The Forced Inviscid Burgers Equation as a Model for Nonlinear Interactions among Dispersive Waves

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ABSTRACT. The forced inviscid Burgers equation is studied as a model for the nonlinear interaction of dispersive waves. The dependent variable $u(x,t)$ is thought of as an arbitrary mode or set of modes of a general system, and the force is tuned to mimic the effects of other modes, which may be either near or far from resonance with u.

When the force is unimodal, a family of exact traveling waves fully describes the asymptotic behavior of the system. When the force is multimodal, with the frequencies of the various modes close to each other, the asymptotic solution is quasi-stationary, punctuated by faster intermittent events. The existence of these "storms" may have significant implications for energy transfer among modes in more general systems.

1. Introduction.

The nonlinear interaction among a large set of waves is a complex phenomenon. Among the issues involved are the tuning (or detuning) of sets of modes, depending of how far they are from perfect resonance. This issue is subtle though, since it depends on the time scale of the nonlinear interaction, which itself depends on the degree of tuning among modes.

A particularly subtle issue appears in the transition from discrete to continuous sets of waves: how to add up the effects of very many near resonances. Do they interfere destructively or constructively? In most theories of continuous spectra, the former is usually assumed, to the point of suppressing altogether the leading order effect of resonances, pushing them to higher orders than those appearing in discrete systems.

In systems that are both forced and damped, statistical cascades often appear, carrying energy among scales, from the scales associated with the forcing, to those where dissipation transfers the energy out of the system. When the scales of forcing

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and dissipation are many decades apart, the intermediate scales span the so–called *turbulent inertial range*, where the system behaves effectively as a Hamiltonian one, and self-similar energy spectra are often observed. Attempts to understand the nature of these self-similar cascades gave rise to the theory of *Wave Turbulence* (e.g., see [**1**], [**2**], [**5**], and [**11**].) In order to close a very complex system, this theory makes a number of assumptions, such as phase decorrelation among the various modes, scale separation between linear and nonlinear phenomena, and infinite size of the system, giving rise to a continuum of modes. These assumptions are hard to justify, and the theory, despite its beauty, has a number of problems, such as internal inconsistencies — as when it predicts upscale energy fluxes– and a mixed record of agreement with observations and numerical simulations (e.g., see [**3**] and [**6**].)

In this work, we consider a simple model, where surrogates for resonances and near resonances can easily be built in: the forced inviscid Burgers equation,

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = f(x,\,t)\,, \tag{1.1.1}$$

where $f = f(x,\,t)$ is some smooth forcing, and both f and u are periodic (of period 2π) in space and have vanishing mean.

Here the dependent variable $u(x,t)$ represents a mode (or set of modes) with linear frequency $\omega = 0$ (as follows from the zero mean condition.) On the other hand, the externally imposed force $f(x,t)$ represents other modes of the system, which (depending on the scale of their dependence on time) will be close or far from resonance with u.

A vastly different reduced model for resonant energy transfer among modes was developed in [**7**]. It it interesting to note that both models, though completely different in conception and structure, contain intermittent regimes — these are strong in [**7**] and much milder in the present work. It appears that intermittence is a natural occurrence in models of turbulent energy cascades.

The nonlinear term in (1.1.1) has two combined functions: to transfer energy among the various (Fourier) components of u, and to dissipate energy at shocks. Thus the "inertial cascade" and the system's dissipation are modeled by a single term. This not only implies a big gain in simplicity, but could also in fact be a rather realistic model for fluid systems, where dissipation is almost invariably associated with some form of wave breaking.

The model equation (1.1.1) above is a simplified version of the equations describing the interaction among resonant triads involving a nondispersive wave [**8**]. The simplification consists in freezing the two dispersive members of the triad, thus making them act as a prescribed force on $u(x,t)$.

It would seem that a more general model, with a non–zero linear frequency ω, is the one given by the equation

$$u_t + \omega\,u_x + \left(\frac{1}{2}\,u^2\right)_x = f\,, \tag{1.1.2}$$

which is equivalent to considering non-zero mean solutions to (1.1.1) — i.e.: write $u = \omega + \tilde{u}$, where ω is the mean[1] of u. However, this last equation can be reduced to equation (1.1.1) by the introduction of the new independent variable $x' = x - \omega\, t$.

A resonant force $f(x,\, t)$ in (1.1.1) is one that does not depend on time; a near–resonant one, on the other hand, evolves slowly. More precisely, a near resonance should be modeled by

$$u_t + \left(\frac{1}{2}\, u^2\right)_x = \epsilon^2 f(x,\, \epsilon t)\,, \tag{1.1.3}$$

where $0 < \epsilon \ll 1$ measures the degree of non–resonance. The reason for the factor ϵ^2 in front of the forcing term follows from considering a quasi–steady approximation to the solution to (1.1.3), namely:

$$u(x) \approx \epsilon \sqrt{2 \int^x f(s,\, \epsilon t)\, ds}\,.$$

This is not quite right, but indicates the correct result: a (slow) forcing of size $O(\epsilon^2)$ in (1.1.3), generally induces a response of amplitude ϵ in u. This shows that any force stronger than ϵ^2 in (1.1.3) would render its own modulation irrelevant, since the induced nonlinearity would act on a much faster time scale than that of the modulation.

Interestingly, the ϵ's above in (1.1.3) can be scaled out by a simple transformation: let $\tilde{t} = \epsilon\, t$, and write $u = \epsilon \tilde{u}$. Then, in terms of these new variables, (1.1.3) becomes (1.1.1). Hence **near-resonances in (1.1.1) cannot be defined as an asymptotic limit involving a small parameter ϵ;** if there is a distinction between near resonant and nonresonant forces, it will have to arise from a finite bifurcation in the behavior of the solutions to (1.1.1) — which in fact occurs, as we will show in section 2.

The fact pointed out in the prior paragraph is part of a more general property of equation (1.1.1), namely: it is a canonical system. For consider a model involving a more general nonlinearity, such as:

$$u_t + N(u)_x = f(x,\, t)\,, \tag{1.1.4}$$

where, generically, we can assume that $N(u) = O(u^2)$ (since any linear term can be eliminated by a Galilean transformation.) Now consider a weakly nonlinear, nearly resonant, situation, where u is small, the force is small and the time scales are slow. Thus, take: $u = \epsilon \tilde{u}$, $f = \epsilon^2 \tilde{f}$, with the time dependence via $\tilde{t} = \epsilon t$, and $0 < \epsilon \ll 1$. Then it is easy to see that, in terms of $\tilde{u}$, $\tilde{f}$, $\tilde{t}$, and x, the leading order system is (1.1.1) — except for, possibly, a constant other than $1/2$ in front of the nonlinear term. In fact, this reduction of the equations to (1.1.1) will occur even if we have a system (i.e.: u in (1.1.4) is a vector), as long as there is a single force (lined up with a single mode of the system.)

[1] Note that, **because f has a vanishing mean, the mean of the solutions to (1.1.1) is a constant.**

Notation and general properties.

It is well known that the inviscid Burgers equation develops shocks. Throughout this work, we shall use the following notation for various quantities at shocks:

- A plus (respectively, minus) superscript (or subscript) stands for the value of the corresponding variable ahead (respectively, behind) of the shock.
- Brackets stand for the jump across the shock of the enclosed expression. Specifically: the value ahead (to the right) minus the value behind (to the left). Thus, for example:

$$[u] = u^+ - u^- ,$$

  is the jump in u across the shock.

Shocks obey the following rules:

- The shock speed is given by the **Rankine-Hugoniot jump conditions,** namely:

$$S = \bar{u} = \frac{1}{2}\left(u^+ + u^-\right) .$$

  where $\bar{u}$ is the arithmetic mean of u at the shock, and S is the shock speed.
- Shocks must satisfy an **entropy condition**, which for (1.1.1) states that u should jump down across shocks, i.e.:

$$[u] \leq 0 .$$

Finally, equation (1.1.1) has an energy

$$E(t) = \int_0^{2\pi} \frac{1}{2} u^2(x,\, t) dx ,$$

which satisfies the equation

$$\frac{dE}{dt} - \sum \left(\frac{1}{12}[u]^3 \right) = \int_0^{2\pi} u(x,t)\, f(x,t)\, dx , \tag{1.1.5}$$

where the sum on the left is over all the shocks in the solution. This sum accounts for the dissipation of energy at the shocks, and the right hand side represents the work by the forcing term f. That is, we have:

$$E_d = -\sum (\frac{1}{12}[u]^3) > 0 \quad \text{and} \quad W_f = \int_0^{2\pi} u\, f\, dx ,$$

where E_d is the energy dissipated per unit time and W_f is the work done by the forcing.

Contents and plan of the paper.

Much of the contents of this paper will be concerned with the study of the energetic interplay between the dissipation at shocks and the work done by the forcing function. Our interest lies mostly in situations where $E(t)$ is close to stationary, so that the work done by the forcing and the dissipation at shocks approximately balance each other. Either of these then represents the amount of energy flowing through the system, and the dependence of this flux on the characteristics of $f(x,t)$ will teach us something about the nature of the energy exchange among near resonant modes.

This paper is organized as follows. In section 2, we study the asymptotic, long time solutions to (1.1.1), under unimodal forces $f(x,t) = f(x - \omega t)$. This long time asymptotic behavior is given by a family of exact traveling wave solutions. We observe an interesting bifurcation between near resonant and nonresonant behavior, taking place at a critical value of ω (that depends on the form and size of the forcing f.) If $|\omega|$ is smaller than this critical value, the forcing does work on the solution; if $|\omega|$ is bigger, on the other hand, it does not. We sketch proofs for these results, which use a novel combination of Hamiltonian formalism with breaking waves.

In section 3, we study two-modal forcings, in which $f = g_1(x) + g_2(x - \omega\, t)$. For large values of ω, g_2 has vanishingly small effect on the asymptotic, long time, solution $u(x,\, t)$. For small values of ω, on the other hand, quasi-steady solutions $u = u(x,\, \tau)$ arise (where τ is a slow time), punctuated by intermittent events (which we call "storms") with enhanced rates of energy exchange between the forcing and the solution, at an intermediate time scale (slow, but faster than τ.) The abnormal rate of energy exchange during storms hints at the possibility that nonlinear wave systems may have regimes where the energy exchange among modes is strongly influenced by fast, intermittent events, involving coherent phase and amplitude adjustments of the full spectrum, rather than by the slow evolution of individual resonant sets.

We can make the statement in the prior paragraph more precise, or at least more suggestive, as follows: We show in section 3 that the extra (integrated) energy exchange during a storm scales like $\omega^{-1/2}$. In our model, however, there are only a finite number of storms per period, which itself scales like ω^{-1}. Thus the average energetic impact of the storms is of order $\omega^{1/2}$, vanishing with ω. Yet, in more complex systems, there are a number of likely scenarios (involving, for instance, random or pseudo-random events), in which the number of storms per period will increase as the period does, at least as $\omega^{-1/2}$. When this is the case, the energetic impact of storms will be at least comparable to that of the quasi-steady parts of the solution, and we will find ourselves at the threshold of an energy cascade driven by intermittent events.

2. A Single Forcing Mode.

In this section, we consider the equation

$$u_t + \left(\frac{1}{2}u^2\right)_x = f(x - \omega t)\,, \tag{2.2.1}$$

where $f = f(z)$ and $u = u(x,t)$ are 2π-periodic in space, real functions, with zero mean. Here we will assume that f is a sufficiently smooth function, and that the initial conditions are such that the solution is, at all times, piece-wise smooth, with a finite number of shocks.

As explained earlier, the forcing term is resonant if $\omega = 0$, near resonant if ω is small and far from resonant if ω is big. Notice though that, from the argument above equation (1.1.4) in the introduction, we cannot give an asymptotic meaning to this distinction through the introduction of a small parameter, since this equation is the canonical model for the description of systems with a fluid-like nonlinearity, weakly forced near resonance. Interestingly, as we shall see below (remark 2.2), the equation admits exact solutions where there is a sharp transition between resonant and non-resonant behavior, at a critical value of the frequency $\omega = \omega_c$.

Below we consider a special set of solutions to equation (2.2.1), given by traveling waves. These solutions not only can be written exactly in closed form, but have special significance — since they describe the long time behavior for the general solution. In subsections 2.2 and 2.3 we show –numerically and analytically– that the solution to the initial value problem for (2.2.1) above converges to the traveling wave solution as $t \to \infty$, which is generically unique.

