Resonant wave interaction with random forcing and dissipation

Paul A. Milewski

Department of Mathematics

University of Wisconsin, Madison

Esteban G. Tabak and Eric Vanden Eijnden

Courant Institute of Mathematical Sciences

New York University

December 14, 2000

Abstract

A new model for studying energy transfer is introduced. It consists of a "resonant duo" –a resonant quartet where extra symmetries support a reduced subsystem with only two degrees of freedom–, where one mode is forced by white noise and the other one is damped. This system has a single free parameter: the quotient of the damping coefficient to the amplitude of the forcing times the square root of the strength of the nonlinearity. As this parameter varies, a transition takes place from a Gaussian, high-temperature, near equilibrium regime, to one highly intermittent and non Gaussian. Both regimes can be understood in terms of appropriate Fokker-Planck equations.

Keywords: Resonance, intermittency, wave turbulence, invariant measures

1 Introduction

Many systems in Nature receive and dissipate energy in very different scales, behaving like conservative systems in between. Hence energy is permanently transferred through this intermediate, *inertial* range, which often includes many decades of spatial and temporal scales. Typically, the path among scales that this flux of energy adopts is not deterministic and orderly, but rather chaotic and noisy. When this is the case, the flux is said to occur through a *turbulent cascade* of energy. The most famous example of such a cascade lies in the field of isotropic fluid turbulence, widely studied since the pioneering work of Kolmogorov. Many others, however, take place in Nature. The ocean, for instance, displays a very rich set of highly anisotropic cascades, whose variation with scale is thought to be determined by a changing balance between the effects of rotation, stratification, and wave breaking.

For dispersive systems, such as surface and internal waves in the ocean, energy transfer among scales is thought to occur largely through *resonant sets*, typically triads or quartets. A rich body of theory has been developed concerning such systems, under the name of Wave (or Weak) Turbulence. The theory predicts kinetic equations for the evolution of the energy spectrum, and self-similar stationary solutions to these equations ([1], [2], [3], [4].) However, the complexity of the systems under study make many of the hypothesis underlying these results necessarily heuristic. In fact, recent studies of a simple, one-dimensional model for dispersive waves, show an incredibly rich behavior, with a number of [often coexistent] self-similar spectra, some of them apparently inconsistent with the existing theory ([5], [6]).

Our purpose here is to consider even simpler models of energy transfer, involving as few modes as possible, in order to isolate the roots of this variety of regimes. To this end, starting from a relatively general dispersive system, we single out a resonant quartet, with two modes forced by white noise and two damped. Then we invoke a symmetry of the quartet equations to reduce the system even further, to a system that we call a "forced and damped resonant duo", which exhibits, when freed from forces and dissipation, the Hamiltonian structure and conserved quantities that characterize far more complex dispersive systems. This system is so reduced though, that its numerical solution is quite straightforward, and much can be understood even on purely theoretical grounds. Yet the system's behavior is surprisingly rich, with a transition between Gaussian, near-equilibrium behavior, to highly intermittent. This suggests strong analogies to similar, largely unexplained transitions in much more complex systems.

The plan of this paper is the following. After this introduction, in section 2, we describe the main features of a resonant quartet, and justify the introduction of white noise as the most controllable energy source. In section 3, we reduce the system even further, to a forced and damped resonant duo, and derive some expected values and bounds for its statistically steady states. In section 4, we solve the reduced system numerically, and show the existence of two distinct regimes: one Gaussian and close to equilibrium, the other highly intermittent and non Gaussian. In sections 5 and 6, the [approximate] invariant measures for both regimes are explicitly obtained as [near] solution to the system's Fokker-Planck equation. The mechanisms underlying transport of energy in the two regimes are discussed in section 7. Finally, in section 8, we summarize our conclusions, and suggest some further work.

2 A Forced and Damped Resonant Quartet

For concreteness, we shall start with a one-dimensional partial integro-differential equation of the form

$$i\,\frac{\partial\Psi}{\partial t} = \mathcal{L}\Psi + \gamma\,|\Psi|^2\,\Psi \quad \text{plus forcing and dissipation,} \tag{2.1}$$

where $\mathcal{L}$ is an Hermitian linear operator with symbol $\hat{\mathcal{L}} = \omega(k)$. In the inertial range, this system can be written in the Hamiltonian form

$$i\,\frac{\partial\Psi}{\partial t} = \frac{\delta H}{\delta\bar{\Psi}}$$

where

$$H = \int \omega(k)|\hat{\Psi}(k)|^2 \mathrm{dk} + \frac{\gamma}{2}\int |\Psi(\mathrm{x})|^4 \mathrm{dx}.$$

Even this relatively simple one-dimensional system, with $\omega = |k|^{1/2}$ and a slightly more general nonlinearity, has been recently shown to display a very rich and puzzling phenomenology, with a number of self-similar statistically steady states [6]. These states often coexist, occupying disjoint ranges in Fourier space, while sometimes one of them takes over the whole inertial range. The underlying bifurcations appear to depend very delicately on the nature and strength of the forcing and dissipation, on the sign of the parameter γ tuning the nonlinearity, and, since all numerical experiments take place in finite domains, on the size of these domains (or, correspondingly, on the spacing between modes in Fourier space.)

Our goal here is to isolate a simpler subsystem of (2.1) where the issue of energy transfer among modes is more transparent. To this end, we shall consider a single resonant quartet, i.e. a set of four modes $\hat{\Psi}_j$, such that the resonant conditions

$$\begin{aligned} k_1 + k_4 &= k_2 + k_3 \\ \omega_1 + \omega_4 &= \omega_2 + \omega_3 \end{aligned}$$

are satisfied. When this is the case, and those four modes are the only ones excited –at least to leading order– in the initial conditions, the $\hat{\Psi}_j$'s are approximated, in the limit of small amplitudes, by $\hat{\Psi}_j(t) = \epsilon a_j(\tau) e^{-i(\omega_j - 2\epsilon\gamma m)t}$, where $\tau = \epsilon^2 t$, $m = \sum_{j=1}^{4} |a_j|^2$ and the a's obey the *resonant* equations ([7], [8])

$$\begin{aligned} i\frac{\mathrm{da}_1}{\mathrm{d}\tau} &= 2\gamma\, \bar{a}_4 a_2 a_3 - \gamma |a_1|^2 a_1 \\ i\frac{\mathrm{da}_2}{\mathrm{d}\tau} &= 2\gamma\, \bar{a}_3 a_4 a_1 - \gamma |a_2|^2 a_2 \\ i\frac{\mathrm{da}_3}{\mathrm{d}\tau} &= 2\gamma\, \bar{a}_2 a_4 a_1 - \gamma |a_3|^2 a_3 \\ i\frac{\mathrm{da}_4}{\mathrm{d}\tau} &= 2\gamma\, \bar{a}_1 a_2 a_3 - \gamma |a_4|^2 a_4 \end{aligned} \tag{2.2}$$

These equations are also Hamiltonian, with

$$H = 4\gamma\, \rho_1\rho_2\rho_3\rho_4 \cos(\Delta\theta) - \frac{\gamma}{2}\left(\rho_1^4 + \rho_2^4 + \rho_3^4 + \rho_4^4\right). \tag{2.3}$$

