{R}_\alpha$ of body B_α is

$$M_{\alpha i} = \int_{\partial B_\alpha} p\mathbf{e}_i\cdot(\mathbf{x}-\mathbf{R}_\alpha)\times\mathbf{n} + \mu\int_{\partial B_\alpha}\mathbf{e}_i\cdot(\mathbf{x}-\mathbf{R}_\alpha)\times[\mathbf{n}\times(\omega-2\mathbf{\Omega})]. \tag{14}$$

To find expressions for the torque, we do not use the auxiliary potential function $\phi_{\alpha i}$ of the previous section, but instead the potential $\chi_{\alpha i}$, which is the solution to the following problem:

$$\begin{aligned}&\Delta\chi_{\alpha i}(\mathbf{x},t)=0 \quad \text{for } \mathbf{x}\in Fl,\\&\nabla\chi_{\alpha i}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t)=\mathbf{e}_i\times(x-\mathbf{R}_\alpha)\cdot\mathbf{n}(\mathbf{x},t) \quad \text{for } \mathbf{x}\in\partial B_\alpha,\\&\nabla\chi_{\alpha i}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t)=0 \quad \text{for } \mathbf{x}\in\partial B_\beta,\ \beta\neq\alpha,\\&|\nabla\chi_{\alpha i}(\mathbf{x},t)|=\mathcal{O}(1/|\mathbf{x}|^3)\to 0 \quad \text{as } |\mathbf{x}|\to\infty\end{aligned} \tag{15}$$

where the last condition applies to the case in which Fl is unbounded. To handle the pressure term in Eq. (14), we proceed as in the previous section, multiplying the Navier-Stokes equation (4) by $\nabla\chi_{\alpha i}$ and integrate over Fl,

$$\rho\int_{\text{Fl}}\nabla\chi_{\alpha i}\cdot\partial_t\mathbf{v}+\int_{\text{Fl}}\nabla\chi_{\alpha i}\cdot\nabla\left(p+\frac{\rho}{2}|\mathbf{v}|^2\right)=\rho\int_{\text{Fl}}\nabla\chi_{\alpha i}\cdot\mathbf{v}\times\omega-\mu\int_{\text{Fl}}\nabla\chi_{\alpha i}\cdot\nabla\times\omega. \tag{16}$$

Each term in this equation can be handled in a way similar to that in the previous section. For instance, the pressure part of (14) can be written as

$$\int_{\text{Fl}}\nabla\chi_{\alpha i}\cdot\nabla p=\int_{\text{Fl}}\text{div}(p\nabla\chi_{\alpha i})=\int_{\partial B_\alpha}p\mathbf{e}_i\times(x-\mathbf{R}_\alpha)\cdot\mathbf{n}.$$

Hence, essentially repeating the same steps that led to expression (6), we obtain

$$\begin{aligned}M_{\alpha i}=&-\rho\partial_t\left\{\sum_{\beta=1}^{N}\int_{\partial B_\beta}\chi_{\alpha i}\mathbf{v}\cdot\mathbf{n}\right\}+\rho\sum_{\beta=1}^{N}\int_{\partial B_\beta}(\partial_t\chi_{\alpha i}+\mathbf{v}\cdot\nabla\chi_{\alpha i})\mathbf{v}\cdot n\\&-\frac{\rho}{2}\int_{\partial B_\alpha}|\mathbf{v}|^2\mathbf{e}_i\cdot(\mathbf{x}-\mathbf{R}_\alpha)\times n\\&+\rho\int_{\text{Fl}}\nabla\chi_{\alpha i}\cdot\mathbf{v}\times\omega+\mu\sum_{\beta=1}^{N}\int_{\partial B_\beta}\nabla\chi_{\alpha i}\times\omega\cdot\mathbf{n}\\&+\mu\int_{\partial B_\alpha}(\omega-2\mathbf{\Omega})\times[\mathbf{e}_i\times(\mathbf{x}-\mathbf{R}_\alpha)]\cdot\mathbf{n},\end{aligned} \tag{17}$$

where the last term is the one proportional to μ in Eq. (14), written in a different way.