2.1. Exact Traveling Solutions. We shall seek traveling wave solutions to (2.2.1) of the form

$$u(x,t) = G(z)\,, \tag{2.2.2}$$

where $z = x - \omega t$. Then equation (2.2.1) becomes the O.D.E.

$$\frac{d}{dz}\left(\frac{1}{2}(G-\omega)^2\right) = f(z)\,. \tag{2.2.3}$$

This has the solutions

$$G(z) = \omega \pm \sqrt{2F(z)}\,, \tag{2.2.4}$$

where $F = \int^z f(s)ds$, with the constant of integration selected so that $F(z) \geq 0$ everywhere.

(2.2.5) We shall **define** F_{cr} to be the choice of F such that $\min(F_{cr}) = 0$.

These solutions can be used to produce exact (periodic) traveling wave solutions that both have a vanishing mean, and satisfy the entropy condition when they include shocks. Generically, three distinct cases can arise, with the **solution determined uniquely by ω if F_{cr} has a single minimum per period.** See figure **1** for illustrative examples.

Case 1. If F is strictly larger than F_{cr}, then the solution must be smooth — with only one sign selected in (2.2.4). This follows because the entropy condition for shocks only allows downward jumps; thus only jumps from the positive to the negative root are allowed. Given that G must be periodic, no shocks are possible when $\min(F) > 0$. In this case, the sign of the square root, and the value of the integration constant defining F, follow upon imposing the condition that the mean of $u = G$ must vanish.
We can write the solutions corresponding to this case as two families of solutions (one for $\omega > 0$ and another one for $\omega < 0$) parameterized by a single parameter $\delta > 0$, as follows:

$$u = \omega - \mathrm{sign}(\omega)\sqrt{2\,(\delta + F_{cr}(z))}\,, \tag{2.2.6}$$

$$\omega = \pm\frac{1}{2\pi}\int_0^{2\pi} \sqrt{2(\delta + F_{cr}(z))}\,dz\,, \tag{2.2.7}$$

where $z = x - \omega t$ and F (as defined above) is given by $F(z) = \delta + F_{cr}(z)$. These formulas show that

> For $|\omega| > \omega_{cr}$, where ω_{cr} is defined below in equation (2.2.8), the traveling wave solutions are as smooth as F_{cr}. Furthermore, they are uniquely determined by the function f and the frequency ω.

Notice that **uniqueness, in this case where $|\omega| > \omega_{cr}$, does not depend at all on F_{cr} having a single minimum per period.**

Case 2. For values of $|\omega|$ smaller than

$$\omega_{cr} = \frac{1}{2\pi}\int_0^{2\pi} \sqrt{2F_{cr}(z)}\,dz\,, \tag{2.2.8}$$

no F can be found that will satisfy the zero mean condition. When this is the case, one must take $F = F_{cr}$, and allow the solution to jump between the positive and the negative roots. Then the solution can return (smoothly) to the positive root through the point where $F_{cr} = 0$, which is generically unique. In this case, the tunable parameter that one can use to adjust the solution so that $u = G$ has a vanishing mean, is the position $z = s$ of the shock (notice that, because u at the shock jumps from $u_- = \omega + \sqrt{2F_{cr}(s)}$ to $u_+ = \omega - \sqrt{2F_{cr}(s)}$, the shock's velocity is ω — therefore it remains fixed in the frame moving with the traveling wave.)

To be specific, **assume that F_{cr} has a single minimum per period,** and let $z = z_m$ be the position of the minimum. Then the equation for the shock position $z = s$ is given by:

$$\omega = -\frac{1}{2\pi}\int_{z_m}^{s} \sqrt{2F_{cr}(z)}\,dz + \frac{1}{2\pi}\int_{s}^{z_m+2\pi} \sqrt{2F_{cr}(z)}\,dz\,, \tag{2.2.9}$$

where $z_m \leq s < z_m + 2\pi$. Since the right hand side in this equation is a (strictly) monotone decreasing function of s (with the values ω_{cr} for $s = z_m$, and $-\omega_{cr}$ for $s = z_m + 2\pi$) there is a unique solution for s. Thus **the traveling wave solution is unique.**

REMARK 2.1. Notice that, in the (non-generic) case when F_{cr} has more than one minimum per period, uniqueness is lost when $\omega \leq \omega_{cr}$. This is because, in this case, there is more than one possible point where a smooth switch from the negative to the positive root in equation (2.2.4) can occur. This feature is at the root of the behavior reported in section 3 for the response to forcings with more than one frequency.

Case 3. In the limiting case when $\omega = \omega_{cr}$, the shock and the smooth transition from negative to positive root coalesce and disappear, leaving a corner moving at speed ω_{cr} as the only singularity of the solution.

Simple example; single harmonic forcing.

Consider the case with a single sinusoidal forcing: $f = f(z) = \sin z$ (with $z = x - \omega t$), for which $F_{cr} = 1 - \cos(z) = 2\sin^2(z/2)$. Then the critical value for ω is given by:

$$\omega_{cr} = \frac{1}{2\pi}\int_0^{2\pi} 2\left|\sin(\frac{z}{2})\right|\,dz = \frac{4}{\pi}\,, \tag{2.2.10}$$

and we have:

Solutions with shocks for the simple example: These occur for $|\omega| < \omega_{cr} = 4/\pi$, and have the form

$$u(x,t) = G(z) = \omega \pm 2\left|\sin(\frac{x-\omega t}{2})\right|\,. \tag{2.2.11}$$

In each period (say $0 \leq z < 2\pi$) there is a (continuous) switch from the minus to the plus sign as z crosses $z = 0$, and a switch from the plus to the minus sign (across a shock) at some position $z = s$. The position of this (single) shock follows from the zero mean condition (i.e.: equation (2.2.9) for this simple example)

$$0 = \int_0^{s} G^+(z)\,dz + \int_s^{2\pi} G^-(z)\,dz = 2\pi\omega - 8\,\cos(s/2)\,, \tag{2.2.12}$$

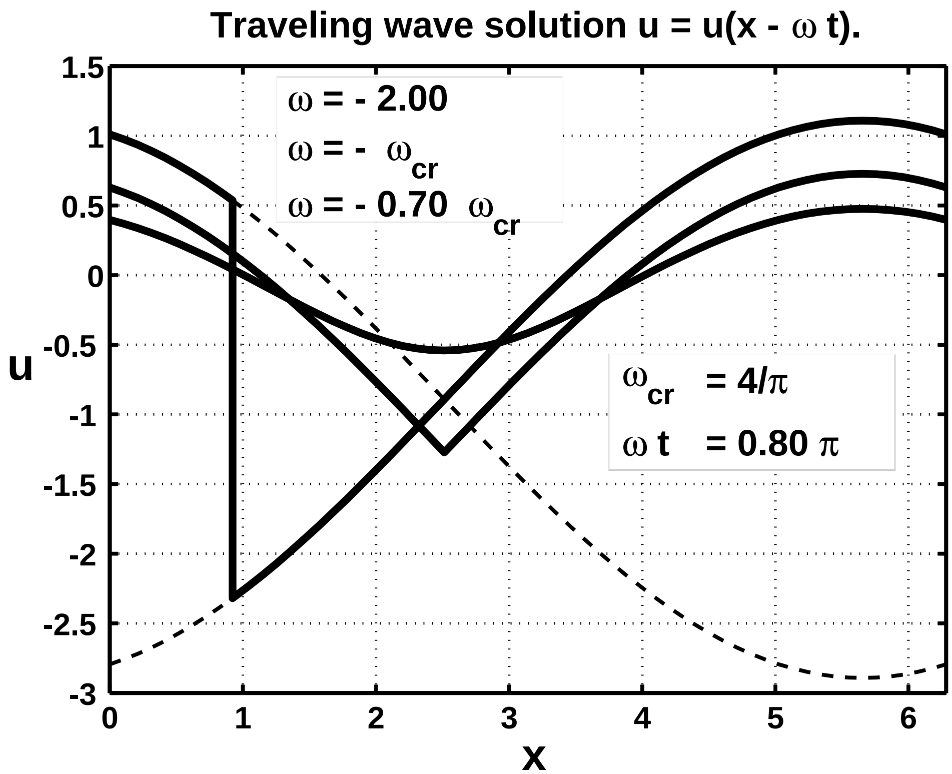

FIGURE 1. Examples of traveling waves for the equation $u_t + (0.5\,u^2)_x = \sin(x - \omega\, t)$. Three solutions are shown:
(a) Smooth solution, for $\omega = -2.00 < -\omega_{cr}$.
(b) Critical solution, with a corner, for $\omega = -\omega_{cr} = -4/\pi$.
(c) Solution with a shock, for $\omega = -0.70\,\omega_{cr}$.
The "envelope" for the solution with a shock, given by $u = \omega \pm \sqrt{2\,F}$ is also shown (dashed line.) In each case, the solution is plotted for a time t such that $\omega\, t = 0.80\,\pi$.

where $G^+ = \omega + 2\,|\sin(z/2)|$, and $G^- = \omega - 2\,|\sin(z/2)|$. Thus

$$s = 2\arccos\left(\frac{\pi}{4}\omega\right), \quad \text{with} \quad 0 \le s < 2\pi\,. \tag{2.2.13}$$

We can also compute the work per unit time W_f done by the external force f on this exact solution. Since this work must agree with the energy E_d dissipated at the shock (given that the solution is steady), we have:

$$W_f = \int_0^{2\pi} f\, u\, dx = E_d = -\frac{1}{12}\,[u]^3 = \frac{16}{3}\left\{1 - \left(\frac{\pi\omega}{4}\right)^2\right\}^{3/2}. \tag{2.2.14}$$

A plot of the work done by the forcing is shown in figure **2**.

REMARK 2.2. For $|\omega| \geq \omega_{cr} = 4/\pi$ the solutions have no shocks (see below) and there is no energy dissipation, nor work done by the forcing (i.e.: the solution is orthogonal to the forcing.) This indicates a rather abrupt change in behavior, which we interpret as the **boundary of resonance.** That is, a **sharp transition from resonant behavior** (with the forcing continuously pumping energy into the system, which is then dissipated by a shock) **to non-resonant behavior** (with no work done by the forcing) occurs at $|\omega| = \omega_{cr}$. It is easy to see that this behavior is general, and not particular to the special harmonic forcing of this simple example. It will occur for the traveling waves produced by any forcing of the form in equation (2.2.1). It is interesting to note that this behavior is analogous to a third order phase transition, with the collective behavior of the modes making up the solution switching from a dissipative configuration to a non-dissipative one.

Smooth solutions for the simple example: These occur for $|\omega| > \omega_{cr} = 4/\pi$, and have the form

$$u(x,t) = \omega \pm \sqrt{2\,(D - \cos(x - \omega t))}\,, \tag{2.2.15}$$

where $D > 1$ and the sign of the square root follow from the condition on the mean:

$$\int_0^{2\pi} u(x,t)\, dx = 0\,.$$

It is actually easier to write ω as a function of D, as follows

$$\omega = \omega(D) = \pm\frac{1}{2\pi}\int_0^{2\pi} \sqrt{2(D - \cos z)}\, dz\,, \quad \text{where} \quad D \geq 1\,. \tag{2.2.16}$$

This equation is the same as (2.2.7) for this simple example, with $D = 1 + \delta$.

Critical solutions for the simple example: These occur for $|\omega| = \omega_{cr} = 4/\pi$, and are given by:

$$u = \pm\left\{\omega_{cr} - 2\left|\sin\left(\frac{x \mp \omega_{cr} t}{2}\right)\right|\right\}. \tag{2.2.17}$$

2.2. Numerical Experiments. In subsection 2.3 we will show (analytically) that the solution to the general initial value problem for equation (2.2.1) converges asymptotically (for large times) to the traveling wave solution — at least in the case where the traveling wave solution is unique, i.e.: F_{cr}, as defined in (2.2.5), has a single minimum per period. In this section, we show the result of a numerical calculation illustrating the convergence process.

