A PSEUDOSPECTRAL PROCEDURE FOR THE SOLUTION OF NONLINEAR WAVE EQUATIONS WITH EXAMPLES FROM FREE-SURFACE FLOWS*

PAUL A. MILEWSKI[†] AND ESTEBAN G. TABAK[‡]

Abstract. An algorithm for the solution of general isotropic nonlinear wave equations is presented. The algorithm is based on a symmetric factorization of the linear part of the wave operator, followed by its exact integration through an integrating factor in spectral space. The remaining nonlinear and forcing terms can be handled with any standard pseudospectral procedure. Solving the linear part of the wave operator exactly effectively eliminates the stiffness of the original problem, characterized by a wide range of temporal scales. The algorithm is tested and applied to several problems of three-dimensional long surface waves: solitary wave propagation, interaction, diffraction, and the generation of waves by flow over slowly varying bottom topography. Other potential applications include waves in rotating and stratified flows and wave interaction with more pronounced topographic features.

Key words. pseudospectral methods, stiffness, nonlinear waves, free-surface flows

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1. Introduction. Wave phenomena in nature often involve relatively small perturbations of equilibrium states. These small perturbations can be described successfully by linear theory over short time intervals. For longer times, however, nonlinear effects accumulate and substantially affect the evolution of the waves. The effect of such nonlinearity is varied: Discrete sets of waves may interact resonantly, either directly or through a slowly changing medium; a single wave may steepen up and break; the nonlinear steepening may be balanced by small dispersive effects, yielding coherent structures such as solitons; and, in higher dimensions, nonlinearity may delay, prevent, or drastically change wave focusing. The mathematical description of these *weakly nonlinear* waves typically involves the linear structure, characterized by a *dispersion relation*, and small nonlinearities which effectively *modulate* the linear solutions over long times (see, for instance, [1] and [12]).

Examples of weakly nonlinear waves in geophysics are the very long waves in the atmosphere and the ocean. Here the equilibrium state is one of uniform rotation; deviations from this state give rise to a wide variety of waves, with a significant impact on our climate. On smaller scales, surface and internal waves in lakes and oceans usually consist of relatively small deviations from a state of rest, so they also can be modeled accurately as weakly nonlinear phenomena.

The temporal scales associated with the linear and nonlinear components of these models typically are very different. The linear part involves a huge range of scales, from the very slow to the very fast, while the effects of nonlinearity are felt only over long time intervals and couple the various linear modes. Thus the numerical solution of the resulting equations encounters the problem of *stiffness*: one is normally

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[†]Department of Mathematics, University of Wisconsin, Madison, WI 53706 (milewski@ math.wisc.edu). The work of this author was partially supported by NSF grant DMS 9401405.

[‡]Courant Institute of Mathematical Sciences, New York University, New York, 10012 (tabak@cims.nyu.edu). The work of this author was partially supported by NSF grant DMS-9501073.

interested in phenomena taking place in the nonlinear time scale, but one has to resolve the much faster linear frequencies in order to reach these long times. When the phenomena under study involve the nonlinear interaction of only a few waves, an asymptotic analysis may filter the linear frequencies, leaving equations that are no longer stiff for the slow evolution of the wave amplitudes. When these interactions are more complex, however, and involve a wide range of linear modes, no such filtered models exist a priori and one has to solve numerically the full set of weakly nonlinear equations.

In this work, we describe an effective procedure to overcome stiffness and solve numerically a class of nonlinear wave equations occurring often in the description of waves in isotropic media. The method is based on the explicit analytical integration of the linear part of the equation, through an integrating factor. The idea of exactly integrating a stiff linear part has been developed before in various contexts. Hou, Lowengrub, and Shelley [6] used this idea to remove the stiffness arising from surface tension in interfacial flows; Rogallo [11] applied it to the numerical solution of the Navier–Stokes equations. For the Korteweg–de Vries (KdV) equation, the implicit pseudospectral methods of Chen and Kerkhoven [3] also aim to minimize the problems of stiffness encountered in the often used method by Fornberg and Whitham [4].