Here $a_j = \rho_j e^{\theta_j}$ and $\Delta\theta = \theta_1 + \theta_4 - \theta_2 - \theta_3$. In addition to preserving H, the solutions satisfy the "Manley-Rowe" relations

$$\frac{\mathrm{d}|\mathrm{a}_1|^2}{\mathrm{d}\tau} = \frac{\mathrm{d}|\mathrm{a}_4|^2}{\mathrm{d}\tau} = -\frac{\mathrm{d}|\mathrm{a}_2|^2}{\mathrm{d}\tau} = -\frac{\mathrm{d}|\mathrm{a}_3|^2}{\mathrm{d}\tau}, \tag{2.4}$$

from which conservation of mass $m = \sum_{j=1}^{4} |a_j|^2$, momentum $p = \sum_{j=1}^{4} k_j |a_j|^2$ and linear energy $e = \sum_{j=1}^{4} \omega_j |a_j|^2$, follow. In fact, one can solve these equations analytically. Setting

$$\begin{aligned} |a_1|^2(\tau) &= X(\tau) + c_1, \\ |a_2|^2(\tau) &= X(\tau) + c_2, \\ |a_3|^2(\tau) &= -X(\tau) + c_3, \\ |a_1|^2(\tau) &= -X(\tau) + c_4, \end{aligned}$$

where $c_1 + c_2 - c_3 - c_4 = 0$, one obtains, after some manipulation,

$$X(\tau) = \alpha \cos(\Omega\tau + \phi) + \beta \,,$$

with α, β, Ω and ϕ determined by the initial data.

In order to study energy transfer, one needs to force some of the modes of the system (2.2), and dissipate some others. In order to control the amount of energy going through the system, it is best to force it through white noise, since, when a system is forced deterministically, it is not clear *a priori* how much energy the forces provide. Typically, such systems will reach equilibrium even in the absence of dissipation. A single nonlinear oscillator forced sinusoidally, for instance, will reach a steady state by detuning from the frequency of the forcing. Hence thinking of deterministic forces as permanent energy sources is not necessarily accurate. On the other hand, when either the forces or the system become more irregular (the latter case arising when the system's internal dynamics is chaotic), the forces do behave systematically as an energy source. In the extreme example provided by white noise, the amount of energy input to the system is strictly controllable, as the following theorem shows:

Theorem: Consider a dynamical system of the form

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = F(u,t) + \sigma_i \, \dot{w}_i \,,$$

where $\dot{w}_i$ stands for white-noise. Then the energy of the system, $E = \sum_i |u_i|^2$, evolves in the following, separable way:

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle E \rangle = \langle E_d(u,t) \rangle + E_w \,,$$

where $E_d(u,t)$ is the deterministic rate of energy change, that would take place even without the white noise, and $E_w = \sum \sigma_i^2$ (the brackets in the expressions above represent ensemble averages). Thus, for instance, if the unforced system is conservative, E_d is zero, and the energy grows linearly in time. If, on the other hand, the unforced system includes dissipation, and if a state of statistical equilibrium is achieved, then we must have $\langle E_d(u,t) \rangle + E_w = 0$.

Proof: This theorem is a simple corollary of Ito Calculus. For readers unfamiliar with it, we prove a discrete analogue for the system

$$u_i^{n+1} - u_i^n = F_i(u^n, n) \, \Delta t + \sigma_i \, w_i^n \sqrt{\Delta t} \,, \qquad (2.5)$$

where $\langle w_i^n \rangle = 0$, $\left\langle w_i^n \overline{w_j^m} \right\rangle = \delta_i^j \delta_n^m$, and the total energy is defined to be

$$E^n = \sum_i |u_i^n|^2 \,. \tag{2.6}$$

From (2.5) and (2.6), we obtain

$$\begin{aligned}
\left\langle E^{n+1} - E^n \right\rangle &= \mathrm{Re} \left\langle \sum_i \left(u_i^{n+1} - u_i^n \right) \overline{\left(u_i^{n+1} + u_i^n \right)} \right\rangle \\
&= \mathrm{Re} \left\langle \sum_i \left(F_i(u^n, n)\, \Delta t + \sigma_i\, w_i^n \sqrt{\Delta t} \right) \overline{\left(2\, u_i^n + F_i(u^n, n)\, \Delta t + \sigma_i\, w_i^n \sqrt{\Delta t} \right)} \right\rangle \\
&= \sum_i \left(\mathrm{Re} \left(\left\langle F_i(u^n, n) \overline{(2\, u_i^n + \Delta t F_i(u^n, n))} \right\rangle \right) + \sigma_i^2 \right) \Delta t \\
&= (\langle E_d(u,t) \rangle + E_w) \Delta t \,,
\end{aligned}$$

which concludes the proof. For instance, if $F_i(u^n, n)$ consists of an energy preserving part plus dissipative terms of the form $-\nu_i\, u_i^n$, then

$$\langle E^{n+1} - E^n \rangle = \sum_i \left(\sigma_i^2 - 2\nu_i |u_i|^2 \right) \, \Delta t + \sum_i \nu_i^2 |u_i|^2 \, (\Delta t)^2$$

and, in the continuous limit,

$$\frac{\mathrm{d} \langle E \rangle}{\mathrm{d}t} = \sum_i \left(\sigma_i^2 - 2\nu_i |u_i|^2 \right) \,.$$

Thus, for a single, nonlinear damped oscillator forced by white noise,

$$i \frac{\mathrm{d}\psi}{\mathrm{d}t} = \left(\omega + F(|\psi|) - i\, \nu \right) \psi + \sigma\, \dot{w} \,, \tag{2.7}$$

(F real), if a statistically steady state is reached, it necessarily satisfies

$$\left\langle |\psi|^2 \right\rangle = \frac{\sigma^2}{2\nu}.$$

Based on these considerations, we shall study the following generalization of (2.2):

$$\begin{aligned}
i \frac{\mathrm{d}a_1}{\mathrm{d}t} &= 2\gamma\, \bar{a}_4 a_2 a_3 - \gamma |a_1|^2 a_1 + \sigma\, \dot{w}_1(t), \\
i \frac{\mathrm{d}a_2}{\mathrm{d}t} &= 2\gamma\, \bar{a}_3 a_4 a_1 - \gamma |a_2|^2 a_2 - i\nu\, a_2, \\
i \frac{\mathrm{d}a_3}{\mathrm{d}t} &= 2\gamma\, \bar{a}_2 a_4 a_1 - \gamma |a_3|^2 a_3 - i\nu\, a_3, \\
i \frac{\mathrm{d}a_4}{\mathrm{d}t} &= 2\gamma\, \bar{a}_1 a_2 a_3 - \gamma |a_4|^2 a_4 + \sigma\, \dot{w}_4(t) \,,
\end{aligned} \tag{2.8}$$

where $\dot{w}_1(t)$ and $\dot{w}_4(t)$ represent white noise. The reasoning behind the new terms added to (2.2) is the following: Once one has decided to force one of the modes –say a_1 for concreteness– with white noise, then a_4 needs to be forced also –and with white noise of the same amplitude– if there is to be any hope for the system to reach statistical equilibrium (for otherwise the Manley–Rowe relations (2.4), combined with the theorem above applied to the equations for a_1 and a_4 separately, imply that $\langle |a_1|^2 - |a_4|^2 \rangle$ will necessarily diverge.) Similarly, once a_2 is damped, so should a_3. Here it is not crucial that the two damping coefficients be equal, but there does not seem to be much point in breaking the system's symmetry by deciding otherwise (in fact, we shall use this symmetry below to reduce our model even further.)