Remark: Using the no-slip boundary condition, we can write

$$-\frac{\rho}{2}\int_{\partial B_\alpha}|\mathbf{v}|^2\mathbf{e}_i\cdot(\mathbf{x}-\mathbf{R}_\alpha)\times n=\rho\mathbf{e}_i\cdot(\dot{\mathbf{R}}_\alpha\times\mathbf{\Omega}_\alpha)\times\boldsymbol{\xi}_\alpha+\rho\mathbf{e}_i\cdot\mathbf{\Omega}_\alpha\times(I_\alpha\mathbf{\Omega}_\alpha), \tag{18}$$

where ξ_α is defined in (11) and I_α is the moment of inertia operator given by

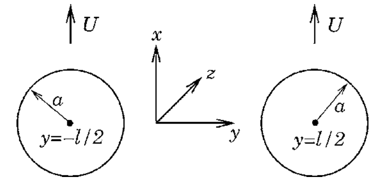

FIG. 1. Diagram showing the coordinates used to describe the two-bubble system.

$$I_\alpha\mathbf{\Omega}_\alpha=\int_{B_\alpha}(\mathbf{x}-\mathbf{R}_\alpha)\times[\mathbf{\Omega}_\alpha\times(\mathbf{x}-\mathbf{R}_\alpha)].$$

□

Remark: In the case of a single body, the following identity holds [it is the analogue to Eq. (12)]:

$$\int_{\partial B_\alpha}(\partial_t\chi_{\alpha i}+\mathbf{v}\cdot\nabla\chi_{\alpha i})\mathbf{v}\cdot n=\epsilon_{ijk}\Omega_j\int_{\partial B_\alpha}\chi_{\alpha k}(\mathbf{x},t)\mathbf{v}\cdot n,$$

an integral that again can be rewritten in terms of the body's added-mass coefficients. □

IV. A PAIR OF BUBBLES RISING SIDE BY SIDE AT HIGH REYNOLDS NUMBER

Our goal in this section is to illustrate the use of the force formula obtained in Sec. II. The system that we will consider consists of two spherical bubbles rising with uniform velocity side by side. This problem was recently computationally studied by Legendre *et al.*[12] This detailed and rather complete paper guided us on the subject. The difference between this system and that of two solid spheres lies in the boundary conditions, the no-slip for the rigid spheres, and the no-tangential-stress for the bubbles. This second boundary condition is much milder in the sense of generating vorticity.

The geometry of the problem is the following. Consider a pair of spherical bubbles of radius a centered, respectively, at $y=-l/2$ and $l/2$ with respect to a Cartesian reference frame $\{\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z\}$, see Fig. 1. We will denote the bubble at $y=-l/2$ by B_α and that at $y=l/2$ by B_β. The bubbles move with constant and equal velocity $U\mathbf{e}_x$. The force on each bubble has two components: a drag in the direction of the motion, $\mathbf{e}_x$, and a lift in the direction transversal to the motion, $\mathbf{e}_y$. These forces are usually given in dimensionless form as drag and lift coefficients, C_D and C_L, respectively, obtained dividing the forces by $\pi a^2\rho U^2/2$. Here we will be interested in the limit of high Reynolds number, $\text{Re}=2a\rho U/\mu$, and far apart bubbles, namely $S=l/a$ large. In these limits, the leading-order terms of C_D and C_L on bubble B_β are, as given in Ref. 12,

$$C_D=-\frac{48}{\text{Re}}\left(1+\frac{1}{S^3}\right), \tag{19}$$

$$C_L=-\frac{6}{S^4}\left(1-\frac{40}{\text{Re}}\right). \tag{20}$$

The term in C_D, which is independent of the distance, was first obtained by Levich,[20] and that proportional to S^{-3} was obtained by Kok.[21] Both Levich and Kok obtained their results equating the viscous power dissipated by the irrotational flow to the work done by the drag. Here both terms in (19) will be obtained from the surface integrals in the last line of Eq. (6). The term in C_L that is independent of the Reynolds number, an attractive force, is due to the irrotational velocity field of the two spheres. This term can be found, for instance, in Ref. 22. Here it is obtained from the integrals on the second line of (6), when only the potential velocity field is taken into account. The term proportional to Re^{-1} in C_L, as far as we know, was not analytically derived yet. In Ref. 12, Legendre *et al.* estimated this term numerically, after arguing that it should be of this form based on an order of magnitude analysis like that in Moore.[23] In principle, it is possible to obtain this part of C_L from Eq. (6). It comes from surface integrals in the second line of (6) and from the volume integral in the third line of (6). The computations are troublesome and it will not be pursued here.