The isotropic wave equations with which we concern ourselves here admit a symmetric factorization which makes their numerical solution particularly efficient. This factorization reduces a second order real equation to a first order complex one, for which integrating factors are readily available.

This paper is structured as follows. In section 2, we describe the basic procedure for the numerical solution of a broad class of nonlinear waves. In section 3, we apply this procedure to the study of long surface waves, in the limit in which the dispersive and the nonlinear effects have the same order of magnitude. We simulate the propagation of a single solitary wave, the interaction of two solitary waves at small angles, the diffraction of a solitary wave past a strait or off the end of a topographic feature, and the emission of waves by a mean current flowing over topography. Finally, further applications and generalizations are proposed along with some concluding remarks.

2. The basic procedure. In this section, we describe the basic procedure as it applies to partial differential equations of the form

(1)
$$\left(\frac{\partial^2}{\partial t^2} + L^2\right)\phi = G(\phi, \phi_t, \nabla\phi),$$

where $\phi(x,t)$ is a real function, L^2 is a positive-definite operator normally involving spatial derivatives, and G is an arbitrary, not necessarily local function. The most frequent example for L^2 is minus the Laplacian, which turns (1) into a nonlinear wave equation. Other examples we are interested in are

$$L^{2} = (-\Delta)^{\frac{1}{2}} \tanh\left[(-\Delta)^{\frac{1}{2}}\right],$$

the general linear wave operator for surface waves in water of arbitrary depth, and

$$L^2 = -\Delta + f^2$$

which appears often in the study of geophysical waves. Here Δ is the two-dimensional Laplacian and f is the local Coriolis parameter $2\Omega \sin(\alpha)$, where Ω is the angular

velocity of the Earth and α is the local latitude. Other examples are presented in the next section.

The present procedure can be generalized easily to settings different from (1), such as systems of equations. For concreteness, however, we prefer to develop it in this particular context.

A difficulty usually encountered in numerically solving equations like (1) is stiffness: the operator L^2 often gives rise to a wide range of time scales, and one is normally interested in the much slower time scale associated with the comparatively small nonlinearity G. In order to study this slow evolution, however, one needs to resolve all the fast frequencies associated with L^2 . Doing this numerically is impractical. The algorithm presented here circumvents this difficulty by resolving the high frequencies analytically. We now describe the method as it applies to (1).

First we reduce (1) to a first order system. To this end, we introduce the new dependent variables

(2)
$$u = \left(\frac{\partial}{\partial t} + iL\right)\phi,$$
$$v = \left(\frac{\partial}{\partial t} - iL\right)\phi,$$

in terms of which (1) becomes

(3)
$$\left(\frac{\partial}{\partial t} - iL\right)u = G(\phi, \phi_t, \nabla\phi)$$

(4)
$$\left(\frac{\partial}{\partial t} + iL\right)v = G(\phi, \phi_t, \nabla\phi).$$

Notice that, under this symmetric factorization of the wave equation (1), since L is real, u and v are complex conjugates and (3) and (4) are actually the same equation. In addition, ϕ and ϕ_t can be computed from the real and imaginary part of u,

D ()

(5)
$$\phi_t = \operatorname{Re}(u), L\phi = \operatorname{Im}(u),$$

so the right-hand side of (3) can be written in terms of u. (When L is not fully invertible, some further work may be required; we will discuss this issue below.) Then, instead of the system (3), (4), one is left with the single complex-valued equation (3).

If the model equation is first order in time, it is already in the required form (3), and the same method applies. In this context, the method has been applied to a fifth order, forced KdV-like equation in [9]. Since that problem is fifth order, it is crucial to be able to circumvent the stiffness of the problem, as we will describe below. The procedure has also been applied in [8] to the study of turbulent cascades in a one-dimensional weakly nonlinear dispersive system. Here the stiffness was due to the necessity to resolve a range of scales wide enough so that a self-similar cascade could be sustained.