Numerical code: In both the calculations shown in this subsection, and those in section 3, we use a (second order in time) Strang [**9**] splitting technique, separating the equation into

$$u_t = f \quad \text{and} \quad u_t + \left(\frac{1}{2}\,u^2\right)_x = 0\,.$$

Then we use a second order Runge–Kutta ODE solver for the first equation. For the second equation we use a second order Godunov [**4**] scheme, with the van Leer

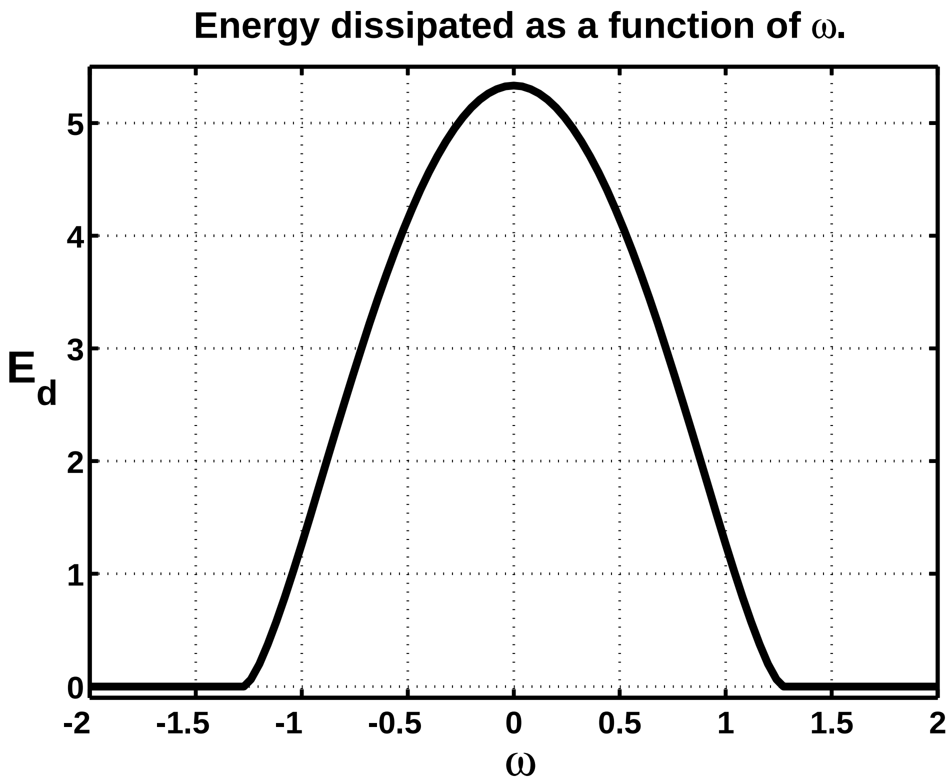

FIGURE 2. Energy dissipated (as a function of ω) by the traveling waves. The energy dissipated E_d (equal to the work W_f done by the external force f) for the equation $u_t + (0.5\, u^2)_x = \sin(x - \omega\, t)$, is shown as a function of ω. Notice the sharp cutoff at $\omega = \pm\, \omega_{cr}$, beyond which no work is done by the force f. This result is general and does not depend on the particular forcing $\sin(x - \omega\, t)$ — it will occur for any forcing of the form $f = f(x - \omega\, t)$.

[**10**] monotonicity switches. This yields a fairly simple and robust shock capturing (second order, both in time and space) algorithm.

Example: Figure **3** shows an example of how a solution to equation (2.2.1) (with $f = \sin(x - \omega\, t)$ and "arbitrary" initial data) converges to a traveling wave as $t \to \infty$. Specifically, we take $\omega = 0.5\, \omega_{cr}$ — where ω_{cr} is given in equation (2.2.10) — and $u(x,\, 0) = \sin(x)$.
The convergence to the traveling wave solution is done via the formation of shocks (only one in this case) that dissipate energy and force the solution to converge to its limiting shape. The arguments in subsection 2.3 give a more precise description of this process. Notice how fast the convergence is: the period in time of the forcing function is $T = 2\pi/\omega$, and (even though the initial condition is $O(1)$ away from

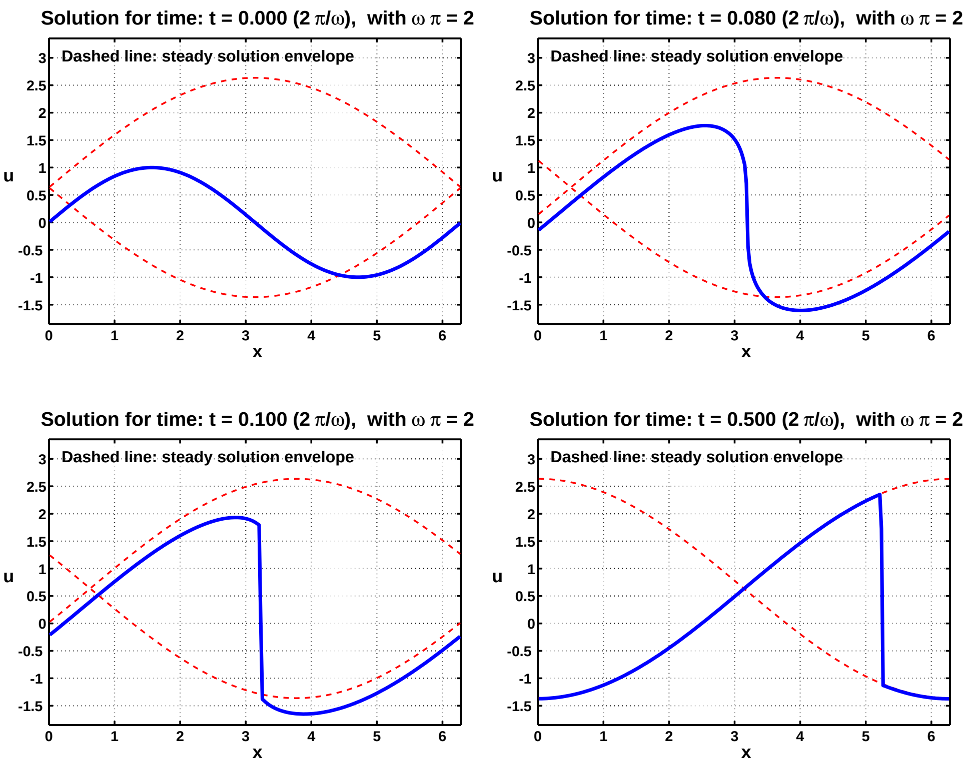

FIGURE 3. Forcing $f = \sin z$ in the equation, with $z = x - \omega t$ and $\omega = \omega_{cr}/2 = 2/\pi$. As $t \to \infty$ the solution converges to the traveling wave $u = \omega \pm 2\,\sin(z/2)$, where the sign switch occurs at the shock position. The traveling wave has period $2\pi/\omega$ in time (same as the forcing.) From left to right and from top to bottom, we plot the solution and the "envelope" $\omega \pm 2\,\sin(z/2)$ (dashed line) for the traveling wave: **(a)** Initial conditions for $t = 0$. **(b)** Time $t = 0.08\,(2\pi/\omega)$, shortly before the formation of the shock. **(c)** Time $t = 0.10\,(2\pi/\omega)$, shortly after the formation of the shock. **(d)** Time $t = 0.50\,(2\pi/\omega)$, once the solution has converged to the traveling wave.

the traveling wave) by $t = T/2$ the solution is indistinguishable from the traveling wave.

2.3. Asymptotic behavior of the solutions. In this subsection, we show (analytically) that the exact solutions of subsection 2.1 yield the large time ($t \to \infty$) asymptotic behavior of the solutions to equation (2.2.1), for any initial data. First, we show that this result holds for $|\omega| > \omega_{cr}$, and initial data relatively close to the corresponding smooth exact solution. Next we present arguments for the case $|\omega| \leq \omega_{cr}$, and arbitrary initial data. The same type of reasoning that we use in this second part can be used to show global convergence when $|\omega| > \omega_{cr}$, but we shall not carry out the details of such argument here.

The main mechanism involved in the convergence (as $t \to \infty$) to the traveling wave solution when $|\omega| \leq \omega_{cr}$ (see remark 2.5) is much more "efficient" than the mechanism involved in the case when $|\omega| > \omega_{cr}$. This is because the first mechanism involves $O(1)$ shocks at all times, while the second has shocks of vanishing amplitude as $t \to \infty$. The calculation displayed in figure **3** illustrates how efficient the mechanism in the case $|\omega| \leq \omega_{cr}$ is, as pointed out at the end of subsection 2.2. By contrast, in numerical calculations done for $|\omega| > \omega_{cr}$, we observed a very slow approach to the limiting behavior.

REMARK 2.3. We shall restrict our arguments to the case when F_{cr}, given by equation (2.2.5), has only two extremal points per period: a single zero, and a single maximum. The single zero condition guarantees a unique traveling wave solution. The single maximum condition is technical and simplifies the arguments — it is not really needed, as we point out later (see remark 2.6.)

When the single zero condition is not satisfied, so that the traveling wave solution for $|\omega| \leq \omega_{cr}$ is not unique, our numerical experiments still show convergence to a traveling wave as $t \to \infty$, which depends on the initial conditions. Actually, the arguments in this section for the case $|\omega| > \omega_{cr}$ do not depend on F_{cr} having a single zero, while the arguments for $|\omega| \leq \omega_{cr}$ indicate convergence to some traveling wave, even if it is not unique.

Case: $|\omega| > \omega_{cr}$. Local convergence to the smooth exact solution.

Preliminaries.

Let $u_{ex} = u_{ex}(z)$ (where $z = x - \omega t$) be the exact solution introduced in subsection 2.1 — see equations (2.2.6 – 2.2.7). Then equation (2.2.1) can be written in the form

$$v_t + \left(\frac{1}{2}v^2 - F\right)_z = 0\,, \tag{2.2.18}$$

where $v = v(z,\, t) = u - \omega$, and $F = F(z) = \frac{1}{2}\left(u_{ex} - \omega\right)^2$ is the function defined earlier in subsection 2.1. We note that

$$\min(F) > 0\,, \qquad \frac{dF}{dz} = f\,,$$

and that v must be a 2π-periodic function of z, with $\mathsf{Mean}(v) = -\omega$. Finally, let

$$v_{ex} = u_{ex} - \omega = -\,\mathrm{sign}(\omega)\,\sqrt{2F(z)}$$

be the exact solution of equation (2.2.18) corresponding to u_{ex} — i.e.: the traveling wave. We shall now **assume that** $\omega < -\omega_{cr}$, since the case $\omega > \omega_{cr}$ can be obtained from the symmetry, in equation (2.2.1), given by: $\omega \to -\omega$, $u \to -u$, $f(z) \to -f(-z)$, and $x \to -x$. Then we can write

$$v_{ex}(z) = \sqrt{2F}\,. \tag{2.2.19}$$

The argument of convergence to the traveling wave solution as $t \to \infty$, in this $|\omega| > \omega_{cr}$ case, is divided in two parts. In part I we show that the initial data can be restricted so that the solution remains positive (and bounded away from zero) for all times, and that such solutions always break and develop shocks — with the sole exception of the traveling wave (which has no shocks.) In part II we construct a (convex) Hamiltonian functional, which decreases for solutions with shocks, and is minimized by the traveling wave solution.

Case: $|\omega| > \omega_{cr}$. Part I.

Along characteristics, equation (2.2.18) takes the form

$$\frac{dz}{dt} = v\,, \qquad \text{and} \qquad \frac{dv}{dt} = \frac{dF}{dz}(z)\,. \tag{2.2.20}$$

This can be written in the Hamiltonian form

$$\frac{dz}{dt} = \frac{\partial h}{\partial v}\,, \qquad \text{and} \qquad \frac{dv}{dt} = -\,\frac{\partial h}{\partial z}\,, \tag{2.2.21}$$

with Hamiltonian

$$h = h(z, v) = \frac{1}{2}v^2 - F(z)\,. \tag{2.2.22}$$

This is the standard Hamiltonian for a particle in a one–dimensional potential field $V(z) = -F(z)$. We shall use this Hamiltonian formulation to derive the following two results:

(2.2.23)
- **I.** We can constrain the initial data $v = v(z, 0)$, so that $v = v(z, t)$ remains positive (and bounded away from zero) for all times $t \geq 0$.
- **II.** Smooth initial data $v = v(z, 0)$, different from v_{ex} (but constrained so that $v > 0$ for all $t \geq 0$), necessarily develop shocks.

These two results will be used in Part II to show convergence of v to v_{ex}, as $t \to \infty$.

For both results, we turn to the phase plane for the Hamiltonian system (2.2.21) (see figure **4**). To prove the first result, notice that, because the characteristic evolution is given by this Hamiltonian system, if the initial data are such that $v(z, 0) \geq v_0(z)$ (where $v = v_0(z) > 0$ is an orbit for the system), then $v \geq v_0$ for all times (hence it remains positive and bounded away from zero.) This is easy to see from the example in figure **4**. Therefore, we shall (from now on) **assume that v is greater than zero.**

To prove the second result, notice that, because here we consider only values of v greater than zero, all centers are excluded, and the trajectories of (2.2.21) are all open in the plane (though, of course, closed in the cylinder defined by the 2π periodicity of z). Several such trajectories, corresponding to different values of h, are displayed in figure **4**. The point we need to make is that any two such trajectories always have different periods, for the period $T(h)$ is given by

$$T(h) = \frac{1}{\sqrt{2}} \int_0^{2\pi} \frac{dz}{\sqrt{h + F(z)}}\,,$$

so that

$$\frac{dT}{dh} = -\,\frac{1}{2\sqrt{2}} \int_0^{2\pi} \frac{dz}{(h + F(z))^{3/2}} < 0\,.$$

This means that any two characteristics with different values of h necessarily meet (the one with the shorter period catches up to the other one.) Therefore a shock must form (since two values of h at a single position z imply two distinct values of v.)