We point out that with the use of Eq. (6), we were able to obtain in a unified way all known results on this problem, which were previously obtained using very different arguments. Moreover, along the computations in Sec. IV C, one can see that we are able to separate the different contributions of the normal viscous stress and the pressure stress to the drag. This cannot be done through power dissipation methods, and it was previously known only in the case of a single bubble.[24] This illustrates one of the main advantages of Howe's formula, namely the identification of different effects on the overall forces.

A. Boundary conditions and velocity potentials

The idea on how to study the flow around bubbles at high Reynolds number[20] (Sec. 5.14) (Refs. 23–26) can be briefly described as follows. At high Reynolds number, the flow outside a system of bubbles is essentially irrotational except for a boundary layer around each bubble of thickness $\sim\mathrm{Re}^{-1/2}$ and a wake of breadth $\sim\mathrm{Re}^{-1/4}$. If the velocity potential of the system is denoted by φ, the velocity field of the fluid can be written as $v=\nabla\varphi+u$, where the velocity correction u is essentially different from zero only inside the vortical region. At the boundary of each bubble, a free-tangential-stress boundary condition must be satisfied. If only the potential part of the flow is taken into account, this condition is violated. In fact, in order to satisfy the free-tangential-stress boundary condition, it is necessary that $\omega=\nabla\times\mathbf{u}$ satisfies

$$\omega=2[(\mathbf{n}\cdot\nabla)\nabla\varphi]\times\mathbf{n} \tag{21}$$

on the surface of each bubble.

In order to proceed, we need the velocity potential φ of a pair of spheres as shown in Fig. 1, as well as the potentials defined by Eq. (1). Velocity potentials for a pair of spheres moving at arbitrary angle with respect to their line of center can be found in Refs. 27 and 28, and approximations, such as that presented below, can be found in Refs. 13 and 14. Up to order $1/l^4$ and in a neighborhood of the sphere B_β, the function φ is given by

$$\varphi=U(A_0+A_3+A_4), \tag{22}$$

where

$$A_0=-\frac{1}{2}\frac{a^3}{r_\beta^3}x_\beta=-\frac{1}{2}\frac{a^3}{r_\beta^2}\cos\theta_\beta,$$

$$A_3=-\frac{1}{2}\frac{a^3}{l^3}\left(1+\frac{1}{2}\frac{a^3}{r_\beta^3}\right)x_\beta=-\frac{1}{2}\frac{a^3}{l^3}\left(r_\beta+\frac{1}{2}\frac{a^3}{r_\beta^2}\right)\cos\theta_\beta, \tag{23}$$

$$A_4=\frac{1}{2}\frac{a^3}{l^4}\left(3+2\frac{a^5}{r_\beta^5}\right)x_\beta y_\beta =\frac{1}{2}\frac{a^3}{l^4}\left(3r_\beta^2+2\frac{a^5}{r_\beta^3}\right)\cos\theta_\beta\sin\theta_\beta\cos\phi_\beta.$$

Here $\{x_\beta,y_\beta,z_\beta\}$ are Cartesian coordinates with origin at the center of sphere B_β and axis parallel to those of system $\{x,y,z\}$ shown in Fig.1, and $\{r_\beta,\theta_\beta,\phi_\beta\}$ are spherical coordinates centered at sphere B_β, with

$$x_\beta=r_\beta\cos\theta_\beta,\quad y_\beta=r_\beta\sin\theta_\beta\cos\phi_\beta,\quad z_\beta=r_\beta\sin\theta_\beta\sin\phi_\beta.$$