The next step involves resolving the stiffness associated with (3) through the exact integration of its linear part. How this is done depends strongly on the spatial domain, the boundary conditions, and the form of the operator L. We shall consider here the simplest case of a periodic domain—the torus T^n —and an operator L which

is diagonal in Fourier space, as are the examples presented above. We introduce $\hat{u}_i(t) = \hat{u}(k_i, t)$, the Fourier transform of u(x, t):

$$\hat{u}_j(t) = \mathcal{F}[u](k_j, t).$$

Taking the Fourier transform of (3) yields

(6)
$$\left(\frac{\partial}{\partial t} - i\hat{L}(k_j)\right)\hat{u}_j(t) = \mathcal{F}\left[G(\phi, \phi_t, \nabla\phi)\right](k_j, t),$$

which is a system of ordinary differential equations for the \hat{u}_j , where $\hat{L}(k_j)$ is the Fourier symbol of the operator L. Multiplying (6) by the integrating factor $e^{-i\hat{L}(k_j)(t-t_n)}$, one obtains

(7)
$$\frac{\mathrm{d}U_j}{\mathrm{d}t} = e^{-i\hat{L}(k_j)(t-t_n)}\mathcal{F}\left[G(\phi,\phi_t,\nabla\phi)\right](k_j,t),$$

where

8)
$$\hat{U}_j(t) = e^{-iL(k_j)(t-t_n)}\hat{u}_j(t).$$

The constants t_n will be useful when (7) is discretized in time. Including these constants in the integrating factor makes the algorithm look identical at each time step, so a number of necessary coefficients can be computed only once.

Now the equations in (7) form a system of ordinary differential equations coupled by the right-hand side, which normally involves convolutions of the \hat{U}_j . These convolutions are not computed directly but rather handled as products in physical space, a standard procedure in pseudospectral methods. Then the system (7) can be solved with any algorithm for integrating ordinary differential equations; we chose a fourth order Runge–Kutta method for the examples in the following section. Notice that if G = 0, then \hat{U}_j is time independent, and we recover the exact solution to (1).

How was the stiffness issue resolved? In considering (7) instead of (3), we have switched from a problem of *stability* to one of *accuracy*: the homogeneous solutions to (3) involve very high frequencies which, if not appropriately resolved with correspondingly small time intervals, yield catastrophic numerical instabilities. In (7) instead, the high frequencies have been resolved exactly and are now in the right-hand side, where they do not affect the stability of the method. It may appear that, by taking long time steps associated with the nonlinear time scale, this oscillatory right-hand side is not properly resolved. Yet this is not the case: we show below that when the nonlinear effects are important in (1), long time steps in (7) will be sufficient to resolve them. The only underresolved nonlinear effects are those that are unimportant in (1). Furthermore, these underresolved nonlinear terms will be averaged to zero by the method.

To see this, consider a general dispersive equation of the form

(9)
$$u_t + iL(u) = G(u)$$

with Fourier transform

(10)
$$\hat{u}_t + i\,\hat{L}(k)\,\hat{u} = \mathcal{F}(G(u))\,,$$

where $\hat{L}(k)$ grows rapidly with k. Introducing \hat{U}_i as in (8), we obtain

(11)
$$\frac{\mathrm{d}U_j}{\mathrm{d}t} = e^{-i\hat{L}(k_j)(t-t_n)}\mathcal{F}(G(u)).$$

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If G(u) is a linear function, with Fourier transform $\hat{G}(k)\hat{u}$, then the right-hand side of (11) takes the simple form $\hat{G}(k_j)\hat{U}_j$. In other words, the rapidly oscillating factor on the right-hand side cancels out, and our method has no accuracy problem. If instead G(u) is quadratic, with Fourier transform $\sum \hat{G}(k,l)\hat{u}_l\hat{u}_{k-l}$, the right-hand side of (11) takes the form

$$\sum_{l} \hat{G}(k,l) \hat{U}_{l} \hat{U}_{k-l} e^{i \left(\hat{L}_{l} + \hat{L}_{k-l} - \hat{L}_{k} \right) (t-t_{n})},$$

where we have simplified the notation by taking $\hat{L}(k_j) = \hat{L}_j$.