3 Reduction to a Forced and Damped Duo

The symmetries between a_1 and a_4 and between a_2 and a_3 in the system (2.8) suggest a further reduction to two variables, by using a single random process for $w_1(t)$ and $w_4(t)$, and considering initial data such that $a_1 = a_4$ and $a_2 = a_3$. After renaming the variables, one ends up with the system for a *resonant duo*:

$$i\frac{\mathrm{d}a_1}{\mathrm{d}t} = 2\gamma\, a_2^2 \bar{a}_1 - \gamma |a_1|^2 a_1 + \sigma\, \dot{w}(t), \tag{3.1}$$

$$i\frac{\mathrm{d}a_2}{\mathrm{d}t} = 2\gamma\, a_1^2 \bar{a}_2 - \gamma |a_2|^2 a_2 - i\nu\, a_2\,. \tag{3.2}$$

A disclaimer seems appropriate here: resonant quartet equations such as (2.8) arise naturally as asymptotic reductions of larger systems, when only a handful of modes is initially excited. The resonant duo proposed here, on the other hand, represents only a particularly symmetric instance of a resonant quartet. It should not be considered as a reduced model for the interaction among only two modes in a larger system, since the implication would be that the two modes have the same wavenumber and frequency. This is not totally unthinkable, since many systems are indexed by wavenumber (with corresponding frequency) and something else, but it is certainly not the case of models such as (2.1), where each wavenumber points to a single degree of freedom.

When both σ and ν are set to zero, the system in (3.1, 3.2) is Hamiltonian, with

$$H = \gamma \left(a_2^2 \bar{a}_1^2 + a_1^2 \bar{a}_2^2 \right) - \frac{\gamma}{2} \left(|a_1|^4 + |a_2|^4 \right) ,$$

and it has exact solutions of the form

$$\begin{aligned} |a_1|^2 &= \alpha\cos(\Omega t+\phi)+\beta_1 \\ |a_2|^2 &= -\alpha\cos(\Omega t+\phi)+\beta_2 \end{aligned}$$

Notice that there is a single non-dimensional parameter in (3.1, 3.2):

$$D = \frac{\nu}{\sigma\sqrt{\gamma}}. \tag{3.3}$$

Thus all but one of γ, σ, or ν can be made equal to one by a suitable rescaling of time and amplitudes, and D serves as a single control parameter for the system.

In the remaining of this paper, we shall study the properties of the statistically steady solutions to (3.1, 3.2). If such state is achievable, the theorem in the previous section, applied to the full system, implies that

$$\left\langle |a_2|^2 \right\rangle = \frac{\sigma^2}{2\nu}, \tag{3.4}$$

which states that the system's energy input σ^2 needs to be fully dissipated by the damping of the second oscillator. The same theorem applied to either of the two equations alone, on the other hand, yields

$$-4\gamma \left\langle |a_1|^2|a_2|^2 \sin(2\Delta\theta) \right\rangle = \sigma^2, \tag{3.5}$$

where $\Delta\theta = \theta_2 - \theta_1$, and we are writing $a_j = \rho_j\, e^{i\theta_j}$. The left hand side of this equation represents nonlinear energy transfer among the two modes, which has to equal the total energy input from white noise. To obtain a lower bound for $|a_1|$, one derives, from equation (3.2), the identity

$$\frac{\mathrm{d}}{\mathrm{dt}} \log\left(|a_2|^2\right) = -4\gamma\, |a_1|^2 \sin(2\Delta\theta) - 2\nu$$

which, after taking averages and looking for a statistically steady state, yields

$$\left\langle |a_1|^2 \right\rangle \geq \frac{\nu}{2\gamma}. \tag{3.6}$$

Another conjectured lower bound for a_1, of a more "thermodynamical" nature, states that

$$\left\langle |a_1|^2 \right\rangle \geq \left\langle |a_2|^2 \right\rangle = \frac{\sigma^2}{2\nu}, \tag{3.7}$$

since a situation in which forcing and dissipation on an otherwise symmetric duo should yield a higher mean square amplitude for the dissipated mode than for the forced one would contradict the second principle of thermodynamics (energy would be transferred "up" from the less to the more excited state).

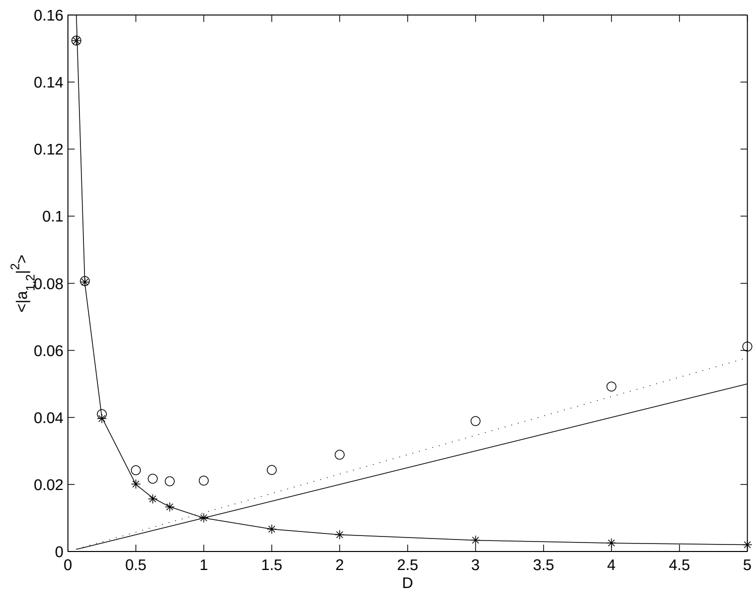

Figure 1: $\langle|a_1|^2\rangle$ (stars) and $\langle|a_2|^2\rangle$ (circles) as functions of $D = \nu/\sigma\sqrt{\gamma}$. Also plotted are the two lower bounds (3.6) and (3.7) in solid line, and $\langle|a_1|^2\rangle = \nu/\sqrt{3}\gamma$ (dotted). The results are averages over time and about 800 realizations of white noise from fixed initial data $a_1 = 0.1 + 0.2i$, $a_2 = 0.15 - 0.1i$, with time averaging from $t = 2500$ to $t = 5000$.