The same φ is given in a neighborhood of sphere B_α by

$$\varphi=U(A_0+A_3-A_4), \tag{24}$$

where A_0, A_3, and A_4 are the same functions given above but evaluated at $\{x_\alpha,y_\alpha,z_\alpha\}$ and $\{r_\alpha,\theta_\alpha,\phi_\alpha\}$, which are Cartesian and spherical coordinate systems, respectively, with the origin at the center of sphere α. Notice that $\nabla\varphi\cdot\mathbf{e}_{r_\alpha}=U\mathbf{e}_x\cdot\mathbf{e}_{r_\alpha}$ and $\nabla\varphi\cdot\mathbf{e}_{r_\beta}=U\mathbf{e}_x\cdot\mathbf{e}_{r_\beta}$. The solution to problem (1) for $\phi_{\beta 1}$ (1, 2, and 3 stand for the directions x,y,z, respectively) in a neighborhood of B_β and up to order $1/l^4$ is given by

$$\phi_{\beta 1}(x_\beta,y_\beta,z_\beta)=A_0 \tag{25}$$

and near B_α by

$$\phi_{\beta 1}(x_\alpha,y_\alpha,z_\alpha)=A_3-A_4. \tag{26}$$

Near B_α, $\phi_{\alpha 1}$ is given by

$$\phi_{\alpha 1}(x_\alpha,y_\alpha,z_\alpha)=A_0, \tag{27}$$

and near B_β, it is given by

$$\phi_{\alpha 1}(x_\beta,y_\beta,z_\beta)=A_3+A_4. \tag{28}$$

Near B_β, the solution of (1) for $\phi_{\beta 2}$ up to order $1/l^4$ is given by

$$\phi_{\beta 2}(x_\beta,y_\beta,z_\beta)=E_0 \tag{29}$$

and, near B_α, by

$$\phi_{\beta 2}(x_\alpha,y_\alpha,z_\alpha)=E_3+E_4, \tag{30}$$

where

$$E_0=-\frac{1}{2}\frac{a^3}{r_\beta^3}y_\beta=-\frac{1}{2}\frac{a^3}{r_\beta^2}\sin\theta_\beta\cos\phi_\beta,$$

$$E_3=\frac{1}{2}\frac{a^3}{l^3}\left(2+\frac{a^3}{r_\beta^3}\right)y_\beta=\frac{1}{2}\frac{a^3}{l^3}\left(2r_\beta+\frac{a^3}{r_\beta^2}\right)\sin\theta_\beta\cos\phi_\beta, \tag{31}$$

$$E_4 = \frac{1}{2}\frac{a^3}{l^4}\left(\frac{3}{2} + \frac{a^5}{r_\beta^5}\right)(3y_\beta^2 - r_\beta^2)$$
$$= \frac{1}{2}\frac{a^3}{l^4}\left(\frac{3r_\beta^2}{2} + \frac{a^5}{r_\beta^3}\right)(3\sin^2\theta_\beta\cos^2\phi_\beta - 1),$$

and again, near B_α, we must replace $(x_\beta, y_\beta, z_\beta)$ by $(x_\alpha, y_\alpha, z_\alpha)$, etc. For $\phi_{\alpha 2}$, similar expressions hold.

B. The pure irrotational part of the forces

We will just compute the forces on bubble B_β (those on B_α then follow from the symmetry of the problem). If the flow is irrotational, only the first two lines of Eq.(6) contribute to the force. In fact, since the velocity of the bubbles is constant, only the second line of Eq. (6) is nonzero. The irrotational contribution to the force in the direction of the motion $\mathbf{e}_x$ is null. This is D'Alambert's paradox. In the context of Eq.(6) with $i=1$, the best way to see this is to consider both bubbles as a single rigid body. By symmetry, the force on the pair is twice the force in each one. But for a single rigid body, the second line of Eq. (6) is zero due to Eq.(8).