If the modes l, l-k, and k are in resonance, that is, if $\hat{L}_l + \hat{L}_{k-l} = \hat{L}_k$, then again the oscillations on the right-hand side cancel out; if they are close to resonant, the oscillations are slow. In either case, the methodology presented here has no accuracy problem when the timestep is long compared to \hat{L}_k . Thus, the method resolves the nonlinearities when they are important.

If the modes are far from resonant, their nonlinear interaction is very weak; thus by taking long time steps we under-resolve, and in a stable way, only unimportant nonlinearities. The only issue left to check is whether by under-resolving weak interactions, we are not making these interactions effectively stronger. This is the case only if the nonlinear terms are sampled at an integer multiple of their period. This can happen if the timestep Δt satisfies

$$\left(\hat{L}_l + \hat{L}_{k-l} - \hat{L}_k\right) \Delta t = 2n\pi \,,$$

where n is an integer. Then, the numerical method would make the modes l and k-l resonate with k, although they do not in the original equation. Even such fictitious resonances should not contaminate the solution: these would occur only to high-frequency modes with little energy and, if the equation is conservative, would only exchange energy conservatively between the modes. In our numerical experiments, we never observed the effects of such resonances. If they ever become an issue, simple and inexpensive solutions, such as randomly alternating between two irrational time-steps, may be devised. If the time step Δt does not satisfy this resonant condition, then it is easy to see that the underresolved terms average to zero.

The argument above extends straightforwardly to general nonlinearities. In fact, it can be made very precise; a more detailed discussion, however, goes beyond the scope of this article.

Returning to the basic procedure, in order to solve the system (7), one needs to compute the right-hand side in terms of the $\hat{U}_j(t)$. Clearly $\hat{u}_j(t)$ can be computed from (8). The computation of $\phi(x,t)$ from (5) is done in Fourier space, where one obtains $\hat{\phi}(k,t) = \mathcal{F}[\phi(x,t)]$ and $\hat{\phi}_t(k,t) = \mathcal{F}[\phi_t(x,t)]$ from the Fourier transform of (2):

$$\begin{split} \hat{u}(k,t) &= \hat{\phi}_t(k,t) + i\hat{L}(k)\,\hat{\phi}(k,t), \\ \hat{u}(-k,t) &= \hat{\phi}_t^*(k,t) + i\hat{L}(-k)\,\hat{\phi}^*(k,t) \end{split}$$

where we have used the reality of ϕ to replace $\hat{\phi}(-k,t)$ by $\hat{\phi}^*(k,t)$. Then

(12)
$$\hat{\phi}(k,t) = \frac{\hat{u}(k,t) - \hat{u}^*(-k,t)}{i\left(\hat{L}(k) + \hat{L}(-k)\right)}$$

(13)
$$\hat{\phi}_t(k,t) = \frac{\hat{u}(k,t) + \hat{u}^*(-k,t)}{2}.$$

Equation (13) determines all the modes of $\hat{\phi}_t$. However, the computation of $\hat{\phi}$ from (12) may be complicated by a lack of invertibility of L. For instance, when $L = -\Delta$, $\hat{\phi}(0,t)$ cannot be obtained from (12), and one needs to compute this mode separately. This is achieved by adding to the system of ordinary differential equations (7) for the $\hat{U}_j(t)$ another differential equation for $\hat{\phi}(0,t)$. This equation is obtained by evaluating (13) at k = 0:

(14)
$$\hat{\phi}_t(0,t) = \operatorname{Re}\left(\hat{u}(0,t)\right).$$

With this issue resolved, the description of the basic procedure is complete. In the next section, we apply it to examples of scientific interest, extending the procedure along the way to deal with situations slightly more general than the one presented in this section.