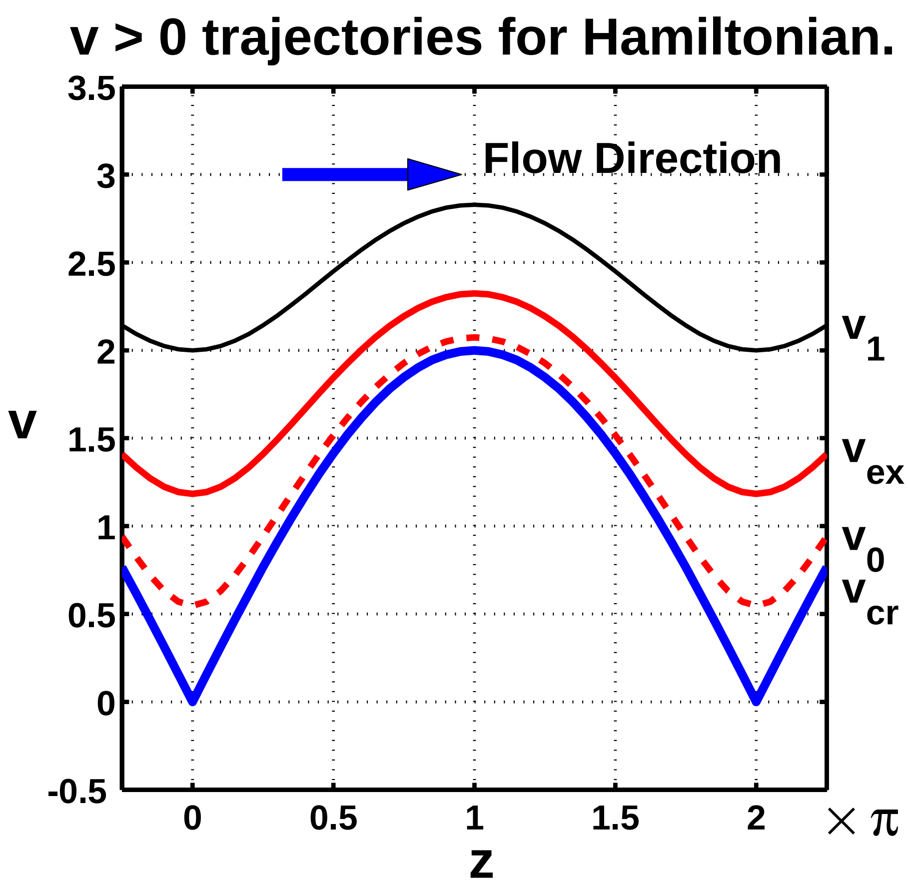

FIGURE 4. Unbounded, $v > 0$ trajectories, for the system with Hamiltonian $h = \frac{1}{2}v^2 + \cos z - 1$. These occur for $v > v_{cr}(z) = \sqrt{2\,F_{cr}(z)}$, where $F_{cr} = (1 - cosz)$. **(a)** Critical trajectory v_{cr}, connecting the saddle points (thick solid line.) **(b)** A trajectory v_0, slightly above critical (dashed line.) **(c)** Two typical trajectories v_{ex} and v_1 (solid lines.) For the P.D.E. $v_t + (\frac{1}{2}v^2 + \cos z)_z = 0$, whose characteristic form is given by this Hamiltonian system, it is clear that: If the initial data for v are above a curve such as v_0, then the solution remains above v_0 for all times.

Hence the only case in which characteristics do not cross is the one in which the initial data lies on a contour line $h =$ constant. However, such initial data is given by

$$v(z, 0) = \sqrt{2(F + h)}\,,$$

and the condition that the average of v be equal to $-\omega$ implies that $h = 0$, i.e., $v = v_{ex}$. Hence all initial data different from v_{ex} develop shocks.

Case: $|\omega| > \omega_{cr}$. Part II.

For the second part of the argument, we need a different Hamiltonian structure, the one given by the integral

$$H = \int_0^{2\pi} \left(\frac{1}{6}v^3 - v\,F\right)\,dz\,. \tag{2.2.24}$$

Using H, equation (2.2.18) can be written in the Hamiltonian form

$$v_t = -\frac{\partial}{\partial z}\left(\frac{\delta H}{\delta v}\right)\,. \tag{2.2.25}$$

We notice that H is convex for positive functions $v = v(z)$ (of mean equal to $-\omega$), with a unique absolute minimum given by $v(z) = \sqrt{2F} = v_{ex}$. Furthermore, because of the Hamiltonian structure (2.2.25), H is conserved while $v(z,t)$ remains smooth. On the other hand (see equation (2.2.27) below) H is dissipated at shocks. Hence, since all v's different from v_{ex} develop shocks (as shown earlier in part I), H cannot settle down until it reaches its minimum value, corresponding to $v = v_{ex}$.

We show now that H decreases when there are shocks. We have:

$$\frac{dH}{dt} = \sum_{j=1}^{N}\left(\int_{s_j}^{s_{j+1}} \frac{\delta H}{\delta v} v_t\,dz - \frac{ds_j}{dt}\left[\frac{1}{6}v^3 - v\,F\right]_j\right)\,, \tag{2.2.26}$$

where we have assumed that there are N shocks per period (for some N), with $s_j = s_j(t)$ the position of the j-th shock, $s_1 < s_2 < \cdots < s_N$, and $s_{N+1} = s_1 + 2\pi$ (from the periodicity.) Substitute now into this last expression v_t from (2.2.25), and the Rankine-Hugoniot jump conditions for the shock velocities:

$$\frac{ds_j}{dt} = \frac{1}{2}\left(v_j^- + v_j^+\right)\,,$$

where $v_j^\pm$ denote the values of v ahead and behind the shock. Then

$$\begin{aligned}
\frac{dH}{dt} &= -\frac{1}{2}\sum_{j=1}^{N}\left(\int_{s_j}^{s_{j+1}}\left(\left(\frac{\delta H}{\delta v}\right)^2\right)_z dz + \left(v_j^- + v_j^+\right)\left[\frac{1}{6}\,v^3 - v\,F\right]_j\right) \\
&= \frac{1}{2}\sum_{j=1}^{N}\left(\left[\left(\frac{\delta H}{\delta v}\right)^2 - \frac{1}{6}\,v^4 + v^2\,F\right]_j - \frac{1}{6}\,v_j^-\,v_j^+\,\left[v^2\right]_j\right) \\
&= \frac{1}{12}\sum_{j=1}^{N}\left(\left[\frac{1}{2}\,v^4\right]_j - v_j^-\,v_j^+\,\left[v^2\right]_j\right) \\
&= \frac{1}{24}\sum_{j=1}^{N}[v]_j^2\,\left[v^2\right]_j < 0\,.
\end{aligned} \tag{2.2.27}$$

The last inequality follows from the entropy condition, that states that v (therefore v^2, since $v > 0$) decreases from left to right across shocks. This concludes the argument of convergence to the traveling wave in the case $|\omega| > \omega_{cr}$.

Case: $|\omega| \leq \omega_{cr}$. Global convergence to the non-smooth traveling waves.

Here we shall consider the case with $|\omega| \leq \omega_{cr}$ (when $F = F_{cr} \geq 0$) and general initial data. We shall require that the zero value of F_{cr} be achieved only once per period in z (for concreteness take this value to be $z = 0$.) The reason for this restriction — which is satisfied by generic functions $F_{cr} = F_{cr}(z)$ — is that: when F_{cr} vanishes at more than one position per period, there is more than one steady solution to equation (2.2.1), and this renders the issue of ultimate convergence to one of the steady solutions more cumbersome. We shall also require that F_{cr} have only one local maximum per period, since this simplifies the arguments (but this condition is not strictly needed, as we point out later, in remark 2.6.)

We shall use the following functional G, a modification of the Hamiltonian H in (2.2.24):

$$G = \int_0^{2\pi} \left(\frac{1}{6}|v|^3 - |v|\, F_{cr}\right) dz\,. \tag{2.2.28}$$

Notice that G is minimized pointwise by functions of the form $v(z) = \pm\sqrt{2F_{cr}} = \pm|v_{ex}|$. In particular, the exact steady solution to equation (2.2.1) is the only minimizer of G consistent with the entropy condition for shocks (v never jumps upwards) and with the requirement that its average be equal to $-\omega$. This follows from the condition that there is only one point where $F_{cr}(z) = 0$, which is the only place at which $v(z)$ can switch smoothly from negative to positive. Hence there can be only one shock switching $v(z)$ back to negative. The position of this shock is then determined by the condition that $v(z) + \omega$ must have a vanishing mean. In particular, for $|\omega| = \omega_{cr}$, this last condition determines that the sign of $v(z)$ never changes, and the only singularity of the solution is a corner at the position of the zero of F_{cr} (with no shocks.)

The **argument for convergence to v_{ex} will be based on the fact that, after an initial transient period, G necessarily decays when $v(z, 0) \neq v_{ex}$.**

Unlike H, G is not a Hamiltonian, yet it allows us to write equation (2.2.18) in the following pseudo-Hamiltonian form (valid wherever $v \neq 0$):

$$v_t = -\,\sigma\,\frac{\partial}{\partial z}\left(\frac{\delta G}{\delta v}\right)\,, \tag{2.2.29}$$

where

$$\sigma = \operatorname{sign}(v) = \begin{cases} 1 & \text{if } v > 0\,. \\ 0 & \text{if } v = 0\,. \\ -1 & \text{if } v < 0\,. \end{cases}$$

Assume now that there are N shocks per period, with $s_j = s_j(t)$ the location of the j-th shock, where $s_1 < s_2 < \cdots < s_N$ and $s_{N+1} = s_1$ (periodicity.) Furthermore, introduce the functions $h = h(z,t) = \frac{1}{2}\,v^2 - F_{cr}$ and $g = \frac{1}{6}\,|v|^3 - |v|\,F_{cr}$. We notice that

$$v_t = -h_z\,, \quad \frac{\partial g}{\partial v} = \sigma\,h\,, \quad \text{and} \quad \sigma\,h\,h_z = \left(\frac{1}{8}\,\sigma\,v^4 - \frac{1}{2}\,\sigma\,v^2\,F_{cr}\right)_z + \frac{1}{2}\,\sigma\,\left(F_{cr}^2\right)_z\,,$$

where the last equation applies away from the shocks (in particular, it is valid when σ is discontinuous due to a zero of v.)

Using the formulas above, we write below an equation for the time evolution of the functional G. Here, as usual, the brackets stand for the jump — front to back — of the enclosed quantities across the shock, the superscripts $\pm$ are used to indicate values immediately ahead and behind a shock, and a subscript j indicates evaluation at the j-th shock. We have:

$$\begin{aligned}
\frac{dG}{dt} &= -\sum_{j=1}^{N}\left(\int_{s_j}^{s_{j+1}} \sigma\, h\, h_z\, dz + \frac{ds_j}{dt}\,[\,g\,]_j\right) \\
&= -\frac{1}{2}\int_0^{2\pi} \sigma\,\left(F_{cr}^2\right)_z\, dz - \frac{1}{2}\sum_{j=1}^{N}\left[\sigma\, F_{cr}^2\right]_j + \\
&\quad \sum_{j=1}^{N}\left(\left[\frac{1}{8}\,\sigma\, v^4 - \frac{1}{2}\,\sigma\, v^2\, F_{cr} + \frac{1}{2}\,\sigma\, F_{cr}^2\right]_j - \frac{1}{2}\,\left(v_j^+ + v_j^-\right)\,[\,g\,]_j\right) \\
&= -\frac{1}{2}\int_0^{2\pi} \sigma\,\left(F_{cr}^2\right)_z\, dz - \frac{1}{2}\sum_{j=1}^{N}\left[\sigma\, F_{cr}^2\right]_j + \\
&\quad \sum_{j=1}^{N}\left(\left[\sigma\,\left(\frac{1}{24}\,v^4 + \frac{1}{2}\,F_{cr}^2\right)\right] - v^+\, v^-\,\left[\sigma\,\left(\frac{1}{12}\,v^2 - \frac{1}{2}\,F_{cr}\right)\right]\right)_j \\
&= S_1 + S_2 + S_3 + S_4\,, \qquad (2.2.30)
\end{aligned}$$

where

$$\begin{aligned}
S_1 &= -\frac{1}{2}\int_0^{2\pi} \sigma\,\left(F_{cr}^2\right)_z\, dz - \frac{1}{2}\sum_{j=1}^{N}\left[\sigma\, F_{cr}^2\right]_j\,, \\
S_2 &= \frac{1}{24}\sum_{\text{local}}\left(\sigma[v]^2\,\left[v^2\right]\right)_j\,, \\
S_3 &= \sum_{\text{transonic}}\left\{v^+\, v^-\,\left(\frac{1}{6}\,\overline{v^2} - F_{cr}\right) - \left(\frac{1}{12}\,\overline{v^4} + F_{cr}^2\right)\right\}_j\,,
\end{aligned}$$

and

$$S_4 = \sum_{\text{to zero}}\left[\sigma\,\left(\frac{1}{24}\,v^4 + \frac{1}{2}F_{cr}^2\right)\right]_j\,.$$

Here the sum S_2 is carried over all the "local" shocks (where v^+ and v^- both have the same sign: $v^+\, v^- > 0$), the sum S_3 is carried over all the "transonic" shocks (where $v^+ < 0 < v^-$), the sum S_4 is carried over all shocks where either v^+ or v^- vanishes, and the overbars indicate the average value across the shock of the appropriate quantity.