We turn to the computation of $F_{\beta 2}$. Substituting (22), (24), (29), and (30) in the second line of Eq.(6), neglecting all terms of order larger than $1/l^4$, remembering that $\mathbf{n}=-\mathbf{e}_{r_\beta}$ on ∂B_β and $\mathbf{n}=-\mathbf{e}_{r_\alpha}$ on ∂B_α, and that, since the bubbles are translating with velocity $U\mathbf{e}_x$, we have $\varphi=\varphi(\mathbf{x}-Ut\mathbf{e}_x)$, which implies that $\partial_t\varphi=-U\nabla\varphi\cdot\mathbf{e}_x$ (the same is true for $\partial_t\phi_{\beta 2}$, etc.), we obtain that the irrotational part of C_L is $-6/S^4$, which is the dominant term in the expression (20) for the lift.

C. The drag coefficient C_D

Now we compute the integrals in the last line of Eq. (6). Again we consider just the forces on bubble B_β. From the boundary conditions (21) with $\dot{\mathbf{R}}_\alpha=\dot{\mathbf{R}}_\beta=U\mathbf{e}_x$, we have on the surface of B_β that $\omega=\omega_\theta\mathbf{e}_\theta+\omega_\phi\mathbf{e}_\phi$, where

$$\omega_\theta = -\frac{2}{a^2\sin\theta_\beta}\partial_{\phi_\beta}\varphi,$$
$$\omega_\phi = +\frac{2U}{a}\sin\theta_\beta + \frac{2}{a^2}\partial_{\theta_\beta}\varphi. \qquad (32)$$

The same expressions hold on the surface of B_α, replacing $r_\beta, \theta_\beta, \phi_\beta$ by $r_\alpha, \theta_\alpha, \phi_\alpha$. A simple symmetry argument involving parity with respect to the reflection $\theta\to\pi-\theta$ can be used to conclude that all functions that are being integrated in the last line of Eq. (6), for $i=2$, which corresponds to a lift force, are null. Therefore, there is no contribution of the last line of Eq. (6) to the lift.

Next we compute the contributions of the last line of Eq (6) to the drag $i=1$, namely in the direction $\mathbf{e}_1=\mathbf{e}_x$. First consider the integral

$$+\mu\int_{\partial B_\beta}\mathbf{e}_1\times\omega\cdot\mathbf{e}_{r_\beta}.$$

This term comes from the integral of the viscous stresses on ∂B_β, Eq. (2), through the modification in Eq. (3). For a bubble, the tangential stresses are null. So, although it is not obvious from the expression above, the resulting force obtained from the above integral is due to normal viscous stresses acting on the surface of the bubble. Again, this shows that the stress expression given in Eq. (13) does not hold for the no-tangential-stress boundary condition as already noted in the remark below Eq. (13). After substituting expressions (32) for ω in the above integral, using some parity arguments, and performing some elementary computations, we are led to the following result:

$$\mu\int_{\partial B_\beta}\mathbf{e}_1\times\omega\cdot\mathbf{e}_{r_\beta} = -8\pi Ua\mu - 4\pi U\mu\frac{a^4}{l^3} + \mathcal{O}(1/l^6).$$

The part of this force that is independent of l is a piece of Levich drag, the other is due to the interaction between bubbles. The other integrals in the last line of Eq. (6),

$$-\mu\int_{\partial B_\beta}\nabla\phi_{\beta 1}\times\omega\cdot\mathbf{e}_{r_\beta} - \mu\int_{\partial B_\alpha}\nabla\phi_{\beta 1}\times\omega\cdot\mathbf{e}_{r_\alpha},$$

are due to pressure stresses. Using ω as in Eq. (32), and the expression for $\phi_{\beta 1}$ in (25), we obtain, after some computations,

$$-\mu\int_{\partial B_\beta}\nabla\phi_{\beta 1}\times\omega\cdot\mathbf{e}_{r_\beta} = -4\pi Ua\mu - 2\pi U\mu\frac{a^4}{l^3}.$$

The term $-4\pi Ua\mu$ is the last piece of Levich drag, which comes from integrating pressure stresses. Finally, using ω as in Eq. (32) and $\phi_{\beta 1}$ as in (26), we obtain

$$-\mu\int_{\partial B_\alpha}\nabla\phi_{\beta 1}\times\omega\cdot\mathbf{e}_{r_\alpha} = -6\pi U\mu\frac{a^4}{l^3}.$$

Adding all these contributions and dividing by $\pi a^2\rho U^2/2$, we obtain the expression (19) for C_D.