3. Examples: Long surface waves. Irrotational surface waves can be described in terms of two dependent variables: the potential function $\phi(x, y, z, t)$ and the surface height $\eta(x, y, t)$, measured for convenience as a departure from the mean level H corresponding to equilibrium. Here x and y are horizontal coordinates, z is the vertical coordinate, and t represents time. When the waves are long relative to the mean depth, the potential ϕ does not depend on z to leading order, i.e., the flow is essentially two-dimensional. If, in addition, the waves have small amplitude, their behavior over relatively small—O(1)—time intervals is well described by the wave equation

(15)
$$\Phi_{tt} - \Delta \Phi = 0,$$

where $\Phi(x, y, t) = \phi(x, y, H, t)$ and Δ stands for the two-dimensional (horizontal) Laplacian. The elevation of the free surface η is given to leading order by

(16)
$$\eta = -\Phi_t.$$

Over longer time intervals, however, the effects of nonlinearity and finite wavelength accumulate, affecting the shape of the solutions to (15) at a rate given by the strength of the nonlinearity and the dispersive terms. If we call ϵH the amplitude of the waves and H/μ the typical horizontal wavelength, the critical balance for which the nonlinear and the dispersive effects have the same order of magnitude is given by

$$\epsilon = \mu^2 \ll 1.$$

For one-dimensional waves, this is the regime described by the KdV and Boussinesq equations. The corresponding equation for isotropic waves in two dimensions is [2, 10]

(17)
$$\Phi_{tt} - \Delta \Phi = \epsilon \left[\frac{1}{3} \Delta^2 \Phi - \Phi_t \Delta \Phi - 2\nabla \Phi \nabla \Phi_t \right]$$

This equation contains the KdV and Boussinesq equations when restricted to onedimensional waves, and the Kadomtsev–Petviashvili (KP) equation as a limiting case in nearly one-dimensional situations. However, (17) is isotropic, which makes it suitable for the study of more general two-dimensional wave propagation. Notice that the leading order behavior of the solutions is given by the wave equation (15) and that the next order includes a linear dispersive correction, given by the square of the Laplacian, and two Burgers-like nonlinear terms. (They take exactly the form of the nonlinearity in Burgers if one follows a characteristic of the leading order wave equation.) Now, the free surface displacement is given by $\eta = -\Phi_t + O(\epsilon)$. In all the results presented, we plot $-\Phi_t(x, y)$, that is, the leading order approximation to the free surface at a fixed time.

To apply the procedure of the previous section to (17), we put all linear terms on the left-hand side. The problem with this, however, is that the operator

$$L^2 = -\left(\Delta + \frac{\epsilon}{3}\Delta^2\right).$$

which is positive definite for moderate wavenumbers, becomes negative for large wavenumbers scaling as $1/\sqrt{\epsilon}$. Not only does the numerical procedure fail in this case, but the equation itself becomes ill-posed. Thus something should be done about these short waves. Notice that (17) is derived under the assumption that the waves are long, so it cannot really model short waves. Thus we are free to modify the equation for short waves to make it well behaved. This is done by replacing $\Delta^2 \Phi$ by $\Delta \Phi_{tt}$, using the leading order wave equation for Φ . The $O(\epsilon^2)$ error that this introduces in (17) just adds to the truncation error already there. Thus (17) becomes

(18)
$$\Phi_{tt} - \Delta \Phi - \frac{\epsilon}{3} \Delta \Phi_{tt} = -\epsilon \left[\Phi_t \Delta \Phi + 2 \nabla \Phi \nabla \Phi_t \right],$$

which does not have the ill-posedness problem of (17). In order to get this equation into the form (1), we factor the left-hand side as

$$\left[\left(1 - \frac{\epsilon}{3}\Delta\right)\frac{\partial^2}{\partial t^2} - \Delta \right] \Phi = \left(1 - \frac{\epsilon}{3}\Delta\right) \left(\frac{\partial^2}{\partial t^2} - \frac{\Delta}{1 - \frac{\epsilon}{3}\Delta}\right) \Phi,$$

so (18) may be rewritten as

(19)
$$\left(\frac{\partial^2}{\partial t^2} - \frac{\Delta}{1 - \frac{\epsilon}{3}\Delta}\right)\Phi = -\frac{\epsilon}{1 - \frac{\epsilon}{3}\Delta}\Big[\Phi_t\Delta\Phi + 2\nabla\Phi\nabla\Phi_t\Big],$$

which has the required form (1). Notice that this equation is not only well-posed but also well-behaved for large wavenumbers, with L^2 approaching $3/\epsilon$ and the prefactor of the right-hand side approaching zero.

