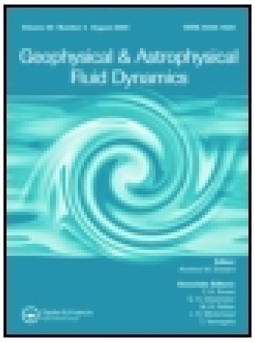

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**P. A. Milewski & E. G. Tabak**

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# A REDUCED MODEL FOR NONLINEAR DISPERSIVE WAVES IN A ROTATING ENVIRONMENT

PAUL A. MILEWSKI[a,]* and ESTEBAN G. TABAK[b]

[a] *Department of Mathematics, University of Wisconsin, Madison;*
[b] *Courant Institute of Mathematical Sciences, New York University*

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The simplest model for geophysical flows is one layer of a constant density fluid with a free surface, where the fluid motions occur on a scale in which the Coriolis force is significant. In the linear shallow water limit, there are non-dispersive Kelvin waves, localized near a boundary or near the equator, and a large family of dispersive waves. We study weakly nonlinear and finite depth corrections to these waves, and derive a reduced system of equations governing the flow. For this system we find approximate solitary Kelvin waves, both for waves traveling along a boundary and along the equator. These waves induce jets perpendicular to their direction of propagation, which may have a role in mixing. We also derive an equivalent reduced system for the evolution of perturbations to a mean geostrophic flow.

*Keywords:* Nonlinear waves; geophysical flows; coastal waves; equatorial waves

## 1. INTRODUCTION

In this paper we present the systematic derivation of reduced equations governing weakly nonlinear long free surface waves in rotating flows. The rotation can vary in one direction to model planetary scale effects. We study in detail two cases: the equatorial waveguide where the Coriolis parameter $f$ is, to leading order, proportional to the distance $y$

*Corresponding author. e-mail: milewski@math.wisc.edu

to the equator (*i.e.*, $f = y$, the equator being at $y = 0$), and shorter scale effects away from the equator where $f$ is approximately constant.

In non-rotating flows, the two-dimensional free surface evolution is governed by an isotropic Boussinesq equation (see Benney and Luke, 1964) which reduces to the Korteweg-de Vries equation for straight wavefronts and to the Kadomtsev–Petviashvilli equation for weakly curved fronts (see Milewski and Keller, 1996).

In the linear limit of rotating shallow water flows, the free surface evolution is composed of steady geostrophic configurations, inertial oscillations and various families of waves: Rossby (or planetary) waves if the variation of $f$ with latitude is taken into account, Poincaré (or gravity) waves, Kelvin waves in the equatorial waveguide and in the presence of boundaries (coastal waves), and mixed Rossby-gravity (or Yanai) equatorial waves. Of the propagating waves, only the Kelvin wave is non-dispersive. Thus, it is interesting to study how finite depth effects (that is, the vertical structure of the flow) affect the evolution of this wave. Furthermore, weak dispersion from finite depth may balance weak nonlinearity and account for the existence of solitary waves.

We derive a system of equations governing these flows, and extend these results to weakly nonlinear, weakly dispersive perturbations to a geostrophic mean flow. Even though the linear structure of this problem is substantially more complex than the case without a mean flow, the derivation of reduced equations follows essentially the same path.

We find approximate solitary Kelvin wave solutions to the reduced equations in the case without a mean flow. These waves are governed by a Korteweg-de Vries equation, and the corresponding three-dimensional flow has counter rotating vortices traveling with the wave. These vortices may contribute to mixing between near shore and offshore waters for coastal waves, and between equatorial and low latitude regions of the atmosphere and the ocean for equatorial waves.

This paper is structured as follows. In Section 2, we formulate the problem and summarize our main results. In Section 3, we derive the reduced equations for the flow, both with and without a mean geostrophic basic flow. In Section 4 we compute approximate solitary wave solutions to these equations. Finally, we present some concluding remarks and some directions for further work.

## 2. FORMULATION AND SUMMARY OF RESULTS

Surface waves propagating in a rotating environment are characterized by several dimensionless parameters. The ratio $\varepsilon$ of the amplitude of the wave to the characteristic depth of the fluid parametrizes nonlinearity. Finite depth effects of the vertical structure of the flow are proportional to the ratio $\mu$ of the depth of the fluid to the typical length of the waves. The relative importance of rotation as compared to the gravity restoring force is parametrized by the ratio of the frequencies of the gravity waves and the rotation of the Earth. This can be expressed as the ratio of the Froude number Fr and the Rossby number Ro. Throughout, we assume $\mathrm{Fr}/\mathrm{Ro} \approx 1$. Lastly, the propagation of the waves depends on the curvature of the wavefronts. In the derivation of the reduced equations we make no special assumption for this parameter; that is, we do not assume a preferred horizontal direction.

Given these scalings, the non-dimensional equations governing the free surface flow of a three-dimensional fluid domain with a flat bottom, gravity pointing in the $-z$ direction, and Coriolis force varying in $y$ are

$$u_t - f(y)\,v + \varepsilon(uu_x + vu_y + wu_z) = -p_x, \tag{2.1}$$

$$v_t + f(y)\,u + \varepsilon(uv_x + vv_y + wv_z) = -p_y, \tag{2.2}$$

$$\mu^2 w_t + \varepsilon\mu^2(uw_x + vw_y + ww_z) = -p_z, \tag{2.3}$$

$$u_x + v_y + w_z = 0. \tag{2.4}$$

The boundary conditions are, on the lower boundary,

$$w = 0, \quad z = -1, \tag{2.5}$$

and on the free surface,

$$p = \eta, \quad z = \varepsilon\eta, \tag{2.6}$$

$$\eta_t + \varepsilon(u\eta_x + v\eta_y) = w, \quad z = \varepsilon\eta. \tag{2.7}$$

Here, $p$ is the deviation from hydrostatic pressure. The dimensional quantities (primed below) can be recovered by the scalings

$$(x',y') = L(x,y), \quad z' = Hz, \quad t' = \tfrac{L}{\sqrt{gH}}t,$$
$$(u',v') = \varepsilon\sqrt{gH}(u,v), \quad w' = \varepsilon\mu\sqrt{gH}w, \quad p' = \varepsilon\rho gHp, \quad \eta' = \varepsilon H\eta,$$

where $H$ is the undisturbed fluid depth, $L$ is the horizontal length scale of motion, $\mu = H/L$, and $\varepsilon = A/H$, where $A$ is the typical amplitude of the wave motion. The Coriolis parameter $f$ in this non-dimensionalization is given by

$$f = \frac{\mathrm{Fr}}{\mathrm{Ro}} = \frac{2|\Omega|\sin(\theta)L}{\sqrt{gH}},$$

where $|\Omega| = (2\pi/24\,\mathrm{hs})$ is the rotation frequency of the Earth and $\theta$ is the latitude.