REMARK 2.4. Notice that only the values of F_{cr} at the points where σ jumps, and are not shocks, contribute to S_1 (that is, the places where v changes sign "smoothly".) This is because only the points where σ jumps contribute to the integral that appears in the definition of S_1, with the sum in the same definition subtracting any contributions that arise at the shocks. In fact, (generically) we can write:

$$S_1 = \frac{1}{2}\left(\sum_{n\in D}[\sigma\, F_{cr}^2]_n \;-\; \sum_{j=1}^{N}[\sigma\, F_{cr}^2]_j\right) = \frac{1}{2}\left(\sum_{n\in D}\left([\sigma]\, F_{cr}^2\right)_n \;-\; \sum_{j=1}^{N}\left([\sigma]\, F_{cr}^2\right)_j\right),$$

where D is a set of indexes for all the positions across which v switches sign.

It should be clear that

- S_1 does not have a definite sign, since its overall sign depends on the relative sizes of F_{cr}^2 at the places where v crosses zero upwards (from negative to positive, so that $[\sigma] = 2$) versus the places where it crosses zero downwards, so that $[\sigma] = -2$.
- On the other hand, S_2 **is always non-positive, vanishing only when there are no local shocks** (as it is the case for the exact traveling wave solution v_{ex}.) This follows because, when $v \geq 0$ on both sides of the shock: $\sigma = 1$ and the entropy condition yields $[v^2] < 0$. Similarly, when $\sigma = -1$, the entropy condition yields $[v^2] > 0$.
- S_3 **is always non-positive, vanishing only when** $v^- = \sqrt{2F_{cr}}$ **and** $v^+ = -\sqrt{2F_{cr}}$ **(or there are no transonic shocks),** as it is the case for the exact solution v_{ex}. We shall only need this last result for $v^- \geq \sqrt{2F_{cr}}$ and $v^+ \leq -\sqrt{2F_{cr}}$. In this case, the proof is quite straightforward, since S_3 can be rewritten in the form:

$$S_3 = -\sum_{\text{trans}}\left(\left(a+b+2\sqrt{F_{cr}}\right)\left(a^2+b^2\right)\sqrt{F_{cr}} + \frac{1}{6}\left(a^4+b^4+2ab^3+2a^3b\right)\right) \leq 0\,,$$

  where $a = \dfrac{v^-}{\sqrt{2}} - \sqrt{F_{cr}} \geq 0$, and $b = -\left(\dfrac{v^+}{\sqrt{2}} + \sqrt{F_{cr}}\right) \geq 0$.
- Finally, S_4 **is always non-positive, vanishing only when there are no shocks in the summation.** This is obvious, since each shock in the summation contributes an amount $\sigma_+ A_+ - \sigma_- A_-$, where $A_\pm > 0$ and either: $\sigma_+ = -1$ and $\sigma_- = 0$, or $\sigma_+ = 0$ and $\sigma_- = 1$.

The argument for convergence to the exact traveling wave solution v_{ex} (in this $\omega \leq \omega_{cr}$ case) will be based on the phase plane for the characteristic equations (2.2.21), corresponding to the Hamiltonian h in equation (2.2.22) — with $F = F_{cr}$. This phase plane, displayed again in figure **5**, is partitioned into two domains by the separatrix $h = 0$: a **domain D containing the closed periodic orbits, and its complement $C(D)$ containing the open orbits.** We will assume here that F_{cr} has a single maximum per period, so that there is a single critical point in D (a center), with all the other orbits being closed and periodic (as shown in figure **5**.) We shall first argue that:

(2.2.31) **The asymptotic behavior for the solutions to (2.2.18) cannot include any values in the interior of the domain D.**

The argument for this goes as follows:

A. First we note that: any two characteristics starting in the interior of D cross in finite time — even if they lie on the same contour line (orbit) for h. Furthermore: for any compact subdomain D_c of D, the crossing time can be uniformly bounded.

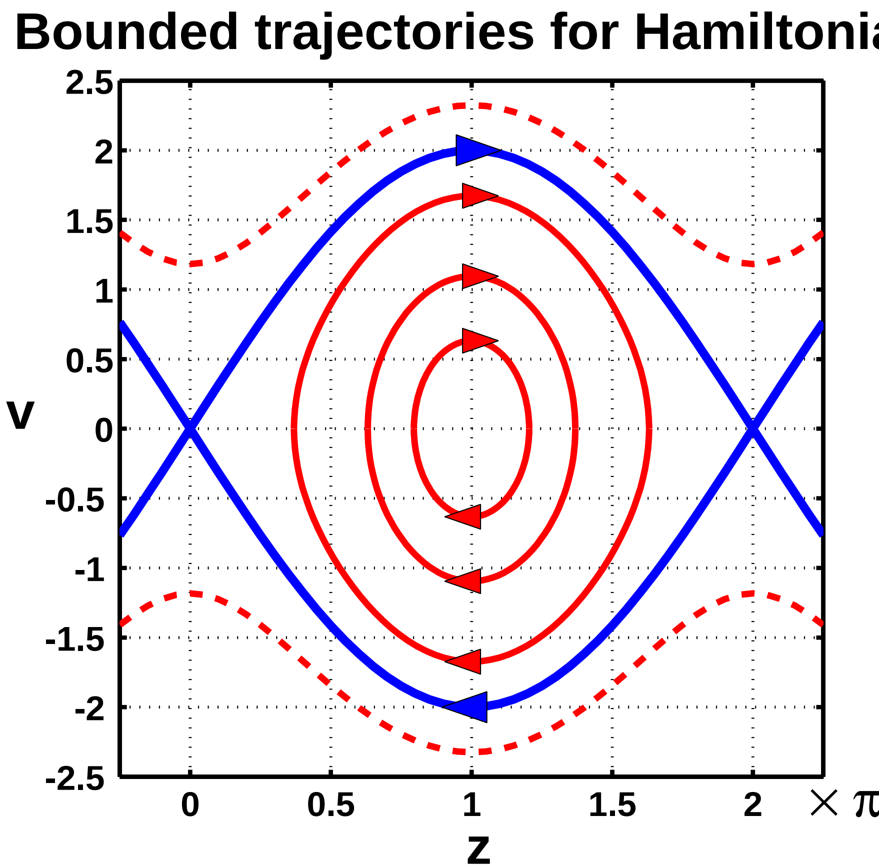

FIGURE 5. Bounded trajectories for the system with Hamiltonian $h = \frac{1}{2}v^2 + \cos z - 1$. These occur in the region $|v| < v_{cr}(z) = \sqrt{2\,F_{cr}(z)}$, where $F_{cr} = (1 - cosz)$. The critical trajectories $v = \pm v_{cr}$, connecting the saddle points (thick solid lines), and several periodic orbits are shown, in addition to a couple of unbounded orbits (dashed lines.) Notice that, in cases where F_{cr} has more than one maximum per period, the bounded orbit region will be more complicated, with saddles and more than one center in it.

This is obvious from figure **5**. A formal argument goes as follows: let $z_1 = z_1(t)$ and $z_2 = z_2(t)$ be any two characteristics corresponding to orbits in D, with z_1 the characteristic for the outermost orbit in D. Then both z_1 and z_2 are periodic functions of time, with $\max(z_1) \geq \max(z_2)$ and $\min(z_1) \leq \min(z_2)$. Then $\max(z_1 - z_2) \geq \max(z_1) - \max(z_2) \geq 0$ and $\min(z_1 - z_2) \leq \min(z_1) - \min(z_2) \leq 0$, so that $z_1 - z_2$ must vanish somewhere, in fact: at least twice per z_1-period. Thus: a uniform bound on the crossing time is given by the maximum of the orbit periods over the domain D_c. Note that: **as the distance of D_c to the boundary of D gets smaller, the crossing time bound goes to infinity,** because the orbit period grows unboundedly as the separatrix is approached (points on the separatrix take an infinite amount of time to move from saddle to saddle, while points inside D move on orbits with a finite period.)

B. Using the result in **A**, we argue now that any part of the initial data contained in a compact subdomain D_c of D, ceases to influence the solution after a finite time. This second result, of course, implies (2.2.31).

The argument here is as follows: suppose that there is a characteristic connecting some point on the solution with the initial data in D_c. But then some neighborhood of this point (possibly one-sided, if the point is on a shock), connected with the initial data in D_c by a "beam" of characteristics, would exist. This is clearly impossible after the time given by the uniform bound in part **A** above.

REMARK 2.5. The result in (2.2.31) is easy to visualize graphically (in terms of what the solution to equation (2.2.18) does as it evolves in time) using the phase plane for the evolution by characteristics in (2.2.21) — as illustrated by figure **5**. It should be clear that any part of the solution curve $v = v(z, t)$, contained inside D, will be stretched and "rolled up" (as illustrated in figure **6**) by the characteristic evolution along the periodic orbits of the Hamiltonian

$$h = \frac{1}{2}v^2 - F_{cr}(z)\,.$$

This then leads to multiple values, which are resolved by the introduction of shocks. It should also be clear that, in this roll up process, the upper and lower envelope of the solution curve will be produced by stretching of the parts of the initial solution curve closest to the separatrix $h = 0$ — which will then be the only parts surviving after the shocks are introduced.

Notice that this is a very "efficient" mechanism for the elimination of any part of the solution curve contained inside D. For all practical purposes, the elimination of these parts occurs in a finite time (roughly, the average "turn over" time for the periodic orbits), after which only a very small region near the critical level curve $h = 0$ can remain. As pointed out at the beginning of this subsection, this fact is clearly seen in the numerical experiments we conducted, with a very sharp separation of scales between the convergence times for the cases $|\omega| > \omega_{cr}$ and $|\omega| < \omega_{cr}$.

REMARK 2.6. The prior remark should make it clear that the key element in obtaining (2.2.31) is the existence of a small "band" of periodic orbits in D close to the separatrix. This is true even if F_{cr} has more than two extremal points per period — leading to several critical points inside D, not just a center.

The critical thing to notice is that the initial data solution curve $v = v(z, 0)$ must be periodic in z. Thus it is clear that: if any part of this curve ends up inside D, then there will have to be points where the curve crosses the separatrix $h = 0$ going from $C(D)$ to D, and vice versa. The neighborhoods of these points inside D will then be stretched and "rolled up" by the characteristic evolution, so that they are the only surviving parts of the initial data inside D (after some time.) Hence (2.2.31) will be valid, even if F_{cr} has many extremal points.