4 Numerical Simulations

Simulating numerically the system in (3.1, 3.2) is a rather straightforward task. Since our interest here lies in energy transfer over long time intervals, a simplectic procedure appears appropriate. We have adopted a simplectic second order (implicit) Runge–Kutta for the deterministic part of the system, and alternated it with an explicit addition of white noise. Thus the algorithm becomes

$$\begin{aligned} a_1^* &= {a_1}^n - i\,\Delta t\,\left(2\gamma\,\left(a_2^h\right)^2\bar{a^h}_1 - \gamma\,|a_1^h|^2 a_1^h\right) \\ a_2^* &= {a_2}^n - i\,\Delta t\,\left(2\gamma\,\left(a_1^h\right)^2\bar{a^h}_2 - \gamma\,|a_2^h|^2 a_2^h - i\,\nu\,a_2^h\right) \\ a_1^{n+1} &= a_1^* + \sqrt{\Delta t}\,\sigma\,w^n \\ a_2^{n+1} &= a_2^*\,, \end{aligned} \tag{4.1}$$

where $a_j^h = \frac{1}{2}\left(a_j^n + a_j^*\right)$ and w^n is a random complex variable drawn from a Gaussian distribution with variance one.

Figure 1 shows the results of a series of experiments, in which γ has been kept fixed at $\gamma = 1$, the amplitude of the white noise at $\sigma = 0.02$, and ν has been varied from $\nu = 0.00125$ to $\nu = 0.1$,

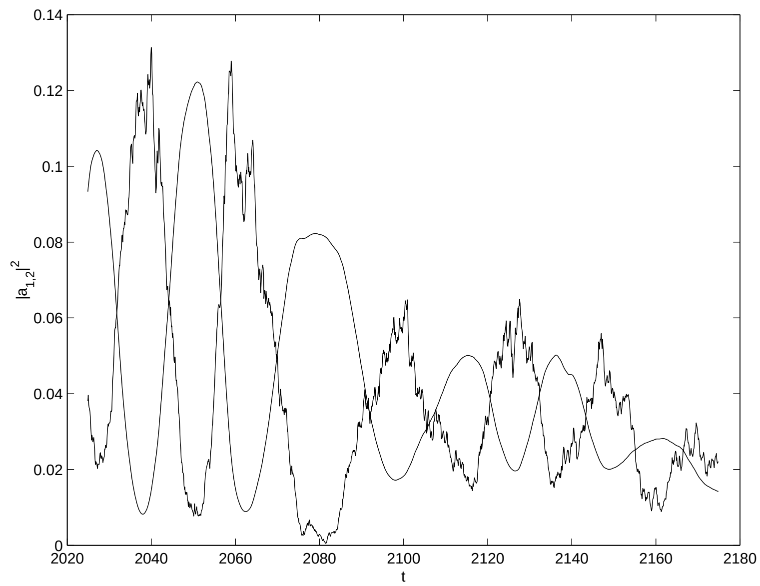

Figure 2: An individual realization of the resonant duo, with $D = 0.25$. The two modes a_1 and a_2 oscillate around each other much as in the unforced, undissipated system, but with less orderly paths.

which corresponds to the non-dimensional parameter D in (3.3) ranging from $D = 0.0625$ to $D = 5$. We have typically run 800 realizations of white noise from fixed initial data ($a_1 = 0.1 + 0.2i$, $a_2 = 0.15 - 0.1i$) up to $t = 5000$, and computed time (as well as ensemble) averages from $t = 2500$ (longer times and more realizations where needed for the very large and very small values of ν, for reasons that will become clear below), with $\Delta t = 0.0025$. We have plotted $\langle |a_1|^2 \rangle$ with stars, $\langle |a_2|^2 \rangle$ with circles, and the two lower bounds for $\langle |a_1|^2 \rangle$ from (3.6) and (3.7) in solid line (the second lower bound agrees, of course, with the expected value of $|a_2|^2$ from (3.4).) In addition, there is a dotted line, corresponding to $\langle |a_1|^2 \rangle = \nu/\sqrt{3}\gamma$, that will be explained below.

We can see the nearly perfect agreement of the numerical results for $\langle |a_2|^2 \rangle$ with their exact value (3.4), and the sharp nature of the two bounds (3.6) and (3.7), which all but define the dependence of $\langle |a_1|^2 \rangle$ on the parameters γ, σ, and ν. There are clearly two different regimes, depending on which lower bound is enforced. For small values of D, $\langle |a_1|^2 \rangle \approx \langle |a_2|^2 \rangle$, and we are close to thermodynamical equilibrium. For large values of D, on the other hand, $\langle |a_1|^2 \rangle \gg \langle |a_2|^2 \rangle$, and the sharp nature of the lower bound (3.7) suggests that the relative phase of the two oscillators is locked near $\Delta\theta = -\pi/4$ whenever a_2 is active. In fact, the dotted line, which fits very well the asymptotic

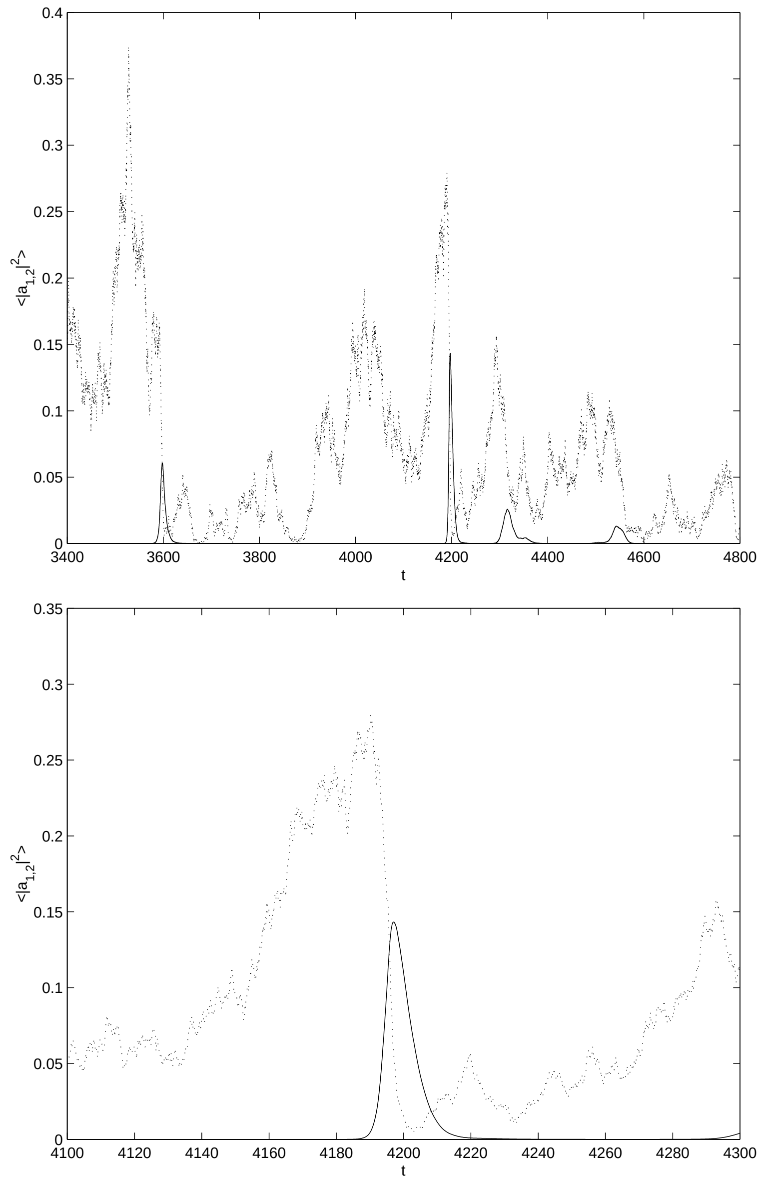

Figure 3: An individual realization of the resonant duo, with $D = 5$. $|a_2|$ is essentially zero most of the time, with intermittent, brief outbursts of energy. a) A relatively long interval, with a few transfer events. b) Detail of one the events.