Still it may be convenient to filter the high wavenumbers of each factor of the right-hand side, since the differentiation of these terms creates large amplitudes that could require small computational time intervals. Since we are allowed to tamper with the wavenumbers \mathbf{k} of order $\epsilon^{-1/2}$, which lie beyond the range of validity of (17), a consistent filter is given by a function of $\sqrt{\epsilon}|\mathbf{k}|$. For example, we have tested

filter =
$$e^{-\alpha \left(\epsilon |\mathbf{k}|^2\right)^2}$$
,

where α is a constant. We multiply the Fourier transform of Φ used to compute the right-hand side of (19) by this filter. This corresponds to a dissipative term $\alpha (\epsilon \Delta)^2$ on the right-hand side of (17). For the experiments reported in this work, we found that filtering the high frequencies was not necessary.

Equations (17), (18), and (19) admit solitary waves in any direction, given by

(20)
$$\eta = -\Phi_t = A(|\mathbf{k}|) \operatorname{sech}^2(\mathbf{k} \cdot \mathbf{x} - C|\mathbf{k}|t),$$

where $A(|\mathbf{k}|)$ and $C(|\mathbf{k}|)$ are given, for (18), (19), by

$$C^2 = \frac{1}{1 - \epsilon \frac{4}{3} |\mathbf{k}|^2},$$



FIG. 1. Propagation of an oblique solitary wave, with parameters $\epsilon = 0.1$ and $\mathbf{k} = \frac{1}{\sqrt{5}}(2,1)$, in a grid with periods $L = 14\pi$ and $H = 28\pi$ and 256×256 points. (a) Initial data, t = 0. (b) t = 10. (c) t = 20. (d) Log-log plot of the normalized error after one tour of the periodic box, as a function of Δt .

(21)
$$A = \frac{4}{3}C^2 |\mathbf{k}|^2.$$

These solitary waves are the main building blocks for the numerical experiments described below. To fit these solitary waves in a periodic box, in principle they have to be replaced by the corresponding cnoidal waves. If the period of the numerical box is large enough, there is no perceptible difference between (20) and the cnoidal waves. We also need to make Φ corresponding to (20) periodic. This is achieved by writing

(22)
$$\Phi = \frac{A}{C|\mathbf{k}|} \tanh(\mathbf{k} \cdot \mathbf{x} - C|\mathbf{k}|t) + \mathbf{B} \cdot \mathbf{x}$$

and choosing the constant ${\bf B}$ such that Φ is periodic.

As a first test of the algorithm, we consider the case of a single solitary wave. Figure 1(a)–(c) shows the numerical evolution of an oblique solitary wave of the form (20), with $\epsilon = 0.1$ and $\mathbf{k} = \frac{1}{\sqrt{5}}(2, 1)$. The x and y periods of the box are given by $L = 14\pi$ and $H = 2L = 28\pi$, respectively, so that the initial data has support surrounding a main diagonal of the numerical domain. The numerical evolution of this data, with $N_x = 256$ and $N_y = 256$ modes in x and y, respectively, is virtually indistinguishable from the exact solution (20). Figure 1(d) shows the L^2 error of the



FIG. 2. Interaction of two solitary waves at a small angle. Each solitary wave has parameters $\epsilon = 0.1$ and $|\mathbf{k}| = 0.5$; they intersect at an angle of $2 \tan^{-1}(0.1)$. The grid has periods $L = 14\pi$ and $H = 140\pi$ and has 256×256 points. (a) Contours of the initial data, t = 0. (b) t = 100. (c) t = 250. (d) Detail of a perspective of the solution at t = 250.

numerical Φ_t after one tour of the periodic box, normalized by $|\Phi_t|_2$, as a function of the time interval Δt . We see that this error scales as $(\Delta t)^4$ and that, for a Δt as large as one, the relative error is about 10 percent, although a typical mode of the computation has a linear frequency close to $\omega = 5$, so we are sampling about one point per period. This is a clear indication that the stiffness of the problem has been duly resolved.