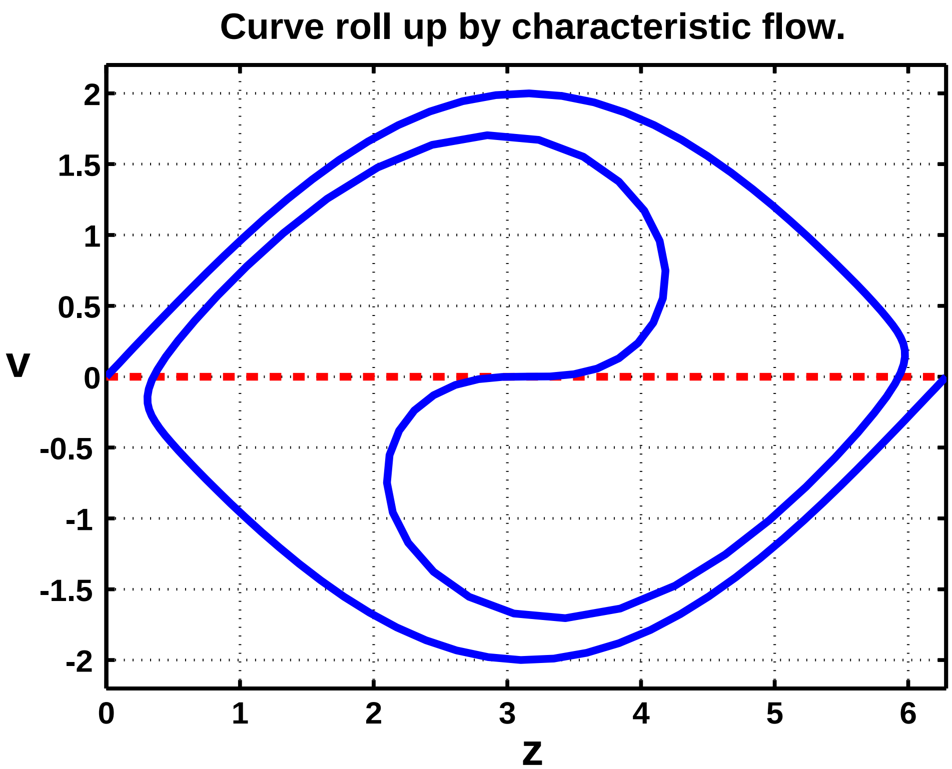

FIGURE 6. Solution curve roll up by the characteristic flow with the Hamiltonian $h = \frac{1}{2}v^2 + \cos z - 1$, and initial condition $v(z,0) = 0$. The figure shows the initial conditions (dashed line) and the curve, as evolved by the characteristic flow, for time $t = 2\pi$. The parts of the curve near the saddles stretch to fill the critical $h = 0$ orbit (as $t \to \infty$), which is the only thing that survives after the shock is put in place.

The result in (2.2.31) shows that, after a long enough time, the solution can cross the line $v = 0$ smoothly **only** in an arbitrarily small neighborhood of $z = 0$, where F_{cr} vanishes (notice that a crossing is needed when $|\omega| < \omega_{cr}$, since the condition $\mathsf{Mean}(v) = -\omega$ cannot be satisfied if either $v \geq v_{cr} = \sqrt{2\,F_{cr}}$ or $v \leq -v_{cr} = -\sqrt{2\,F_{cr}}$.) If there is such a crossing, it must be upwards, with v returning to negative values through a transonic shock. Moreover, the solution needs to lie entirely on $C(D)$ or at most, if within D, in an arbitrarily small neighborhood of the separatrix $h = 0$.

Once in the situation described in the prior paragraph, S_1 in equation (2.2.30) becomes arbitrarily small (see remark 2.4). Since (as shown earlier) S_2, S_3, and S_4 are non-positive, it follows that the functional G, defined in (2.2.28), can no longer increase: it remains constant for solutions that are either smooth or only have shocks from $\sqrt{2F_{cr}}$ to $-\sqrt{2F_{cr}}$, and decreases for all other solutions. Furthermore, the same argument we used for the case $|\omega| > \omega_{cr}$ can be used to show that any parts of the solution (in $C(D)$) not lying on a single contour line $h =$ constant, necessarily break and form shocks. Hence, as long as $v(z,t)$ stays away from v_{ex}, G decreases. Again, we conclude that the long time ($t \to \infty$) asymptotic limit of the solution $v = v(x, t)$ must be given by $v_{ex}(z)$ (which minimizes G.)

3. Two Forcing Modes.

In this section, we study the effects — on the solutions to equation (1.1.1) — of a forcing term consisting of the sum of two traveling waves of different speeds. For concreteness, we shall only consider the case in which one of these speeds is zero, corresponding to a perfect resonance, and we shall observe the changes in behavior as the other speed ranges from zero to infinity. To be specific, we will consider the equation

$$\text{(3.3.1)} \qquad \begin{cases} & u_t + \left(\dfrac{1}{2}u^2\right)_x = f(x,\, t)\,, \\ \text{where} & \\ & f(x,t) = g_1(x) + g_2(x - \Omega\, t) \end{cases}$$

Ω is a constant, g_1 and g_2 are 2π periodic smooth functions with vanishing mean, and $u = u(x, t)$ is 2π periodic in space, with zero mean. We will **assume** $\Omega > 0$, since the case $\Omega < 0$ can be reduced to this one using the symmetry: $x \to -x$, $u \to -u$, and $f \to -f$.

When Ω is small, the two forcing modes oscillate at nearly the same [resonant] frequency. Our interest in this situation arises from the general question of the effects of the superposition of many near resonant interactions in general systems. In order to estimate the combined effect of the interaction of a mode with very many others, one needs to assess the degree of phase coherence among the corresponding forcing terms. Such assessment depends fundamentally on the consideration of three issues:

(3.3.2)

- **1.** How close to each other are the linear frequencies of the forcing modes.
- **2.** How much these linear frequencies are affected ["renormalized"] by nonlinear effects.
- **3.** How often do strong [intermittent] nonlinear events effectively reinitialize the phases of the various forcing modes. Also, do these reinitializations tend to randomize or rather further correlate the various phases?

It should be quite clear that these questions are not easy to answer. Moreover, once answers are assumed (see next paragraph), one has only defined the nature of the forcing; its effects on the evolution of the forced mode still need to be assessed. Furthermore, since the forcing arises from combinations of other modes which are also similarly forced, the problem has and enormously complicated nature.

Attempts to bypass this great complexity often rely on universal assumptions, such as randomization of the phases and separation of the linear and nonlinear scales, which are very difficult to justify. Typically, these closures are sometimes successful — in that their predictions agree with the observed behavior of the system under study — and sometimes a radical failure, with the reasons for this disparity open to debate.

Here we isolate the issue of the response to a given force, by prescribing the form of the forcing term. Moreover, we consider only two forcing modes, and prescribe their form and frequency as if they were not subject to nonlinear interactions themselves. By so reducing the complexity of the problem, we are able to resolve some questions regarding the effects of the degree of coherence of the forcing modes on the behavior of the forced mode.

The plan of this section is the following: First we describe certain general features of the solution to equation (3.3.1). Then we study separately two limiting regimes, corresponding to Ω either very large or very small. In each case we study (analytically and numerically) the behavior of the corresponding solutions.

Let us start by noticing that the forcing term in (3.3.1) is periodic in time, of period $2\pi/\Omega$. Since (3.3.1) is dissipative — though only through shocks, which are not necessarily present all the time — we expect that **the solution $u(x,\, t)$ will converge to a periodic pattern of the same periodicity.** We have checked this numerically, by computing the quantity

$$D = \int_0^{2\pi} \left(u(x,\, t + \frac{2\pi}{\Omega}) - u(x,\, t) \right)^2 dx\,. \tag{3.3.3}$$

In all the numerical experiments that we performed, D decreased rapidly, becoming effectively zero in about one or two periods of the forcing function.

We shall now distinguish two distinct extreme regimes, with the general case behavior interpolating between these two. Of the two terms in the forcing, $g_1 = g_1(x)$ is in resonance with $u(x,t)$, since the latter has a vanishing mean (hence zero linear frequency in the unforced case.) On the other hand, the forcing $g_2 = g_2(x - \Omega\, t)$ will be close to –or far from– resonance depending on the size of Ω. When $\Omega \gg 1$, we expect the leading order effect of $g_2(x - \Omega\, t)$ on $u(x,t)$ to cancel, due to averaging. When $0 < \Omega \ll 1$, on the other hand, the effect of g_2 can no longer be neglected. In this second case we expect g_1 and g_2 to combine into a single, quasi-steady force, yielding a quasi-steady solution $u = u(x,\, \Omega\, t)$ — very much a modulated version of the steady solution to (2.2.1), studied in section 2, for $\omega = 0$. Namely, in this last case we expect:

$$\left(\frac{1}{2} u^2 \right)_x \approx g_1(x) + g_2(x - \Omega\, t)\,. \tag{3.3.4}$$

That this is roughly the case, yet with some interesting qualifications, will become clear in the analysis that follows.

3.1. Case: $\Omega \gg 1$; g_2 far away from resonance.

In this subsection, we show that, when Ω is large, the solution $u = u(x,t)$ to (3.3.1) is close to the solution that one would obtain if the only forcing term were $g_1(x)$ — that is (to leading order) g_2 has no effect. To see this, introduce the small parameter

$$\epsilon = \frac{1}{\Omega}\,.$$

Then equation (3.3.1) takes the form

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = g_1(x) + g_2(x-\tau)\,, \tag{3.3.5}$$

where $\tau = \dfrac{t}{\epsilon}$ is a fast time variable. We now propose the following asymptotic expansion:

$$u = u_0(x) + \epsilon\, u_1(x,\,\tau) + O(\epsilon^2)\,, \tag{3.3.6}$$

where the dependence on τ is 2π–periodic. Then, at leading order, equation (3.3.5) yields

$$\frac{\partial u_1}{\partial \tau} + \left(\frac{1}{2}\,u_0^2\right)_x = g_1(x) + g_2(x-\tau) = \frac{\partial G_1(x)}{\partial x} - \frac{\partial G_2(x-\tau)}{\partial \tau}\,, \tag{3.3.7}$$

where G_1 and G_2 are the integrals of g_1 and g_2, respectively (uniquely defined by the condition that both should have a vanishing mean.) We will assume the (generic) condition that G_1 **has a single minimum per period.**

Integrating equation (3.3.7) over one period (in τ), we obtain

$$\left(\frac{1}{2}\,u_0^2\right)_x = \frac{\partial G_1(x)}{\partial x} \quad\Longrightarrow\quad u_0(x) = \pm\sqrt{2\,(D + G_1(x))}\,, \tag{3.3.8}$$

where $D = -\min(G_1)$, the solution crosses (continuously) from the negative to the positive root at the position of the minimum of G_1, and has a shock (jumping from the positive to the negative root) at a position determined by the requirement that the average of $u_0 = u_0(x)$ should vanish. This leading order solution agrees with the solution that one would obtain if the forcing consisted exclusively of g_1 (see section 2, for $\omega = 0$.)

Substituting (3.3.8) into (3.3.7) we then find that

$$u_1(x,\,\tau) = -\,G_2(x-\tau) + p(x)\,, \tag{3.3.9}$$

where $p = p(x)$ is a 2π–periodic function of vanishing mean, that is determined at the next order in the asymptotic expansion. Numerical experiments — not shown here — corroborate the results of this asymptotic analysis.

3.2. Case: $0 < \Omega \ll 1$; quasi–steady forcing. When $0 < \Omega \ll 1$, we can (in principle) think of the solution to equation (3.3.1) as frozen in time near each value $t = t_0$. This yields a quasi–steady leading order solution $u = u(x,\,\Omega\,t)$, where $u(x,\,\Omega\,t_0)$ is given by the steady state solution (section 2, case $\omega = 0$) to the case with a single forcing mode, with $f = f(x) = g_1(x) + g_2(x - \Omega\,t_0)$. In this subsection, we shall discuss this quasi-steady solution in some detail.

We begin with a simple asymptotic expansion that implements the idea in the paragraph above. Using Ω as the small parameter, we write

$$u(x,\,t) = u_0(x,\,\tau) + \Omega\,u_1(x,\,\tau) + O(\Omega^2)\,, \tag{3.3.10}$$

where the dependence on τ is 2π–periodic and $\tau = \Omega\,t$ is a slow time variable. Then, at leading order, (3.3.1) yields

$$\left(\frac{1}{2}\,u_0^2\right)_x = g_1(x) + g_2(x-\tau)\,. \tag{3.3.11}$$

Thus

$$u_0(x,\,\tau) = \pm\sqrt{2\,G(x,\,\tau)}\,, \tag{3.3.12}$$

where $G = G(x,\,\tau)$ is defined (for each τ) by

$$\frac{\partial G}{\partial x} = g_1(x) + g_2(x-\tau) \quad \text{and} \quad \min_{0\leq x<2\pi}(G) = 0\,. \tag{3.3.13}$$

In each period $0 \leq x < 2\pi$ the solution crosses (continuously) from the negative to the positive root at the point $x = x_m(\tau)$ where $G = 0$, and has a shock (jumping from the positive to the negative root) at a position $x = s(\tau)$, chosen so that the mean of u_0 vanishes.

The solution (3.3.12) above works as long as G has a single minimum per period, in which case $x_m = x_m(\tau)$ and $s = s(\tau)$ are well defined and depend smoothly on τ. However, there will generally be some special times, $\tau = \tau_c$, at which this fails. Generically G will have several local minimums, evolving in time, with one of them smaller than all the others. The (generic) special times occur when two local minimums exchange the property of being the global minimum. At these times x_m ceases to be smooth, jumps discontinuously from one position to another, and the expansion in (3.3.10) becomes inconsistent and fails.

Remark 3.1. As pointed out at the beginning of subsection 2.3 (and remark 2.5) the convergence of the solution to a steady state — when the forcing is time independent — is generally very fast. Thus, we can be pretty sure that (3.3.10) will describe the behavior of the solution away from the critical times τ_c. The question (which we will address below) now becomes: what happens for $\tau \approx \tau_c$?