In Section 3, we derive an asymptotic reduction of these equations. Denoting by $N(x,y,t)$ the leading order term in an expansion for $\eta(x,y,t)$, and by $U(x,y,t)$ and $V(x,y,t)$ the leading order terms for $u(x,y,0,t)$ and $v(x,y,0,t)$ (that is, the velocity measured at the height of the unperturbed free surface), we obtain the system

$$\begin{aligned}
U_t + N_x - fV &= -\varepsilon(UU_x + VU_y),\\
V_t + N_y + fU &= -\varepsilon(UV_x + VV_y),\\
N_t + U_x + V_y &= -\varepsilon((NU)_x + (NV)_y)\\
&\quad + \frac{1}{3}\mu^2((\Delta - f^2)N_t - f'N_x + 2ff'V).
\end{aligned} \tag{2.8}$$

If, on the other hand, we choose to denote by $U(x,y,t)$ and $V(x,y,t)$ the mean values in the vertical of the leading order behavior of the velocities $u(x, y, z, t)$ and $v(x,y,z,t)$, the system becomes

$$\begin{aligned}
U_t + N_x - fV &= -\varepsilon(UU_x + VU_y) + \frac{1}{3}\mu^2(U_x + V_y)_{tx},\\
V_t + N_y + fU &= -\varepsilon(UV_x + VV_y) + \frac{1}{3}\mu^2(U_x + V_y)_{ty},\\
N_t + U_x + V_y &= -\varepsilon[(NU)_x + (NV)_y].
\end{aligned} \tag{2.9}$$

This choice for $U$ and $V$ has the advantage that the third equation (*i.e.*, mass conservation) is exact to all orders in $\varepsilon$ and $\mu$.

If one sets $\varepsilon = 0$ in either set of Eqs. (2.8) and (2.9), one obtains the finite depth corrections to the linear shallow water equations for waves in a rotating environment. For the case of constant $f$, these finite depth effects modify the shallow water dispersion relation, yielding, the geostrophic mode with $\omega = 0$ and the Poincaré waves with

$$\omega = \pm\sqrt{\frac{|\mathbf{k}|^2 + f^2}{1 + (\mu^2|\mathbf{k}|^2)/3}}, \tag{2.10}$$

The Kelvin wave becomes a superposition of modes of the form $\exp(-fky/\omega)\exp[i(kx - \omega t)]$, where

$$\omega = \sqrt{\frac{k^2}{1 + (\mu^2 k^2)/3}}. \tag{2.11}$$

The depth thus introduces weak dispersion and a slight dependence of the decay rate on wavenumber. The solitary wave described below balances these effects with nonlinearity.

The Eqs. (2.8) and (2.9) are generalizations of long wave equations, such as the Benney–Luke (BL) equation for water waves (see Benney and Luke, 1964) to waves in a rotating environment. Thus, by removing rotation, one can obtain as special cases the BL equation, the one-dimensional Korteweg-de Vries (KdV) equation and the weakly two-dimensional Kadomtsev–Petviashvilli (KP) equation, which also combine small nonlinear and dispersive effects. However, the systems (2.8) and (2.9) are two-dimensional and isotropic (except for the possible spatial variation of $f$), while the KdV equation is one-dimensional, and the KP equation is strongly anisotropic, applying only to weak deviations from one-dimensionality. These equations are included in (2.8) and (2.9) as particular limits. Thus, the KdV equation can be obtained by setting $\varepsilon = \mu^2$, $f = 0$, $V = 0$, and considering functions $U(\tau, \theta) = N(\tau, \theta)$, where $\tau = \varepsilon t$ and $\theta = x - t$. Similarly, the KP equation follows from taking $\varepsilon = \mu^2, f = 0, y \to \mu^{-1}y, v \to \mu v$, and $U(\tau, \theta) = N(\tau, \theta)$ as for the KdV.

It is not necessary however to drop the Coriolis effect altogether to obtain variations of these equations. For instance, we can recover the case of weak rotation and weak two-dimensionality, considered in (Grimshaw, 1985; Katsis and Akylas, 1987). Their particular scaling can be obtained by taking the scalings for the KP above, but with $f \to \mu f$, whence one can deduce the modified KP equation

$$\left(U_\tau + \frac{3}{2}UU_\theta + \frac{1}{6}U_{\theta\theta\theta}\right)_\theta + \frac{1}{2}(U_{yy} - f^2 U) = 0. \qquad (2.12)$$

(This equation has a constraint on locally confined solutions (see Grimshaw and Melville, 1989), associated to Poincaré modes that are not treated properly).

A KdV equation follows naturally from the systems (2.8) and (2.9), even with large Coriolis parameter $f$, when one considers the slow evolution of Kelvin waves, given by

$$U = N = g(y)\,q(\theta, \tau), \quad g(y) = \mathrm{e}^{-\int^y f(y')dy'},$$
$$V = 0,$$

where $\theta$ and $\tau$ have the same meaning as above. The corresponding equation for the evolution of $q$, derived in Section 4, is

$$q_\tau + \beta q q_\theta + \frac{1}{6}q_{\theta\theta\theta} = 0, \qquad (2.13)$$

where,

$$\beta = \frac{3}{2}\frac{\int g^3 dy}{\int g^2 dy}.$$

This equation has well known solitary wave solutions, given by

$$q = A\ \mathrm{sech}^2[K(x - Ct)], \quad A = \frac{2}{\beta}K^2, \quad C = 1 + \varepsilon\frac{2}{3}K^2. \qquad (2.14)$$

Figures 1a and 2a show these solitary Kelvin waves in the coastal and equatorial cases. The second order corrections to these flows have a nontrivial $v$-component. These corrections, computed in Section 4, consist mainly of jets perpendicular to the direction of propagation of the waves. These jets, shown in Figures 1b and 2b, may play a role in mixing.