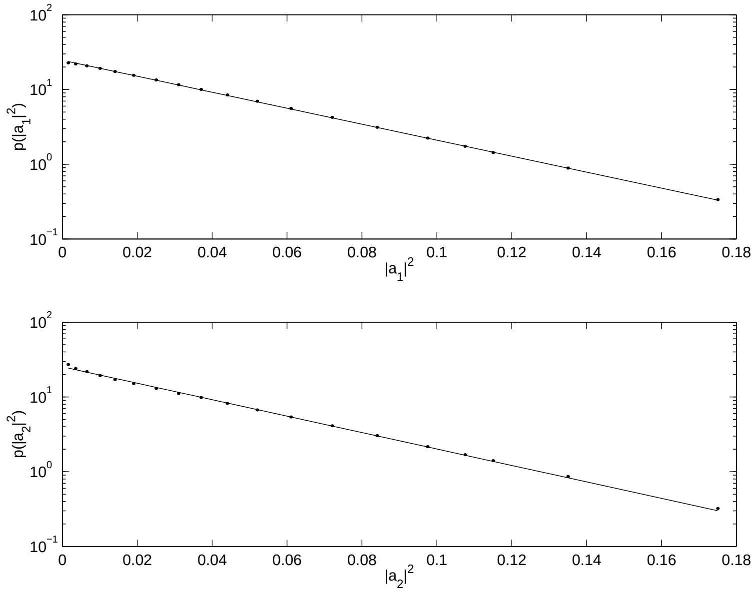

Figure 4: Numerical invariant measure for $D = 0.25$, and Gaussian measures with the same variance. The distributions of both a_1 and a_2 are indistinguishable from [complex] Gaussian, and their variances are equal.

behavior of $\langle |a_1|^2 \rangle$, corresponds to a value $\Delta\theta = -\pi/6$, which will be shown below to arise from simple theoretical considerations.

Figures 2 and 3 display individual realizations of the two regimes, corresponding to $D = 0.25$ and $D = 5$. In the former, the two modes oscillate around each other much as in the unforced, undissipated system, but with more noisy paths. In the latter, $|a_2|$ is essentially zero most of the time, with intermittent, brief outbursts of energy.

Figures 4 and 5 show the (numerical) invariant measures for the same two values of D, contrasted with Gaussian measures with the same variance. We see that, for $D = 0.25$, the distributions of both a_1 and a_2 are indistinguishable from Gaussian, while for $D = 5$, a_1 is still Gaussian, but a_2 is fundamentally different, with a sharp peak at $|a_2| = 0$ and a very long tail.

These behaviors will be fully accounted for in sections 5, 6, and 7.

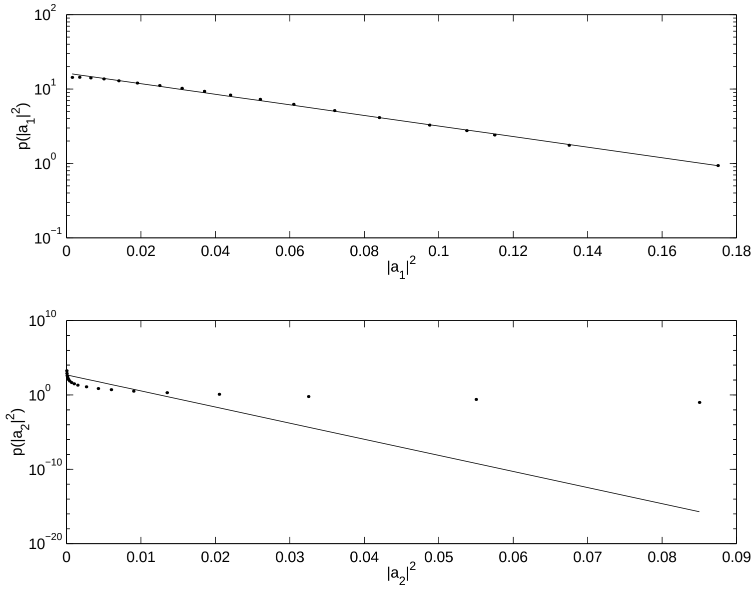

Figure 5: Numerical invariant measure for $D = 5$, contrasted with Gaussian measures with the same variance. a_1 is still Gaussian, but a_2 is fundamentally different, with a sharp peak at $|a_2| = 0$, and a very long tail.

5 The Near-Equilibrium, High-Temperature Regime

The system (3.1, 3.2) has four real degrees of freedom, which can be taken to be the amplitudes and phases of the two modes. However, only the difference between the two phases, $\Delta\theta = \theta_2 - \theta_1$ enters the dynamics of the amplitudes, so the system can be further reduced to the three-dimensional one

$$\frac{\mathrm{d}\rho_1}{\mathrm{d}t} = 2\gamma\,\rho_2^2\rho_1 \sin(2\Delta\theta) + \frac{\sigma^2}{4\rho_1} + \frac{\sigma}{\sqrt{2}}\,\dot{w}_1 \tag{5.1}$$

$$\frac{\mathrm{d}\rho_2}{\mathrm{d}t} = -2\gamma\,\rho_1^2\rho_2 \sin(2\Delta\theta) - \nu\rho_2 \tag{5.2}$$

$$\frac{\mathrm{d}(\Delta\theta)}{\mathrm{d}t} = \gamma\left(\rho_2^2 - \rho_1^2\right)(1 + 2\cos(2\Delta\theta)) + \frac{\sigma}{\sqrt{2}\rho_1}\,\dot{w}_2\,, \tag{5.3}$$

where $\dot{w}_1$ and $\dot{w}_2$ stand for two independent real white-noises.

The corresponding Fokker-Planck operator is

$$L = L_H + L_F \tag{5.4}$$

where

$$
\begin{aligned}
L_H &= 2\gamma\,\rho_1\rho_2 \sin(2\Delta\theta)\left(\rho_2\frac{\partial}{\partial\rho_1} - \rho_1\frac{\partial}{\partial\rho_2}\right) \\
&\quad +\gamma\left(\rho_2^2-\rho_1^2\right)(1+2\cos(2\Delta\theta))\frac{\partial}{\partial\Delta\theta}
\end{aligned} \tag{5.5}
$$

represents the Hamiltonian component of the evolution, and

$$
L_F = \frac{\sigma^2}{4\rho_1}\frac{\partial}{\partial\rho_1} - \nu\rho_2\frac{\partial}{\partial\rho_2} + \frac{\sigma^2}{4}\frac{\partial^2}{\partial\rho_1^2} + \frac{\sigma^2}{4\rho_1^2}\frac{\partial^2}{\partial(\Delta\theta)^2} \tag{5.6}
$$

represents the effects of forcing and damping. An invariant measure of the evolution $f(\rho_1,\rho_2,\Delta\theta)$ solves

$$
L^* f = 0\,, \tag{5.7}
$$

where the operator L^* is the adjoint of L.