Notice that the drag is completely determined by the surface integrals in the last line of Eq. (6), and that these integrals depend only on the vorticity on the surface of both bubbles. This result is in agreement with those of Kang and Leal[24] and Stone.[26] In particular, Stone showed that for any bubble of any shape under steady translation, the drag coefficient up to order Re^{-1} depends only on the vorticity on the bubble surface, and it does not depend on the vorticity within the fluid (this was demonstrated using an identity that relates the energy dissipated by the potential flow to the vorticity on the surface of the bubble). The same applies here, as shown in the following.

After all integrals already computed in Secs. IV B and IV C are subtracted from the force expression in Eq. (6), the following terms are left:

$$-\rho\int_{\partial B_\beta}(\nabla\varphi\cdot\mathbf{u})\nabla\phi_{\beta i}\cdot\mathbf{n} + \rho\int_{\partial B_\beta}(\nabla\phi_{\beta i}\cdot\mathbf{u})\mathbf{v}\cdot\mathbf{n}$$
$$+\rho\int_{\partial B_\alpha}(\nabla\phi_{\beta i}\cdot\mathbf{u})\mathbf{v}\cdot\mathbf{n} + \rho\int_{\mathrm{Fl}}(\nabla\varphi\times\omega)\cdot\nabla\phi_{\beta i}$$
$$-\rho\int_{\partial B_\beta}\frac{|\mathbf{u}|^2}{2}\nabla\phi_{\beta i}\cdot\mathbf{n} + \rho\int_{\mathrm{Fl}}(\mathbf{u}\times\omega)\cdot\nabla\phi_{\beta i}.$$

It is convenient to further transform these integrals: the first plus the second line and the third line of the above expression become, after some computations, the first and the second integral, respectively, of the following:

$$+\rho\int_{\mathrm{Fl}}\mathbf{u}\cdot\nabla\times(\nabla\phi_{\beta i}\times\nabla\varphi)-\rho\int_{\mathrm{Fl}}(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\nabla\phi_{\beta i}. \quad (33)$$

The order of magnitude of each integral above with respect to Re^{-1} can be estimated using the analysis presented by Moore[23] of the flow around a single bubble (see also Ref. 25). The first conclusion we get from this analysis is that up to order Re^{-1}, the contribution of the second integral in (33) can be neglected in comparison to the first one for both components of the force. This is very natural since $\mathbf{u}$ goes to zero as $\mathrm{Re}\to\infty$. So, we are left with the first integral of (33).

Let us consider the first integral in (33) with $i=1$, which corresponds to the drag direction $\mathbf{e}_1=\mathbf{e}_x$. In this case, it is convenient to write $\varphi=U(\phi_{\alpha 1}+\phi_{\beta 1})$, which holds due to the definition of $\phi_{\alpha 1}$ and $\phi_{\beta 1}$ in (1). Then

$$\begin{aligned}\nabla\phi_{\beta 1}\times\nabla\varphi &= U\,\nabla\,\phi_{\beta 1}\times\nabla\phi_{\alpha 1}\\ &=\frac{U}{2}\nabla\,(\phi_{\beta 1}-\phi_{\alpha 1})\times\nabla(\phi_{\alpha 1}+\phi_{\beta 1})\end{aligned}$$

and a parity analysis with respect to the variable y (we recall that $\{x,y,z\}$ are Cartesian coordinates with the origin at the center of volume of the two bubbles) implies that the first integral in (33) is zero. Therefore, at least up to the order of magnitude considered here, there is no contribution to the drag that comes from integrals in Eq. (6) that involve the vorticity inside the fluid.

We can also show that the opposite situation arises for the lift: All integrals in the last line of (6), which involve the vorticity on the bubble surface, do not contribute to the viscous part of the lift. The viscous part of the lift depends only on the vorticity within the fluid.

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