Seldom are geophysical flows small perturbations of a quiescent state. Generally the unperturbed state is one in geostrophic balance, with a nontrivial sheared mean flow $\bar{U}(y)$. The geostrophic balance implies, for the single layer model, a corresponding tilt of the free

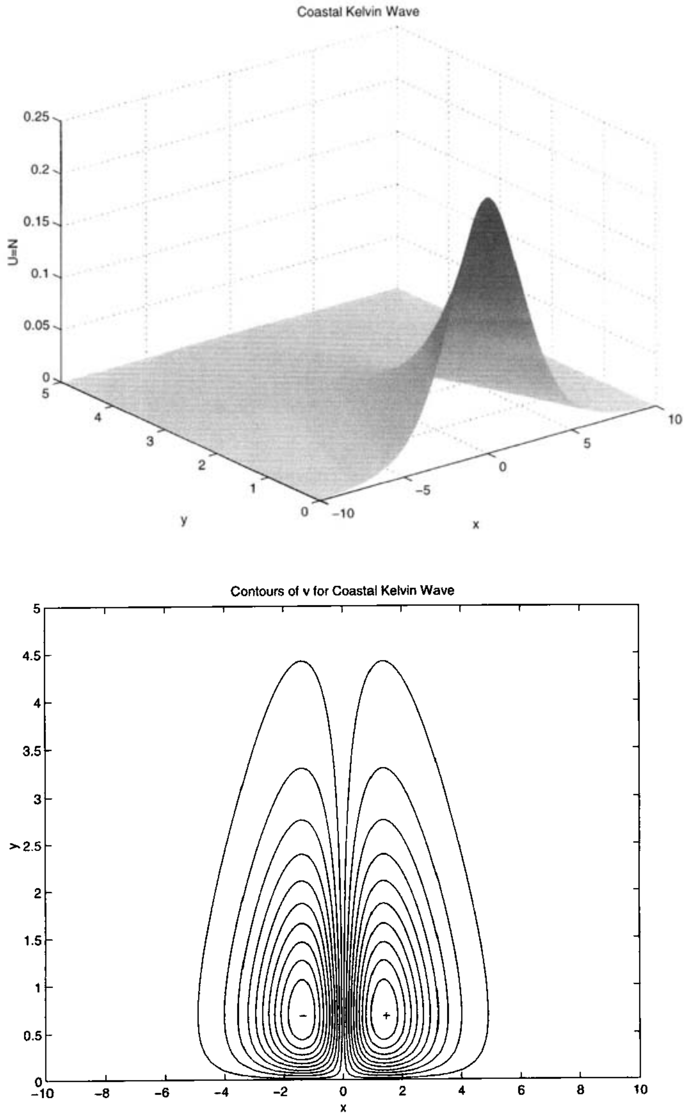
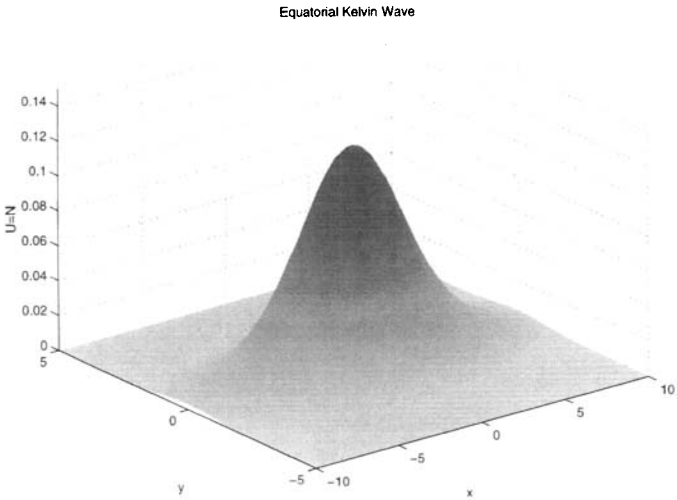



FIGURE 1 Coastal Kelvin wave with $K = 0.35$. (a) Leading order solution. (b) Induced transversal velocity $v$.

surface $\bar{N}$. The corresponding equations for a single layer of homogeneous fluid are

$$(\partial_t + \bar{U}\partial_x)u + (\bar{U}' - f)\, v + \varepsilon(uu_x + vu_y + wu_z) = -p_x, \tag{2.15}$$



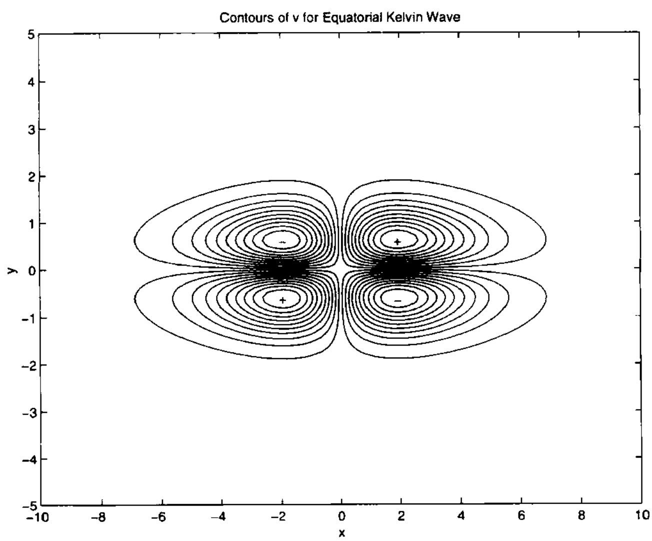




FIGURE 2 Equatorial Kelvin wave with $K = 0.25$. (a) Leading order solution. (b) Induced transversal velocity $v$.

$$(\partial_t + \bar{U}\partial_x)v + fu + \varepsilon(uv_x + vv_y + wv_z) = -p_y, \tag{2.16}$$

$$\mu^2(\partial_t + \bar{U}\partial_x)w + \varepsilon\mu^2(uw_x + vw_y + ww_z) = -p_z, \tag{2.17}$$

$$u_x + v_y + w_z = 0, \tag{2.18}$$

with boundary conditions

$$w = 0, \quad z = -1, \tag{2.19}$$

and

$$p = \eta, \quad z = \bar{N} + \varepsilon\eta, \tag{2.20}$$

$$(\partial_t + \bar{U}\partial_x)\eta - f\bar{U}v + \varepsilon(u\eta_x + v\eta_y) = w, \quad z = \bar{N} + \varepsilon\eta. \tag{2.21}$$