Let us consider first the regime $D \ll 1$, for which the numerical experiments suggest a near Gaussian invariant measure with $\langle\rho_1^2\rangle \approx \langle\rho_2^2\rangle = \sigma^2/2\nu$, that is,

$$
f(\rho_1,\rho_2,\Delta\theta) = C\rho_1\rho_2 e^{-2\nu\left(\rho_1^2+\rho_2^2\right)/\sigma^2}. \tag{5.8}
$$

We now show that the density in (5.8) is indeed the leading order term of an expansion in D. Upon rescaling

$$
\rho_1 \to \frac{\sigma}{\sqrt{\nu}}\,\tilde\rho_1, \qquad \rho_2 \to \frac{\sigma}{\sqrt{\nu}}\,\tilde\rho_2, \qquad t \to \tilde t/\nu,
$$

and dropping the tildes, the equation in (5.7) reduces to

$$
\left(\frac{1}{D^2}\bar L_H^* + \bar L_F^*\right) f = 0\,, \tag{5.9}
$$

where the operators $\bar L_H^*$ and $\bar L_F^*$ are L_H^* and L_F^* with ν, γ, σ set to one. We look for a solution of (5.9) of the form

$$
f = f_0 + D^2 f_1 + O(D^4). \tag{5.10}
$$

Inserting this expansion into (5.9) and equating coefficients of equal powers of D, we obtain

$$
\bar L_H^* f_0 = 0, \tag{5.11}
$$

$$
\bar L_H^* f_1 = -\bar L_F^* f_0. \tag{5.12}
$$

(5.11) implies that f_0 belongs to the null-space of $\bar{L}_H^*$, $f_0 \in \text{Ker}\, \bar{L}_H^*$. Since the Hamiltonian dynamics conserves both the Hamiltonian and the energy

$$H = -\frac{1}{2}\left(\rho_1^4 + \rho_2^4\right) + 2\rho_1^2\rho_2^2\cos(2\Delta\theta), \qquad E = \rho_1^2 + \rho_2^2,$$

the null-space of $\bar{L}_H^*$ is spanned by functions of the type $\rho_1\rho_2$ times an arbitray function of E and H, i.e. (5.11) yields

$$f_0 = \rho_1\rho_2\, g\,(E, H)\,, \tag{5.13}$$

where $g(\cdot)$ is arbitrary except for the boundary conditions for (5.11),

$$\lim_{\rho_1^2+\rho_2^2\to\infty} f_0 = 0, \qquad \lim_{\rho_1\to 0}\rho_1 f_0 = 0, \qquad \lim_{\rho_2\to 0}\rho_2 f_0 = 0.$$

We determine $g(\cdot)$ from the solvability condition for (5.12),

$$\bar{L}_F^* f_0 \in \text{Ran}\, \bar{L}_H^* = (\text{Ker}\, \bar{L}_H)^-. \tag{5.14}$$

By an argument similar to the one above, it follows that the null-space of $\bar{L}_H$ is spanned by the functions of the form $h(E, H)$, where $h(\cdot)$ is arbitrary. Thus, the solvability condition in (5.14) is given by

$$0 = \int_0^{2\pi}\int_{\mathbb{R}_+^2} h(E,H)\bar{L}_F^*\left(\rho_1\rho_2 g\,(E,H)\right)\mathrm{d}\rho_1\mathrm{d}\rho_2\mathrm{d}(\Delta\theta).$$

We claim that this integral equation can be transformed into a partial differential equation in E, H for $g(E, H)$. To see this, change the integration variables in order to integrate first on the regions where E and H are constant, then on E, H. Since both $h(E, H)$ and $g(E, H)$ depend only on E and H, the first integral can be performed explicitly, and the resulting integral equation can be written as

$$0 = \int_0^{\infty}\int_{-3E^2/4}^{E^2/4} J(E,H)h(E,H)L_1 g(E,H)\mathrm{d}H\mathrm{d}E. \tag{5.15}$$

where J is some Jacobian and L_1 is an operator in E, H whose explicit form must be obtained by integrating $\bar{L}_F^*\left(\rho_1\rho_2\ \cdot\right)$ at E, H constant. Since $h(E, H)$ is arbitrary in (5.15), the factor $L_1 g(E, H)$ must be identically zero, which gives indeed a partial differential equation in E, H for $g(E, H)$. The full operator L_1 is rather complicated and we shall not write it here explicitly, since we only need its restriction on functions depending only on E. Indeed, under the consistent assumption that g depends on E alone, the integral in (5.15) reduces to

$$0 = \int_0^{\infty}\bar{h}(E)\Big(4Eg(E) + 2(E+E^2)g'(E) + E^2 g''(E)\Big)\mathrm{d}E, \tag{5.16}$$

where

$$\bar{h}(E) = \int_{-E^2/2}^{E^2/2} h(E,H)J(E,H)dH,$$

Since $\bar{h}$ is arbitrary, (5.16) is equivalent to the differential equation

$$4Eg(E) + 2(E+E^2)g'(E) + E^2 g''(E).$$

whose only bounded solution is $g(E) = Ce^{-2E}$. It follows that

$$f_0 = C\rho_1\rho_2 e^{-2\left(\rho_1^2+\rho_2^2\right)}, \tag{5.17}$$

which in the original dimensional variables yields (5.8).

6 The Intermittent Regime: An Exactly Solvable Model

From the numerical experiments, we know that, when $D \gg 1$, we reach a highly intermittent regime, with $\rho_1 \gg \rho_2$ on the average and a nearly locked phase $\Delta\theta$. Notice that this is consistent with equation (5.3), when it is dominated by the deterministic part. The locked phase then needs to satisfy

$$1 + 2\cos(2\Delta\theta) = 0\,,$$

so

$$\sin(2\Delta\theta) = \pm\frac{\sqrt{3}}{2}\,.$$

Moreover, only the phase yielding the sine with the minus sign is stable under the [deterministic] dynamics of (5.3). This suggests replacing the system (5.1, 5.2, 5.3) by the reduced

$$\frac{\mathrm{d}\rho_1}{\mathrm{d}t} = -2\alpha\gamma\rho_2^2\rho_1 + \frac{\sigma^2}{4\rho_1} + \frac{\sigma}{\sqrt{2}}\dot{w}_1 \tag{6.1}$$

$$\frac{\mathrm{d}\rho_2}{\mathrm{d}t} = 2\alpha\gamma\rho_1^2\rho_2 - \nu\rho_2\,. \tag{6.2}$$

Here the constant α, with $0 \le \alpha \le \sqrt{3}/2$, measures the effectiveness of phase locking, and should approach the value $\sqrt{3}/2$ as D goes to infinity. Equivalently, this amounts to saying that, in the limit of large D, the invariant measure for the original system (5.1, 5.2, 5.3) satisfies

$$f(\rho_1,\rho_2,\Delta\theta) \approx \bar{f}(\rho_1,\rho_2)\delta(\Delta\theta + \pi/6), \tag{6.3}$$

where $\delta(\cdot)$ is the delta function and $\bar{f}(\rho_1, \rho_2)$ is the invariant measure for the approximate system in (6.1, 6.2).