On each side of a critical time τ_c, the expansion in (3.3.10) is valid, but the position of the shock ($x = s(\tau)$) and the zero ($x = x_m(\tau)$) jump across $\tau = \tau_c$, implying a discontinuous global change in the solution u. Hence, there is a set of discrete times when the solution u needs to adjust "rapidly" from one quasi–steady state to another ($O(1)$ away) one. The existence of these **adjustment processes, which we will call "storms",** raises the following questions:

(3.3.14)

1. What is the time–scale (i.e., the duration) of a storm?

2. During a storm: are there significant effects in the energy exchange between the forcing function $f = f(x,\,t)$ in (3.3.1) and the solution $u = u(x,\,t)$? That is to say: is the work per unit time

$$W_f = \int_0^{2\pi} f\,u\,dx\,,$$

done by the external force, significantly affected by the storm?

Remark 3.2. Notice that the total energy (as follows from equation (3.3.12))

$$E \approx \int_0^{2\pi} \frac{1}{2}\,u_0^2(x,\,\tau)\,dx = \int_0^{2\pi} G(x,\tau)\,dx \tag{3.3.15}$$

is a continuous function of τ for the quasi–steady solution, even though u_0 itself is not. This implies that any extra energy exchange between u and the forcing f during a storm will need to be matched by extra dissipation over the course of the storm.

REMARK 3.3. Away from the storms, the asymptotic solution in (3.3.10) shows that there is a leading order balance between the work W_f done by the forcing f on the solution u, and the energy E_d dissipated at the shocks. Namely:

$$E_d - W_f = O(\Omega)\,. \tag{3.3.16}$$

This, of course, is in agreement with the fact that the total energy E is a slow function of time ($E = E(\tau)$, as shown by equation (3.3.15).) This suggests the following extra question, related to **2** in (3.3.14) above: **How is the balance in (3.3.16) affected by a storm?**

Before attempting to answer these questions analytically, let us set up a simple example, that will help both make the discussion concrete, and verify its results through numerical experiments. Let us select a forcing term of the form

$$f(x,t) = \sin(x) + 2\,\sin(2(x - \Omega\,t))\,, \tag{3.3.17}$$

in equation (3.3.1). Then G, as defined in (3.3.13), is given by:

$$G = \int f(x,t)\,dx = C\,(\Omega\,t) - (\cos(x) + \cos(2(x - \Omega\,t)))\;, \tag{3.3.18}$$

where $C = \max_x(\cos(x) + \cos(2(x - \Omega\,t)))$. The critical times at which the zero of G jumps are given by

$$t_n = \frac{(2n+1)\,\pi}{2\,\Omega}\,, \tag{3.3.19}$$

where n is an integer. At these times

$$G(x,\,\Omega\,t_n) \;=\; \frac{1}{8}\,(1 - 4\cos(x))^2\;, \tag{3.3.20}$$

and

$$u_0(x,\,\Omega\,t_n) \;=\; \pm\frac{1}{2}\,|1 - 4\cos(x)|\;, \tag{3.3.21}$$

with two candidate crossings of zero.

At the critical times t_n (of which there is one per period), there are two solutions $u_0(x)$ of the form (3.3.21), in which u_0 switches from negative to positive at one of the zeros, has a corner at the other, and switches once from positive to negative through a shock, at a position determined by the condition that u_0 has a vanishing average. The quasi–steady solution given by the asymptotic expansion in (3.3.10) approaches one (or the other) of these two solutions as $t \to t_n$ from below (or above.)

REMARK 3.4. In addition to the two special solutions mentioned in the prior paragraph, there is a full one–parameter family of solutions (of which the two solutions just described are extreme cases.) In this family, both zeros of u_0 are used for upward (negative to positive) crossings, and there are consequently two shocks switching the solution back to negative. The positions of these two shocks are related by the constraint on the average of u_0, which leaves one free parameter. The relevance of this one parameter family of solutions is that, during a "storm", the actual solution $u(x,t)$ sweeps this family, one member at a time, at an intermediate rate, faster than $O(\Omega\,t)$, but slower than $O(t)$. Before showing this curious result through an asymptotic expansion, we illustrate it with a numerical solution.

Figure **7** displays the [numerical] solution to equation (3.3.1) with the forcing given by (3.3.17), starting from the asymptotic solution shortly before the critical time t_1, for a value of the frequency $\Omega = 0.01$, not exceedingly small. The dotted line gives the envelope for the asymptotic, quasi-steady solution $u_0(x, \tau)$ (i.e.: the curves $u = \pm\sqrt{2G}$.) In this figure we can see the actual solution $u = u(x, t)$ switching its upward crossing point from one zero of $G(x, \Omega t_1)$ to the other, through a relatively fast transition, involving the development, growth, travel and eventual disappearance of a second shock. During this transition, the solution sticks very closely to the envelope of the quasi-steady solution. The slight disagreement, most visible in frame (e), is due to the finite size of Ω: as Ω gets smaller, the full "storm" takes place with the envelope nearly constant, and we should compare it with the "critical" envelope (that has two zero crossings per period.)

Figure **8** shows the total energy of the solution as a function of time, for a full period[2] in time π/Ω, and four values of the frequency, from $\Omega = 1/50$ to $\Omega = 1/400$. Note that this figure shows the energy converging to a function of time with a corner at t_1, as the frequency Ω tends to zero (the limit is the function given by equation (3.3.15), for this special case when G is given by (3.3.18).) Such a cornered energy function corresponds to an instantaneous storm, which changes the phase of the solution discontinuously at $t = t_1$.

A more thorough understanding of the energetics of a storm is gained by looking at either the energy dissipation rate $E_d = E_d(t)$ (caused by the shocks), or the work $W_f = W_f(t)$ done by the forcing (see figures **9** and **10**). Both show a marked spike during the storm, approximately duplicating the regular amount of work and dissipation. The doubling of the energy dissipation rate is easily explained as arising from the appearance of an extra shock during a storm, of a size comparable to the regular one. The close agreement between the energy dissipated and the work performed by the forcing, on the other hand, can be explained by the slow evolution of storms, faster than the regular $O(\Omega t)$ rate, but clearly slower than a $O(t)$ rate. Hence, at any particular time, the energy input and output need to be in balance to leading order. In other words: **even during a storm the solution is quasi–steady** (as we will show below.)

The points just raised bring us back to the natural question of what is the time–scale for a storm (namely, question **1** in (3.3.14).) Quantifying this time–scale will tell us how significant storms are from the viewpoint of energy exchange: fast storms do not have time to affect the energy exchange significantly, while slower storms do. Notice that the storms have a very definite duration in figures **9** and **10**: they start and end rather abruptly. Measuring these durations suggests that they scale with the square–root of the frequency Ω. That this is precisely the case can be inferred from the following asymptotic argument:

Consider, during a storm[3] an asymptotic expansion of the form

$$u(x,t) = u_0(x, T) + \delta\, u_1(x, T) + O(\delta^2)\,, \tag{3.3.22}$$

where $T = \delta\,(t - t_c)$, and $\Omega \ll \delta \ll 1$ is a small parameter to be determined (δ gives the storm time scale.) The right hand side $f = f(x, \tau)$ in equation (3.3.1)

[2] Note that, because g_2 in (3.3.17) has period π, in this case the long time asymptotic solution to (3.3.1) has period π/Ω in time — not $2\pi/\Omega$, as in the general case.

[3] Taking place for $t \approx t_c = \tau_c/\Omega$, where τ_c is defined below equation (3.3.13).

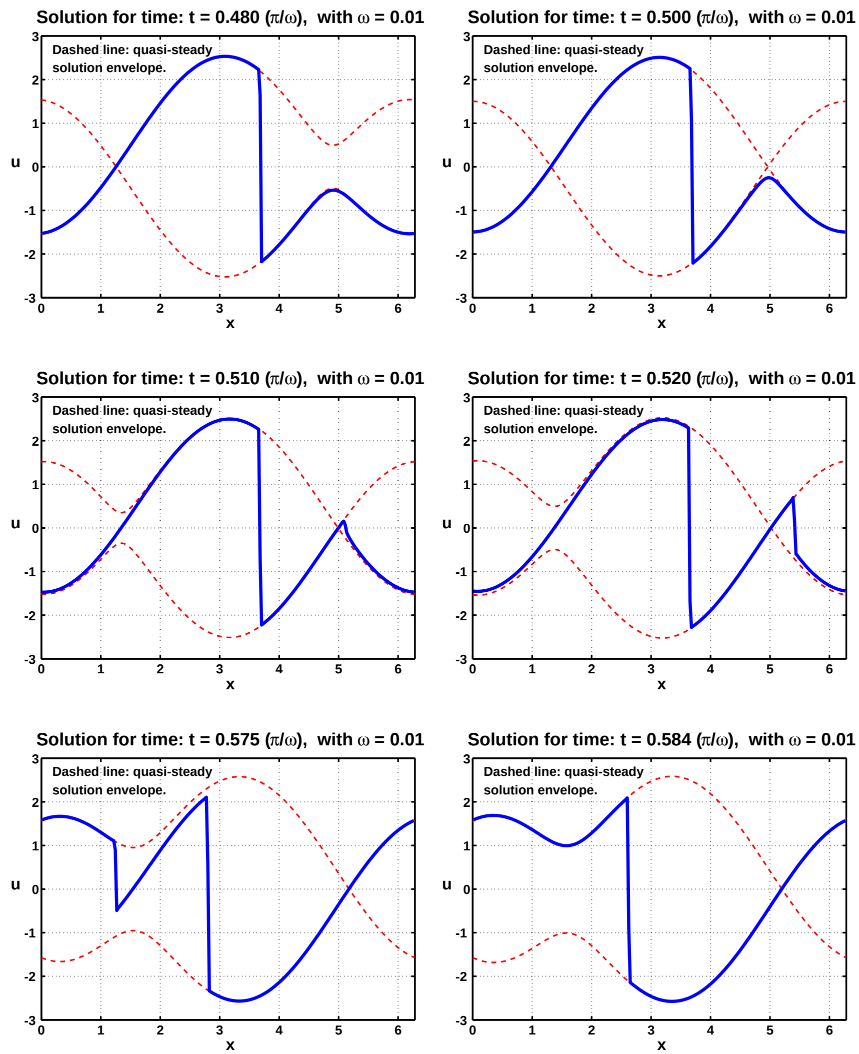

FIGURE **7**. Asymptotic $t \to \infty$ solution to the equation $u_t + (\frac{1}{2}\,u^2)_x = \sin(x) + 2\sin(2\,(x - \Omega\,t))$, with $\Omega = 0.01$. Time slices of the solution are shown for t near the critical time $t_c = \pi/(2\,\Omega)$, when F_{cr} has a double zero. The asymptotic solution is periodic in time, of period π/Ω. Because Ω is small, the solution is quasi-steady at all times. The plots here illustrate the evolution in the time scale $O(\sqrt{\Omega}t)$, with corrections of order $O(\sqrt{\Omega})$, for t near t_c (when a double shock arises.) For t away from t_c the solution is close to the unique quasi-steady solution of the problem. For t close to t_c the solution evolves following the one parameter family of quasi-steady solutions possible when $t = t_c$. Two shocks arise in this stage. Left to right and top to bottom, the figures show the solution (and the envelope $\pm\sqrt{2F_{cr}}$ for the quasi-steady solution, in a dashed line) for the times: **(a)** $t = 0.480\,(\pi/\Omega)$, **(b)** $t = 0.500\,(\pi/\Omega)$, **(c)** $t = 0.510\,(\pi/\Omega)$, **(d)** $t = 0.520\,(\pi/\Omega)$, **(e)** $t = 0.575\,(\pi/\Omega)$, and **(f)** $t = 0.584\,(\pi/\Omega)$.