The tilt $\bar{N}$ of the free surface due to the geostrophic balance is given by

$$\bar{N}(y) = -\int^y f(s)\,\bar{U}(s)\,ds. \tag{2.22}$$

The reduced equations for weakly nonlinear, weakly dispersive flows, computed in Section 3, are

$$\begin{aligned}
&(\partial_t + \bar{U}\partial_x)U + N_x + (\bar{U}' - f)V = -\varepsilon(UU_x + VU_y)\\
&\qquad\qquad + \frac{1}{3}\mu^2(\bar{N}+1)^2\partial_x(\partial_t + \bar{U}\partial_x)(U_x + V_y),\\
&(\partial_t + \bar{U}\partial_x)V + N_y + fU = -\varepsilon(UV_x + VV_y)\\
&\qquad\qquad + \frac{1}{3}\mu^2(\bar{N}+1)^2\partial_y(\partial_t + \bar{U}\partial_x)(U_x + V_y)\\
&\qquad\qquad + \mu^2\bar{N}'(\bar{N}+1)(\partial_t + \bar{U}\partial_x)(U_x + V_y),\\
&(\partial_t + \bar{U}\partial_x)N - f\bar{U}V + (\bar{N}+1)(U_x + V_y)\\
&\qquad\qquad = -\varepsilon[(NU)_x + (NV)_y].
\end{aligned} \tag{2.23}$$

Here $N(x, y, t)$ stands for the leading order behavior of the free surface, and $U(x, y, t)$ and $V(x, y, t)$ for the mean values in the vertical of $u(x, y, z, t)$ and $v(x, y, z, t)$.

## 3. DERIVATION OF THE ASYMPTOTIC EQUATIONS

In the usual weakly nonlinear approximation we expand (2.6), (2.7) about $z = 0$ to obtain

$$p + \varepsilon p_z \eta + O(\varepsilon^2) = \eta, \quad z = 0, \tag{3.1}$$

$$\eta_t + \varepsilon(u\eta_x + v\eta_y) = w + \varepsilon w_z \eta + O(\varepsilon^2), \quad z = 0. \tag{3.2}$$

The leading order solution, obtained by setting $\varepsilon = \mu^2 = 0$, has, from (2.6) and (2.3), $p^{(0)} = \eta^{(0)} = N(x, y, t)$. Therefore, if the initial data does not include $z$-dependent inertial modes, then $(u^{(0)}, v^{(0)}) = [U(x, y, t), V(x, y, t)]$ do not depend on $z$, and (2.4) yields

$$w^{(0)} = -(z + 1)(U_x + V_y). \tag{3.3}$$

Thus, (3.2) completes the system of equations satisfied by the leading order solution

$$\begin{aligned} U_t + N_x - fV &= 0, \\ V_t + N_y + fU &= 0, \\ N_t + U_x + V_y &= 0. \end{aligned} \tag{3.4}$$

These equations contain as solutions steady geostrophic modes, inertial oscillations, Kelvin and Poincaré waves and, if $f$ depends on latitude, Rossby waves (and Yanai waves in the equatorial waveguide). Our goal is to compute corrections to the leading order equations to take into account the nonlinear interaction of these various waves and the effects of a small but finite depth of the fluid.

We compute these corrections by expanding $u$, $v$, $p$, $w$, $\eta$ in power series in $\varepsilon$, $\mu^2$ as follows

$$\begin{aligned} u &= U(x, y, t) + \varepsilon u^{(1,0)} + \mu^2 u^{(0,1)} + \cdots, \\ v &= V(x, y, t) + \varepsilon v^{(1,0)} + \mu^2 v^{(0,1)} + \cdots, \\ w &= -(z + 1)(U_x + V_y) + \varepsilon w^{(1,0)} + \mu^2 w^{(0,1)} + \cdots, \\ p &= N(x, y, t) + \varepsilon p^{(1,0)} + \mu^2 p^{(0,1)} + \cdots, \\ \eta &= N(x, y, t) + \varepsilon \eta^{(1,0)} + \mu^2 \eta^{(0,1)} + \cdots. \end{aligned} \tag{3.5}$$

The first nonlinear corrections $p^{(1,0)} = \eta^{(1,0)}$, $u^{(1,0)}$, $v^{(1,0)}$ are also independent of $z$, and thus $w^{(1,0)} = -(z+1)(u_x^{(1,0)} + v_y^{(1,0)})$. The inhomogeneous equations become

$$\begin{aligned} u_t^{(1,0)} + \eta_x^{(1,0)} - fv^{(1,0)} &= -UU_x - VU_y, \\ v_t^{(1,0)} + \eta_y^{(1,0)} + fu^{(1,0)} &= -UV_x - VV_y, \\ \eta_t^{(1,0)} + u_x^{(1,0)} + v_y^{(1,0)} &= -(UN)_x - (VN)_y. \end{aligned} \tag{3.6}$$

The finite depth corrections have a non-trivial $z$ structure. Using (3.5) in (2.3) and (3.1), we obtain

$$p^{(0,1)} = \left(\frac{1}{2}z^2 + z\right)(U_{xt} + V_{yt}) + \eta^{(0,1)}(x,y,t). \tag{3.7}$$

Then, Eqs. (2.1), (2.2) and (2.4) become

$$\begin{aligned} u_t^{(0,1)} + p_x^{(0,1)} - fv^{(0,1)} &= 0, \\ v_t^{(0,1)} + p_y^{(0,1)} + fu^{(0,1)} &= 0, \\ u_x^{(0,1)} + v_y^{(0,1)} + w_z^{(0,1)} &= 0, \end{aligned} \tag{3.8}$$

with boundary condition

$$\eta_t^{(0,1)} = w^{(0,1)}, \quad z = 0. \tag{3.9}$$

In order to evaluate the boundary condition (3.9), we need $w^{(0,1)}$. It is obtained by integrating the incompressibility equation in $z$, for which we need to compute $u_x^{(0,1)} + v_y^{(0,1)}$. To this end, we decompose $u^{(0,1)}$, $v^{(0,1)}$ into two parts: $(u^{(0,1)}, v^{(0,1)}) = ((1/2)z^2 + z)(u_p^{(0,1)}, v_p^{(0,1)}) + (u_h^{(0,1)}, v_h^{(0,1)})$. Then,

$$\begin{aligned} w^{(0,1)} = &- \left(\frac{1}{6}z^3 + \frac{1}{2}z^2 - \frac{1}{3}\right)(u_{p,x}^{(0,1)} + v_{p,y}^{(0,1)}) \\ &- (z+1)(u_{h,x}^{(0,1)} + v_{h,y}^{(0,1)}). \end{aligned} \tag{3.10}$$