For this model, it is still true that, if a statistically steady state is achieved, it must have

$$\left\langle \rho_2^2 \right\rangle \;=\; \frac{\sigma^2}{2\nu}\,. \tag{6.4}$$

Moreover, from

$$\frac{\mathrm{d}}{\mathrm{dt}} \log\left(\rho_2^2\right) = -4\alpha\gamma\rho_1^2 - 2\nu\,,$$

one obtains, instead of a lower bound for ρ_1 as in (3.6), the sharper result that

$$\left\langle \rho_1^2 \right\rangle \;=\; \frac{\nu}{2\alpha\gamma}\,. \tag{6.5}$$

Finally, looking at the energy transfer among modes, one obtains

$$-4\alpha\gamma \left\langle \rho_1^2 \rho_2^2 \right\rangle \;=\; \sigma^2\,, \tag{6.6}$$

which, together with (6.4) and (6.5), implies that

$$\left\langle \rho_1^2 \rho_2^2 \right\rangle \;=\; \left\langle \rho_1^2 \right\rangle \left\langle \rho_2^2 \right\rangle\,, \tag{6.7}$$

strongly suggesting that ρ_1 and ρ_2 are independent random variables.

In fact, one can find an exact invariant measure for the system in (6.1, 6.2) satisfying these properties. The Fokker-Planck operator for the new system in (6.1, 6.2) is given by

$$L = 2\alpha\gamma\,\rho_1\rho_2\left(\rho_2\frac{\partial}{\partial\rho_1} - \rho_1\frac{\partial}{\partial\rho_2}\right) + \frac{\sigma^2}{4\rho_1}\frac{\partial}{\partial\rho_1} - \nu\rho_2\frac{\partial}{\partial\rho_2} + \frac{\sigma^2}{4}\frac{\partial^2}{\partial\rho_1^2}, \tag{6.8}$$

which, consistently with (6.7), has a separable invariant measure of the form

$$\bar{f}(\rho_1, \rho_2) = C\rho_1\rho_2^{-1+2\alpha\gamma\sigma^2/\nu^2} e^{-2\alpha\gamma\left(\rho_1^2+\rho_2^2\right)/\nu}. \tag{6.9}$$

The higher temperature mode, ρ_1, is Gaussian, as observed in the numerics. For $\rho_2 \ll \sqrt{\nu/2\alpha\gamma}$ the distribution for the lower temperature mode, ρ_2, is essentially a power law with exponent $\beta = -1 + 2\alpha\gamma\sigma^2/\nu^2 = -1 + 2\alpha/D^2$, which approaches -1 as D goes to infinity. This is consistent with the observed sharp peak at $\rho_2 = 0$ and very long tail, since $\nu/2\alpha\gamma \gg \langle\rho_2^2\rangle = \sigma^2/2\nu$ for large D. In fact, this distribution agrees nearly to perfection with the observed one, as the plots in Figure

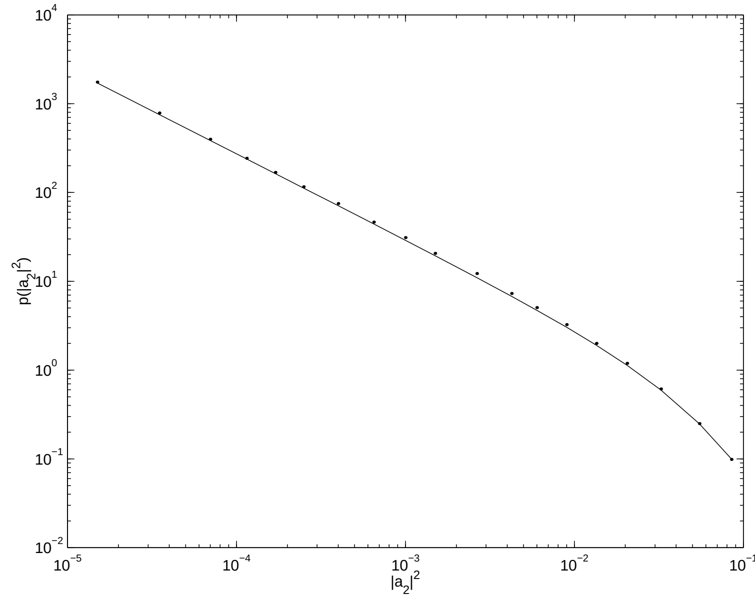

Figure 6: Numerical invariant measure for $|a_2|$, $D = 5$, compared with the theoretical prediction in (6.9). The value of $\alpha \approx 0.82$ was drawn from the observed mean of ρ_1, and then used to build the distribution for ρ_2.

6 show for $D = 5$. Here the value of $\alpha \approx 0.82$ was drawn from the observed mean of ρ_1, and then used to build the distribution for ρ_2.

The intermittent behavior of individual realizations is also easier to understand in the simplified model in (6.1, 6.2). Notice that, whenever ρ_1^2 is smaller than $\nu/2\alpha\gamma$, equation (6.2) predicts damping of ρ_2. Hence the situation is the following: ρ_2 is pinned near zero most of the time by the strong damping, while ρ_1 undergoes free Brownian motion. However, as soon as ρ_1 crosses the threshold of instability $\rho_1^2 = \nu/2\alpha\gamma$, ρ_2 grows explosively, and soon starts to drag ρ_1 down, back to the stable regime. White noise is not important during this bursts of energy transfer, so we can set $\sigma = 0$. With this, the system in (6.1, 6.2) is exactly solvable; its solution for the initial condition $\rho_1(0) = r_1$, $\rho_2(0) = r_2$ is given implicitly by

$$\begin{aligned} t &= \frac{1}{2\alpha\gamma}\int_{\rho_1}^{r_1} \frac{\mathrm{d}z}{z\,(r_2^2 + r_1^2 - z^2 + (\nu/\alpha\gamma)\ln(z/r_1))}, \\ \rho_2^2 &= r_2^2 + r_1^2 - \rho_1^2 + \frac{\nu}{\alpha\gamma}\ln(\rho_1/r_1), \end{aligned} \tag{6.10}$$

A plot comparing this solution with the energy outburst of figure 3b is shown in Figure 7 (we picked the initial values for (6.10) to fit ρ_1 and ρ_2 at time t=4185). The very close agreement between