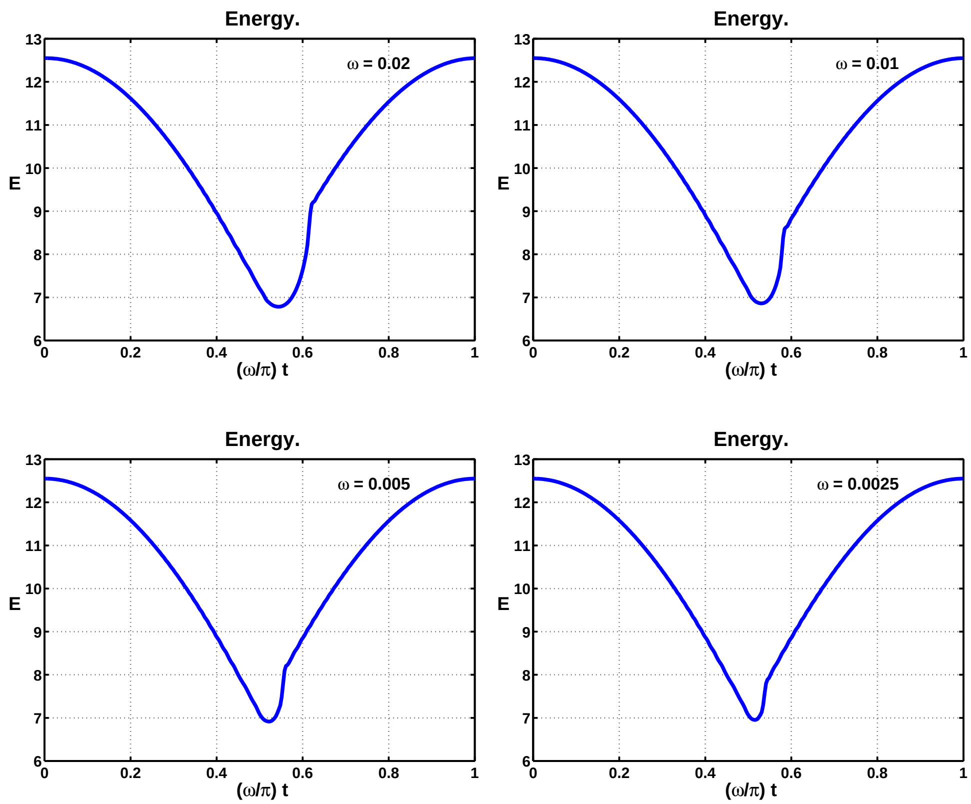

FIGURE 8. Energy $E = E(t)$ — shown over one period $0 \leq (\Omega/\pi)t \leq 1$ in time — for the $t \to \infty$ asymptotic solution for the equation $u_t + (\frac{1}{2}\,u^2)_x = \sin(x) + 2\sin(2\,(x - \Omega\,t))$. Left to right and top to bottom, plots for the cases $\Omega = 1/50$, $\Omega = 1/100$, $\Omega = 1/200$, and $\Omega = 1/400$ are shown.

can expanded in the form

$$f(x,\,\tau) = f(x,\,\tau_c) + \frac{\Omega}{\delta}\,T\,f_\tau(x,\,\tau_c) + \dots\,, \tag{3.3.23}$$

where $\tau = \Omega\,t$, as in the expansion in (3.3.10). Substituting (3.3.22) and (3.3.23) into equation (3.3.1) we obtain, to leading order:

$$u_0(x,\,T) = \pm\sqrt{2\,G_c(x)}\,, \tag{3.3.24}$$

where $G_c = G(x, \tau_c)$, and G is as in (3.3.13). Because $\tau = \tau_c$, generically G_c will have two zeros per period, and the dependence of u_0 on T is through the position of the two shocks in (3.3.24). That is: u_0 must be a member of the one parameter family of solutions that the steady state problem has at the critical times (see remark 3.4), with the parameter a function of T.

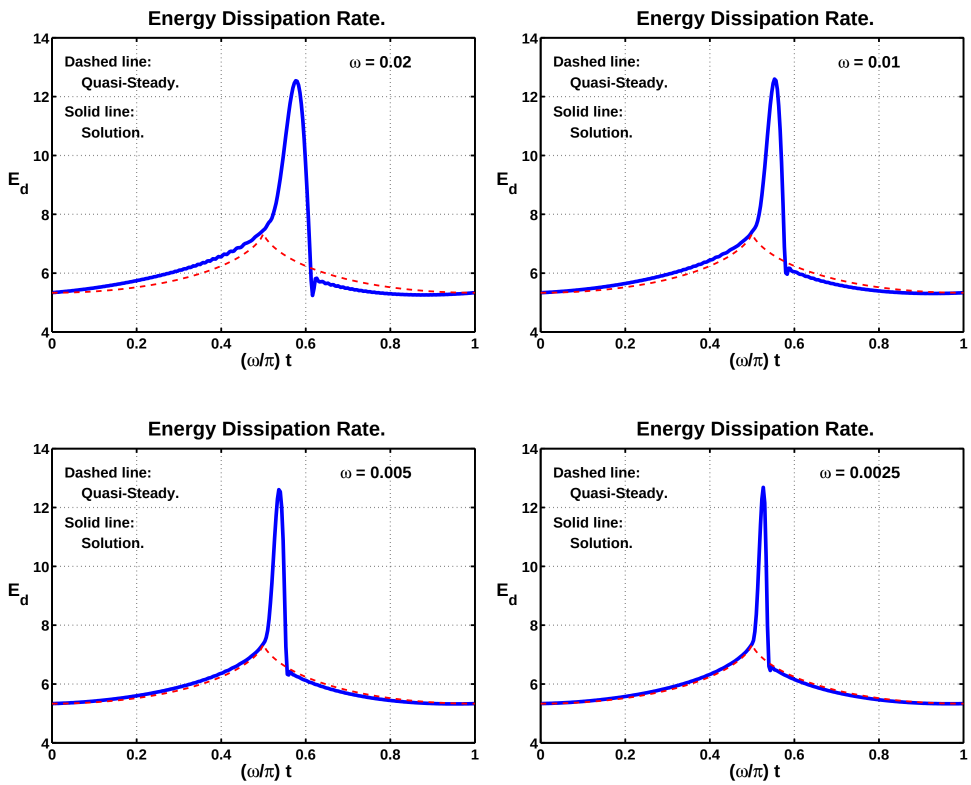

FIGURE 9. Energy dissipation rate $E_d = E_d(t)$ — shown over one period $0 \leq (\Omega/\pi)t \leq 1$ in time — for the $t \to \infty$ asymptotic solution for the equation $u_t + (\frac{1}{2}u^2)_x = \sin(x) + 2\sin(2(x - \Omega t))$. Left to right and top to bottom, plots for the cases $\Omega = 1/50$, $\Omega = 1/100$, $\Omega = 1/200$, and $\Omega = 1/400$ are shown. The width of the dissipation spike near the time where the shock in the quasi-steady solution changes location, behaves like $\Delta t \approx 1/\sqrt{\Omega}$. The energy dissipation rate for the quasi-steady solution is shown by the dotted line.

At the next order in the expansion we have, on each side of the equation:

$$\delta\,\left(u_{0T} + (u_0\,u_1)_x\right) = \frac{\Omega}{\delta}\,T\,f_\tau(x, \tau_c)\,, \tag{3.3.25}$$

which requires $\delta = \sqrt{\Omega}$ in order to balance. Hence the (intermediate) time–scale, valid during storms, is given by $T = \sqrt{\Omega}\,t$, as suggested by the numerical experiments.

Answers to the questions posed earlier.

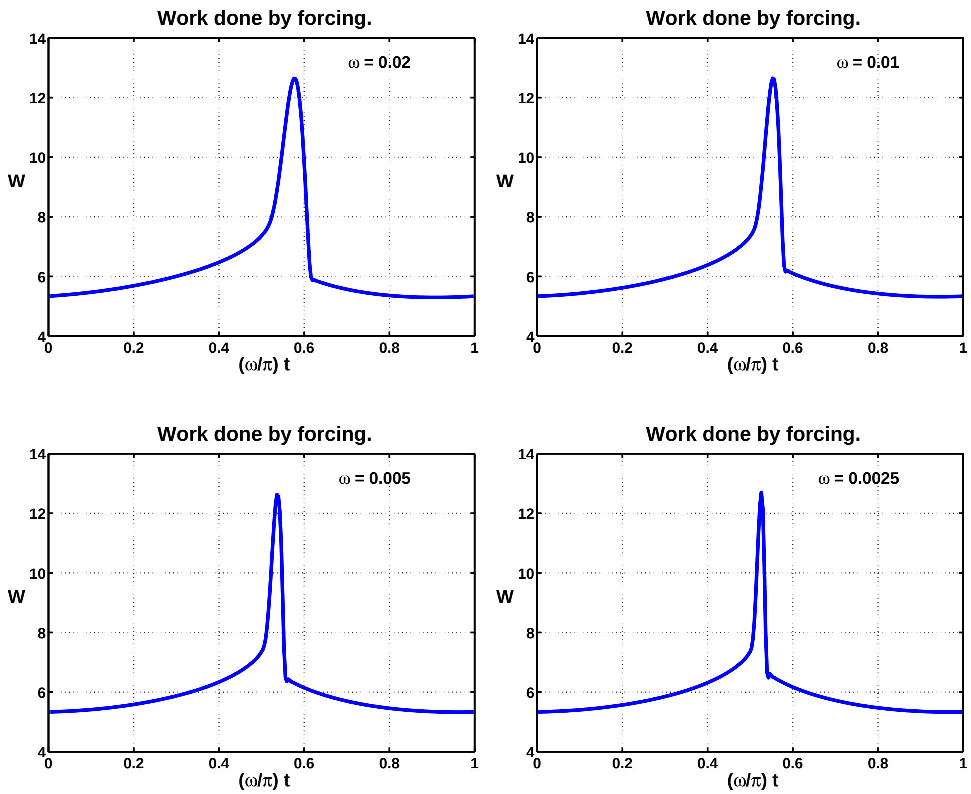

FIGURE **10**. Work done by the forcing $W_f = W_f(t)$ — shown over one period $0 \leq (\Omega/\pi)t \leq 1$ in time — for the $t \to \infty$ asymptotic solution for the equation $u_t + (\frac{1}{2}\,u^2)_x = \sin(x) + 2\sin(2\,(x - \Omega\,t))$. Left to right and top to bottom, plots for the cases $\Omega = 1/50$, $\Omega = 1/100$, $\Omega = 1/200$, and $\Omega = 1/400$ are shown. Note how closely the work done and the dissipation match, as a consequence of the fact that the solution, at all times, is fairly close to a quasi-steady solution.

We can now answer the questions that were posed earlier in this subsection as follows:

(3.3.26)

- **A.** Storms have a typical duration $\Delta t = 1/\sqrt{\Omega}$, evolving on an (intermediate slow) time–scale $T = \sqrt{\Omega}\,t$ (question **1** in (3.3.14).)
- **B.** During a storm both: the work per unit time W_f by the force f, and the energy dissipation rate E_d by the shocks are (roughly) twice as large as their values away from a storm, since the solution has two shocks during a storm, and only one away from it (question **2** in (3.3.14).)
- **C.** Combining the answers in **A** and **B**, we see that the **overall excess dissipation caused by a storm is** $O(1/\sqrt{\Omega})$**.**
- **D.** Storms alter the balance between dissipation and work given by (3.3.16), replacing it by

$$E_d - W_f = O(\sqrt{\Omega})\,.$$

As pointed out in the introduction to this paper, the fact that the effects caused by a storm scale with $\sqrt{\Omega}$, not Ω, may have important consequences when considering the effects of a complex set of near-resonances (something that escapes the scope of this present paper, and which we plan on investigating in future work.)

References

[1] Benney, D.J. and Newell, A.C., "Random wave closures", Stud. Appl. Math, vol. 48 (1969), 29–53.

[2] Benney, D.J., and Saffman, P.G., "Nonlinear interactions of random waves in a dispersive medium", Proc. Roy. Soc. A, vol. 289 (1965), 301.

[3] Cai, D., Majda, A.J., McLaughlin, D.W. and Tabak, E.G., "Spectral Bifurcations in Dispersive Wave Turbulence" PNAS, vol. 96 (1999), 14216–14221.

[4] Godunov, S. K., "A difference scheme for numerical computation of discontinuous solutions of equations of fluid dynamics", Mat. Sb., vol. 47 (1959), 271–306.

[5] Hasselmann, K., "On the nonlinear energy transfer in a gravity wave spectrum. Part I: General theory", J. Fluid Mech., vol. 12 (1962), 481–500.

[6] Majda, A.J., McLaughlin, D.W. and Tabak, E.G, "A one-dimensional model for dispersive wave turbulence", J. Nonlinear Sci., vol. 7 (1997), 9–44.

[7] Milewski, P.A., Tabak, E.G. and Vanden Eijnden, E., "Resonant wave interaction with random forcing and dissipation", to appear in Stud. Appl. Math. (2001).

[8] Rosales, R.R., Tabak, E.G. and Turner, C.V., "Resonant triads involving a nondispersive wave", to appear in Stud. Appl. Math. (2001).

[9] Strang, G., "On the construction and comparison of difference schemes", SIAM J. Num. Anal., vol. 5 (1968), 506–517.

[10] van Leer, B., "Towards the ultimate conservative difference scheme V. A second order sequel to Godunov's method", J. Comput. Phys., vol. 32 (1979), 101–136.

[11] Zakharov, V.E., Lvov, V. and Falkovich, G., **Wave Turbulence**, Springer, New York, 1992.

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