Now, using the third equation in (3.4), $u_p^{(0,1)}$, $v_p^{(0,1)}$ satisfy

$$u_{p,t}^{(0,1)} - f\, v_p^{(0,1)} = N_{xtt}, \tag{3.11}$$

$$v_{p,t}^{(0,1)} + fu_p^{(0,1)} = N_{ytt}. \tag{3.12}$$

Since these equations are linear and inhomogeneous, we can write the divergence of its solution in the form

$$u_{p,x}^{(0,1)} + v_{p,y}^{(0,1)} = \mathcal{L}N_{tt}, \tag{3.13}$$

where $\mathcal{L}$ is a linear operator and where we have used our previous assumption that the initial data does not contain $z$-dependent inertial waves to eliminate the homogeneous solutions to (3.11), (3.12). The operator $\mathcal{L}$ can be computed explicitly; it is given by

$$\mathcal{L} = -(\partial_{tt} + f^2)^{-1}\partial_t\Delta;$$

when $f$ is a constant, and a more convoluted expression when $f$ is a nontrivial function of $y$. However, we will not make use of this explicit form, since the operator $\mathcal{L}$ will be shown below to factor out of the final equations. From (3.10), the boundary condition (3.9) becomes

$$\eta_t^{(0,1)} + u_{h,x}^{(0,1)} + v_{h,y}^{(0,1)} = \frac{1}{3}\mathcal{L}N_{tt}. \tag{3.14}$$

In order to factor out the operator $\mathcal{L}$, we notice that, since the homogeneous part of Eqs. (3.11) and (3.12) has the same form as that of the first two equations in (3.4), we have $U_x + V_y = -\mathcal{L}N$, and thus, (3.14) becomes

$$\eta_t^{(0,1)} + u_{h,x}^{(0,1)} + v_{h,y}^{(0,1)} = -\frac{1}{3}(U_{xtt} + V_{ytt}). \tag{3.15}$$

From (3.4) we obtain $U_{xtt} + V_{ytt} = -(\Delta - f^2)N_t + f'N_x - 2ff'V$ and so

$$\begin{aligned} u_{h,t}^{(0,1)} + \eta_x^{(0,1)} - f\,v_h^{(0,1)} &= 0, \\ v_{h,t}^{(0,1)} + \eta_y^{(0,1)} + f\,u_h^{(0,1)} &= 0, \\ \eta_t^{(0,1)} + u_{h,x}^{(0,1)} + v_{h,y}^{(0,1)} &= \tfrac{1}{3}[(\Delta - f^2)N_t - f'N_x + 2ff'V]. \end{aligned} \tag{3.16}$$

We finally obtain the reduced system of Eqs. (2.8), equivalent to (2.1)–(2.4) to O($\varepsilon$, $\mu^2$) by adding (3.4) to $\varepsilon$ multiplied by (3.6) and $\mu^2$ multiplied by (3.16). The equations in (2.8) describe the evolution of $(\hat{U}, \hat{V}, \hat{N}) = (U, V, N) + \varepsilon(u^{(1,0)}, v^{(1,0)}, \eta^{(1,0)}) + \mu^2(u_h^{(0,1)}, v_h^{(0,1)}, \eta^{(0,1)})$ (dropping the hats). This is a closed system of equations for $U$, $V$ and $N$. In order to complete the solution at all heights consistent to order $\mu^2$, we need to solve these equations and then add the contribution from $u_p^{(0,1)}$, $v_p^{(0,1)}$, obtained by solving (3.11), (3.12).

Note that if $\mu^2 = 0$, (2.8) is exact, that is, the expansion in powers of $\varepsilon$ terminates and one may set $\varepsilon = 1$. This is the usual *shallow water* limit, in which the waves are so long that they do not have any vertical structure; the ratio of depth to wavelength in this approximation is formally zero. The more general scaling in (2.8) corresponds to the *long wave* limit, which includes the leading order corrections to the vertical structure, arising from the small but finite ratio of depth to wavelength.

In the preceding discussion, the primitive variables $U$ and $V$ appearing in (2.8) correspond to the velocities $u$, $v$ evaluated at the undisturbed free surface (notice that $u_p$, $v_p$ are zero at $z = 0$). One can obtain a one-parameter family of models by deriving equations where $U$, $V$ are the velocities at some other reference level $z_0$. Then, the derivation for the dispersive corrections is modified as follows: Introducing the reference level $z_0$ in (3.7) yields

$$p^{(0,1)} = \frac{1}{2}[(z+1)^2 - (z_0+1)^2](U_{xt} + V_{yt}) + p_h^{(0,1)}(x,y,t). \tag{3.17}$$

Then, from (3.1), the definition of $\eta^{(0,1)}$ is changed to

$$p_h^{(0,1)} + \frac{1}{2}[1 - (z_0+1)^2](U_{xt} + V_{yt}) = \eta^{(0,1)}(x,y,t). \tag{3.18}$$

The appropriate decomposition of $u^{(0,1)}$, $v^{(0,1)}$ is now $(u^{(0,1)}, v^{(0,1)}) = (1/2)((z+1)^2 - (z_0+1)^2)(u_p^{(0,1)}, v_p^{(0,1)}) + (u_h^{(0,1)}, v_h^{(0,1)})$, and

$$\begin{aligned} w^{(0,1)} = &-\left[\frac{1}{6}(z+1)^3 - \frac{1}{2}(z+1)(z_0+1)^2\right](u_{p,x}^{(0,1)} + v_{p,y}^{(0,1)}) \\ &- (z+1)(u_{h,x}^{(0,1)} + v_{h,y}^{(0,1)}). \end{aligned} \tag{3.19}$$