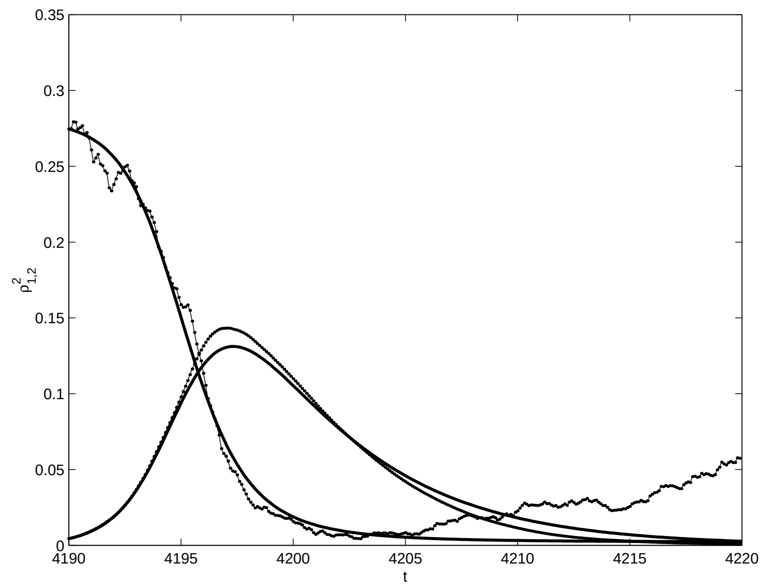

Figure 7: Comparison of the exact solution (6.10) with the energy outburst of figure 3b. The initial values for (6.10) were picked to fit ρ_1 and ρ_2 at time t=4185.

the two curves leaves little doubt that the scenario just described applies to the original equations in (5.1, 5.2, 5.3) as well. Notice incidentally that the threshold of instability $\rho_1^2 = \nu/2\alpha\gamma$ yields the mean value $\langle\rho_1^2\rangle$ in the intermittent regime; hence $\langle\rho_1^2\rangle$ settles at a value such that $\rho_2 = 0$ is neutrally stable.

7 Discussion

The behaviors observed in sections 5 and 6 can be explained by comparing various time-scales or, equivalently, various rates associated with transport of energy in the system. The first one is ν, giving the rate at which energy is removed from the system by means of dissipation. We shall compare ν with

$$\nu^\star = 4\gamma \left\langle |\rho_1^2\rho_2^2 \sin(2\Delta\theta)| \right\rangle / \left\langle \rho_1^2 + \rho_2^2 \right\rangle . \tag{7.1}$$

Since

$$\begin{aligned} \frac{\mathrm{d}\rho_1^2}{dt} &= 4\gamma\rho_1^2\rho_2^2 \sin(2\Delta\theta) + \sigma^2 + \sqrt{2}\sigma\rho_1\dot{w}_1, \\ \frac{\mathrm{d}\rho_2^2}{dt} &= -4\gamma\rho_1^2\rho_2^2 \sin(2\Delta\theta) - 2\nu\rho_2^2, \end{aligned}$$

$\nu^\star$ gives a measure of the [averaged] rate at which energy is transported between the modes by means of the Hamiltonian part of the dynamics, normalized by the total energy of the system.

In the Gaussian regime, we have $\langle\rho_1^2\rangle \approx \langle\rho_2^2\rangle = \sigma^2/2\nu$ and

$$4\gamma\langle|\rho_1^2\rho_2^2\sin(2\Delta\theta)|\rangle \approx 4\gamma\langle\rho_1^2\rangle\langle\rho_2^2\rangle\langle|\sin(2\Delta\theta)|\rangle \approx \frac{2\gamma\sigma^4}{\pi\nu^2},$$

which, since $D \ll 1$, implies

$$\nu^\star \approx \frac{2\gamma\sigma^2}{\pi\nu} \gg \nu.$$

It follows that any blob of energy fed into the system on either of the modes will bounce back and forth between them many times by means of oscillations before being dissipated. In other words, the system is able to "thermalize" the modes, $\langle\rho_1^2\rangle \approx \langle\rho_2^2\rangle$, and the actual temperature is fixed by the amount of forcing and damping applied to the system. In fact, consistently with this picture, the Gaussian measure in (5.8) might also have been predicted by a rough application of equilibrium statistical mechanics as the least biased measure given the information in the conserved quantity, $\rho_1^2 + \rho_2^2$ [9].

In the intermittent regime, one observes

$$\langle\rho_2^2\rangle = \sigma^2/2\nu \ll \langle\rho_1^2\rangle \approx \nu/\sqrt{3}\gamma,$$

$$4\gamma\langle|\rho_1^2\rho_2^2\sin(2\Delta\theta)|\rangle \approx \sigma^2,$$

and $D \gg 1$, so

$$\nu^\star \approx \frac{\sqrt{3}\gamma\sigma^2}{\nu} \ll \nu.$$

In words, any amount of energy transferred from mode a_1 to mode a_2 is dissipated there almost immediately, with no time to backscatter to mode a_1. Notice that, unlike the original system in (5.1,5.2), the approximate system in (6.1,6.2) possesses a Maxwell demon which strictly forbids transfer of energy from mode a_2 to mode a_1. In view of the ordering between the ν's occurring as $D \gg 1$, this leads to no practical difficulty in the regime where the system in (6.1,6.2) is relevant. However, an interesting consequence of the presence of the Maxwell demon is that the system in (6.1,6.2) predicts $\langle\rho_2^2\rangle \gg \langle\rho_1^2\rangle$ in the range $D \ll 1$ (see (6.4) and (6.5)). In other words, the approximate system never thermalizes and the energy keeps flowing the same way from a_1 to a_2 even if a_1 becomes the lower temperature mode.

8 Conclusions and Further Extensions

A system with only two free modes is, presumably, the simplest model for energy transfer. Here we have introduced one such model, with a number of pleasant properties:

- The unforced, undissipated system has the Hamiltonian structure and conserved quantities typical of more general dispersive systems.
- The forcing, in the form of white noise, permits a strict control of the energy flux through the system.
- There is a single free parameter, the quotient of the damping coefficient to the amplitude of the forcing times the square root of the strength of the nonlinearity.
- Many properties of the statistically steady states of the system can be easily estimated. Yet there is one important output –the mean amplitude of the forced mode– which does not fall off a preliminary analysis.

A numerical study of the system shows a clear transition between Gaussian, near equilibrium behavior at high temperatures, to highly intermittent, non Gaussian behavior when the rate of dissipation is high. Both behaviors can be understood even quantitatively from an analysis of the Fokker-Planck equation of the system. We think that this "solvable" model may shed light on similar transitions to intermittency taking place in many [far more complex] turbulent scenarios.

There are two obvious extensions that we plan to investigate in the near future: the inclusion of an intermediate, *inertial* range (modes that are neither forced nor damped), and the effects of non-resonant interactions. Both extensions are necessary if one hopes to fully understand the rich phenomenology of wave turbulence.

Acknowledgments

The work of P. A. Milewski was partially supported by NSF grant DMS-9401405; the work of E. G. Tabak was partially supported by NSF grant DMS-9501073; and the work of E. Vanden Eijnden was partially supported by NSF grant DMS-9510356.

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