The derivation now follows as before, resulting in the replacement of (3.16) with

$$\begin{aligned} u_{h,t}^{(0,1)} + \eta_x^{(0,1)} - f v_h^{(0,1)} &= \frac{1}{2}[1 - (z_0+1)^2](U_{xt} + V_{yt})_x, \\ v_{h,t}^{(0,1)} + \eta_y^{(0,1)} + f u_h^{(0,1)} &= \frac{1}{2}[1 - (z_0+1)^2](U_{xt} + V_{yt})_y, \\ \eta_t^{(0,1)} + u_{h,x}^{(0,1)} + v_{h,y}^{(0,1)} &= \left[\frac{1}{6} - \frac{1}{2}(z_0+1)^2\right](U_{xtt} + V_{ytt}). \end{aligned}$$

Finally, the reduced system of Eqs. (2.8) becomes

$$\begin{aligned} U_t + N_x - f\,V &= -\,\varepsilon(UU_x + VU_y) \\ &\quad + \frac{1}{2}\mu^2[1-(z_0+1)^2](U_x+V_y)_{tx}, \\ V_t + N_y + f\,U &= -\,\varepsilon(UV_x + VV_y) \\ &\quad + \frac{1}{2}\mu^2[1-(z_0+1)^2](U_x+V_y)_{ty}, \\ N_t + U_x + V_y &= -\,\varepsilon[(NU)_x + (NV)_y] \\ &\quad + \mu^2\left[\frac{1}{6} - \frac{1}{2}(z_0+1)^2\right](U_x+V_y)_{tt}. \end{aligned} \tag{3.20}$$

Letting $z_0 = 0$ and using the leading order equations in the $\mu^2$ corrections, one recovers (2.8). Also, letting $(z_0+1)^2 = \frac{1}{3}$ eliminates the $\mu^2$ correction to the surface equation, yielding the system (2.9). It is easy to check that this choice of $z_0$ corresponds to the height at which $u$ and $v$ adopt values equal to their vertical means. The general system (3.20) with arbitrary $z_0$ is the analogue of the various Boussinesq approximations for shallow water. The dispersion relations for Poincaré and Kelvin waves discussed in Section 2 are independent of $z_0$.

Next we consider perturbations to a mean flow in geostrophic balance, where the primitive equations are given by (2.15)–(2.18). Again, we propose an expansion of the form (3.5), where the leading order solution satisfies

$$\begin{aligned} (\partial_t + \bar{U}\partial_x)U + (\bar{U}' - f\,)V + N_x &= 0, \\ (\partial_t + \bar{U}\partial_x)V + f\,U + N_y &= 0, \\ (\partial_t + \bar{U}\partial_x)N - f\bar{U}V + (\bar{N}+1)(U_x+V_y) &= 0. \end{aligned} \tag{3.21}$$

The nonlinear corrections are still independent of $z$; they satisfy:

$$\begin{aligned} (\partial_t + \bar{U}\partial_x)u^{(1,0)} + (\bar{U}' - f\,)v^{(1,0)} + \eta_x^{(1,0)} &= -UU_x - VU_y, \\ (\partial_t + \bar{U}\partial_x)v^{(1,0)} + fu^{(1,0)} + \eta_y^{(1,0)} &= -UV_x - VV_y, \\ (\partial_t + \bar{U}\partial_x)\eta^{(1,0)} - f\bar{U}v^{(1,0)} + (\bar{N}+1)(u_x^{(1,0)} + v_y^{(1,0)}) &= -(UN)_x - (VN)_y. \end{aligned} \tag{3.22}$$

We will proceed in a way similar to the second part of this section, leaving a reference depth free. We allow the reference depth $z_0$ to vary in $y$. The finite depth corrections to the pressure are obtained from (2.17)

$$p^{(0,1)} = \frac{1}{2}[(z+1)^2 - (z_0+1)^2](\partial_t + \bar{U}\partial_x)(U_x + V_y) + p_h^{(0,1)}(x,y,t), \tag{3.23}$$

which implies, from (2.20),

$$p_h^{(0,1)} + \frac{1}{2}[(\bar{N}+1)^2 - (z_0+1)^2](\partial_t + \bar{U}\partial_x)(U_x + V_y) = \eta^{(0,1)}(x,y,t). \tag{3.24}$$

We must now satisfy

$$\begin{aligned}(\partial_t + \bar{U}\partial_x)u^{(0,1)} + (\bar{U}' - f)v^{(0,1)} + p_x^{(0,1)} &= 0,\\ (\partial_t + \bar{U}\partial_x)v^{(0,1)} + f\,u^{(0,1)} + p_y^{(0,1)} &= 0,\\ u_x^{(0,1)} + v_y^{(0,1)} + w_z^{(0,1)} &= 0,\end{aligned}$$

with boundary condition

$$(\partial_t + \bar{U}\partial_x)\eta^{(0,1)} - f\bar{U}v^{(0,1)} = w^{(0,1)}, \quad z = \bar{N}. \tag{3.25}$$

In order to apply (3.25), we need $w^{(0,1)}$ or, from the incompressibility condition, $u_x^{(0,1)} + v_y^{(0,1)}$. As above, we decompose $u^{(0,1)}$, $v^{(0,1)}$ into

$$(u^{(0,1)}, v^{(0,1)}) = \frac{1}{2}[(z+1)^2 - (z_0+1)^2](u_p^{(0,1)}, v_p^{(0,1)}) + (u_h^{(0,1)}, v_h^{(0,1)}).$$

Then, integrating the continuity equation yields

$$\begin{aligned} w^{(0,1)} = &- \left[\frac{1}{6}(z+1)^3 - \frac{1}{2}(z+1)(z_0+1)^2\right](u_{p,x}^{(0,1)} + v_{p,y}^{(0,1)})\\ &+ (z+1)(z_0+1)z_0' v_p^{(0,1)} - (z+1)(u_{h,x}^{(0,1)} + v_{h,y}^{(0,1)}).\end{aligned}$$

There are now, in principle, two systems to be solved: one for the $z$-dependent part of the solution and one for the $z$-independent part.

The first system is

$$
\begin{aligned}
(\partial_t + \bar{U}\partial_x)u_p^{(0,1)} + (\bar{U}' - f)v_p^{(0,1)} &= -\partial_x(\partial_t + \bar{U}\partial_x)(U_x + V_y), \\
(\partial_t + \bar{U}\partial_x)v_p^{(0,1)} + fu_p^{(0,1)} &= -\partial_y(\partial_t + \bar{U}\partial_x)(U_x + V_y),
\end{aligned}
$$

and the system for the $z$-independent solution, forced by the solution to the $z$-dependent system, is

$$
\begin{aligned}
&(\partial_t + \bar{U}\partial_x)u_h^{(0,1)} + \eta_x^{(0,1)} + (\bar{U}' - f)v_h^{(0,1)} \\
&\qquad = \frac{1}{2}[(\bar{N}+1)^2 - (z_0+1)^2]\partial_x(\partial_t + \bar{U}\partial_x)(U_x + V_y), \\
&(\partial_t + \bar{U}\partial_x)v_h^{(0,1)} + \eta_y^{(0,1)} + fu_h^{(0,1)} \\
&\qquad = \frac{1}{2}[(\bar{N}+1)^2 - (z_0+1)^2]\partial_y(\partial_t + \bar{U}\partial_x)(U_x + V_y) \\
&\qquad\quad + \bar{N}'(\bar{N}+1)(\partial_t + \bar{U}\partial_x)(U_x + V_y), \\
&(\partial_t + \bar{U}\partial_x)\eta^{(0,1)} - f\bar{U}v_h^{(0,1)} + (\bar{N}+1)(u_{h,x}^{(0,1)} + v_{h,y}^{(0,1)}) \\
&\qquad = f\,\bar{U}\frac{1}{2}[(\bar{N}+1)^2 - (z_0+1)^2]v_p^{(0,1)} \\
&\qquad\quad + (\bar{N}+1)(z_0+1)z_0'v_p^{(0,1)} \\
&\qquad\quad - \left[\frac{1}{6}(\bar{N}+1)^3 - \frac{1}{2}(\bar{N}+1)(z_0+1)^2\right](u_{p,x}^{(0,1)} + v_{p,y}^{(0,1)}).
\end{aligned}
$$

Choosing $(z_0+1)^2 = \frac{1}{3}(\bar{N}+1)^2$ decouples these two systems, avoiding the need to compute the analogue of the operator $\mathcal{L}$ above. This choice corresponds to, for each $y$, computing $u$ and $v$ at the height where their values agree with their vertical means, and yields

$$
\begin{aligned}
&(\partial_t + \bar{U}\partial_x)u_h^{(0,1)} + \eta_x^{(0,1)} + (\bar{U}' - f)v_h^{(0,1)} \\
&\qquad = \frac{1}{3}(\bar{N}+1)^2\partial_x(\partial_t + \bar{U}\partial_x)(U_x + V_y), \\
&(\partial_t + \bar{U}\partial_x)v_h^{(0,1)} + \eta_y^{(0,1)} + fu_h^{(0,1)} \\
&\qquad = \frac{1}{3}(\bar{N}+1)^2\partial_y(\partial_t + \bar{U}\partial_x)(U_x + V_y) \\
&\qquad\quad + \bar{N}'(\bar{N}+1)(\partial_t + \bar{U}\partial_x)(U_x + V_y), \qquad (3.26) \\
&(\partial_t + \bar{U}\partial_x)\eta^{(0,1)} - f\bar{U}v_h^{(0,1)} + (\bar{N}+1)(u_{h,x}^{(0,1)} + v_{h,y}^{(0,1)}) = 0.
\end{aligned}
$$

If now we add the systems (3.21), (3.22) and (3.26), we obtain the evolution equations in (2.23).

## 4. SOLITARY WAVES

We show now that the systems (2.8) and (2.9) admit approximate solitary waves for the balance $\varepsilon = O(\mu^2)$. Specifically, we are interested in two particular cases: the equatorial waveguide, where $f = y$, and mid-latitude coastal waves, where $f$ is constant and the flow in $y > 0$ has $y = 0$ as a boundary where $V = 0$. In the coastal wave case, solitary waves were found in Grimshaw (1985) for a model of stratified flows. Solitary waves arise because the Kelvin waves are, to leading order, nondispersive, and thus the corrections due to nonlinearity and to finite depth effects can balance each other. For concreteness, we shall work with the system (2.9). The procedure for (2.8) or for any other choice of $z_0$, is identical and leads to exactly the same equations, with small variations in the higher order corrections since the systems use slightly different variables.

The leading order solution for a Kelvin wave with $\varepsilon = \mu = 0$ has $V = 0$ and

$$U = N = g(y)\, q(x-t), \quad g(y) = \exp\left[-\int^y f(y')dy'\right]. \tag{4.1}$$

Thus, equatorial Kelvin waves have $g(y) = \exp(-y^2/2)$ and coastal Kelvin waves have $g(y) = \exp(-y)$. Anticipating that the nonlinear and dispersive corrections will slowly modulate these waves in time, we let $q(\theta, \tau)$ where $\theta = x - t$ and $\tau = \varepsilon t$ (We take $\varepsilon = \mu^2$ throughout this derivation.) Writing,

$$\begin{aligned} U &= g(y)\, q(\theta, \tau) + \varepsilon u, \\ V &= \varepsilon v, \\ N &= g(y)\, q(\theta, \tau) + \varepsilon n, \end{aligned}$$

the equations for the corrections are

$$u_t + n_x - fv = -gq_\tau - g^2 qq_\theta - \frac{1}{3} gq_{\theta\theta\theta}, \tag{4.2}$$

$$v_t + n_y + fu = -\frac{1}{3}g' q_{\theta\theta}, \tag{4.3}$$

$$n_t + u_x + v_y = -g q_T - 2g^2 q q_\theta. \tag{4.4}$$

The solvability condition is obtained by adding (4.2) to (4.4), multiplying by $g$ and integrating over $y$. Defining $(\bar{u}, \bar{n}) = \int (u, n)\, g\, dy$, we obtain

$$(\bar{u} + \bar{n})_t + (\bar{u} + \bar{n})_x = -3\left(\int g^3 dy\right) q q_\theta - \frac{1}{3}\left(\int g^2 dy\right) q_{\theta\theta\theta} - 2\left(\int g^2 dy\right) q_T.$$

To prevent the solution for $(\bar{u} + \bar{n})$ from becoming unbounded in time, thereby disordering the original asymptotic expansion, we impose the solvability condition

$$q_T + \beta q q_\theta + \frac{1}{6} q_{\theta\theta\theta} = 0, \tag{4.5}$$

where,

$$\beta = \frac{3}{2}\frac{\int g^3 dy}{\int g^2 dy}. \tag{4.6}$$

Equation (4.5) is a Korteweg-de Vries equation for the waveform along the direction of propagation. For coastal Kelvin waves, $\beta = 1$, whereas for equatorial waves $\beta = \sqrt{3/2}$. The equation has well known soliton and periodic cnoidal wave solutions (see Whitham, 1974). In particular, the soliton solutions, reverting to the $x$, $t$ variables are

$$q = A \operatorname{sech}^2[K(x - Ct)], \quad A = \frac{2}{\beta}K^2, \quad C = 1 + \varepsilon\frac{2}{3}K^2. \tag{4.7}$$

These solutions are shown in Figures 1a and 2a for the coastal and equatorial waves respectively.

It is interesting to examine the O($\varepsilon$) induced flow resulting from these solutions. These flows contain nontrivial $V$. After imposing the solvability condition (4.5), the Eqs. (4.2)–(4.4) take the form

$$u_t + n_x - fv = (\beta g - g^2)qq_\theta - \frac{1}{6}gq_{\theta\theta\theta} \equiv R_1, \tag{4.8}$$

$$v_t + n_y + fu = -\frac{1}{3}g'q_{\theta\theta}, \tag{4.9}$$

$$n_t + u_x + v_y = (\beta g - 2g^2)qq_\theta + \frac{1}{6}gq_{\theta\theta\theta} \equiv R_3. \tag{4.10}$$

We seek solutions depending only on $\theta = x - t$ and $y$. From (4.8), (4.10), one can calculate $(n - u)$:

$$(n-u)_y - (f + f'/f)(n-u) = -\frac{f'}{2f}\, g(\beta - g)q^2 + \left(\frac{f}{3} + \frac{f'}{6f}\right)gq_{\theta\theta}. \tag{4.11}$$

Next, using (4.8), we obtain for $v$:

$$v = \frac{1}{f}[(n-u)_\theta - R_1], \tag{4.12}$$

and, from (4.9), we calculate $n$:

$$n_y + fn = v_\theta + f(n-u) + \frac{1}{3}fgq_{\theta\theta}. \tag{4.13}$$

For the case of a coastal Kelvin wave, these calculations yield

$$u = \frac{1}{6}(y+1)e^{-y}q_{\theta\theta} - (y + e^{-y})e^{-y}(qq_\theta)_\theta, \tag{4.14}$$

$$v = (e^{-2y} - e^{-y})qq_\theta, \tag{4.15}$$

$$n = \frac{1}{6}ye^{-y}q_{\theta\theta} - (y + e^{-y})e^{-y}(qq_\theta)_\theta. \tag{4.16}$$

Although the leading order solution has $V = 0$, the correction has nonzero velocity in the direction normal to the wall. Figure 1b shows the velocity $v$ corresponding to the solitary wave in Figure 1a. The velocity field consists of two jets, one preceding the wave and pointing offshore and one following the wave pointing towards the coast. Thus, such coastal Kelvin waves in the atmosphere or the ocean could contribute to mixing between near shore and offshore regions. We note that the $v$ correction arises solely from the nonlinear terms, whereas $n$ and $u$ corrections arise from finite-depth dispersive effects.

If one wants to complete the full $z$-dependent solution to this order, one needs to solve (3.11), (3.12) which, with $N$ from (4.1), give

$$u_p^{(0,1)} = -gq_{\theta\theta}, \quad v_p^{(0,1)} = 0. \tag{4.17}$$

For the equatorial Kelvin wave, $v$ can be written down in closed form:

$$v = \sqrt{\frac{3}{2}}\pi e^{-y^2/2}[\mathrm{Erf}(y) - \mathrm{Erf}(\sqrt{2/3}\,y)]qq_\theta, \tag{4.18}$$

where

$$\mathrm{Erf}(y) = \frac{2}{\sqrt{\pi}}\int_0^y e^{-t^2}\,dt.$$

The expressions for $n$ and $u$ involve integrals of the error function, but their difference does not; it is given by

$$n - u = \frac{e^{-y^2/2}}{2}\left(\sqrt{\frac{3}{2}}\left\{\left(1 - \sqrt{2/3}\,e^{-y^2/2}\right)\right.\right. \\ \left.\left. + \sqrt{\pi}y\left[\mathrm{Erf}(y) - \mathrm{Erf}\left(\sqrt{2/3}\,y\right)\right]\right\}q^2 - \frac{1}{3}q_{\theta\theta}\right).$$

The values of $v$ and $n - u$ are the most significant, since they constitute the leading order behavior of the corresponding variables. Figure 2b shows values of $v$ for the solitary wave solution of Figure 2a. The field now consists of four jets, two pointing toward mid-latitudes preceding the waves and two pointing towards the Equator following the wave.

## 5. CONCLUSIONS

A system of equations was derived describing the weakly-nonlinear, weakly-dispersive evolution of long waves in a rotating environment. These equations can be used to study wave interactions and nonlinear corrections to leading order linear waves. In particular, corrections were computed to the nondispersive Kelvin waves, leading to a KdV equation with solitary wave solutions. The corrections include a nontrivial structure for the velocity normal to the direction of propagation of the wave, which can contribute to mixing near shore and offshore waters in the case of coastal waves, and between equatorial and extra-equatorial regions of the atmosphere and the ocean in the equatorial case.

The system was derived for free-surface waves, and then extended to model perturbations to a mean geostrophic flow. Other extensions under development include the more realistic stratified case, which has the added richness of potential nonlinear interaction between the various internal modes. Finally, bottom topography or other inhomogeneities of the medium, such as a variable stratification, can be included in a straightforward manner. When combined with an order one mean flow, these would allow for the study of wave-topographic resonances (Majda *et al.*, 1997) and of the topographic generation of nonlinear waves by flows over topographic disturbances (Milewski and Tabak, 1998; Grimshaw and Yi, 1991).

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