Our second experiment studies the interaction of two solitary waves (20) propagating at small angles to each other. The problem of steady traveling solutions to (17) depending on two phases $\mathbf{k}_j \cdot \mathbf{x} - C_j |\mathbf{k}_j| t$ was studied by Milewski and Keller in [10]. They showed that for large angles, solitary waves will not interact, and, to the approximation implicit in (17), they can be superposed linearly. For small angles between the directions of propagation (less than 40°), the waves interact strongly, and the linear superposition is no longer a solution to (17). In this case Milewski and Keller found steady spatially periodic solutions corresponding to a hexagonal traveling pattern. In an infinite domain the analogous solutions consist of two traveling solitary waves connected by one wave of larger amplitude, similar to a Mach stem in oblique shock reflection.

We present some solutions for the corresponding initial value problem: we study the evolution of initial data consisting of two solitary waves at small angles to each



FIG. 3. Edge diffraction of a truncated solitary wave. We solve 19 on a 128×256 grid (half of which is shown) with $\epsilon = 0.1$ and initial data corresponding to a truncated solitary wave 20 with $|\mathbf{k}| = 1.15$. (a) Initial data, t = 0. (b) t = 15. (c) t = 30.

other. We ran experiments with angles ranging from $\pi/2$, with almost no interaction between the solitary waves, to nearly glancing, with strongly nonlinear interaction. Figure 2(a)–(d) displays the results of a run with an angle of $2 \tan^{-1}(0.1) \sim 12^{\circ}$, where nonlinear effects are important over very long time intervals, thus testing the power of the algorithm most strongly. The initial data have $\epsilon = 0.1$ and |k| = 0.5; the periods in x and y are $L = 14\pi$ and $H = 140\pi$, respectively, and there are 256 grid points both in x and in y. In the plots, we see how the forward half of the two solitary waves detach from the backward halves, creating a single front with a lengthening, large amplitude, one-dimensional midsection (similar to a Mach stem). Because of the detachment of the backward halves of the solitons, it does not appear that this initial data will lead to the hexagonal patterns of [10].

In a third numerical experiment with (19), we studied a model problem corresponding to the diffraction of a solitary wave traveling parallel to a boundary which ends abruptly. Physical examples include waves past straits and waves diffracting off the end of topographical features. To avoid introducing nonperiodic boundary conditions, we took as initial data a solitary wave of the form (20) truncated at the ends with a fast cutoff. This is intended to represent the wave immediately after the passage of a strait and ignores the finite time it would take for a solitary wave to exit the strait. Physically, one could imagine that a solitary wave is traveling parallel to a boundary which is removed at t = 0. Figure 3 displays the numerical solution to (19) at various later times. (The actual computational domain was twice as large to allow for periodicity.) We can see the solitary wave propagating without change and a nearly circular refraction pattern emanating from the end of the truncated nonlinear wave. Since the solitary wave travels at a speed larger than any linear wave, a linear theory would not adequately describe the diffraction pattern off the end of the soliton.

Finally, we experimented with forcing effects on such weakly nonlinear evolution equations. A physical example of such a forcing are the waves generated by a flow with constant nonzero velocity U over a localized topographical feature, such as the ripples observed on the surface of water flowing over a bump. To make the equations isotropic, we will consider a frame moving with the mean velocity of the flow, so that the topography will in fact be moving with speed -U. The limit in which the topography is small $O(\epsilon^2)$ and the velocity U is close to 1 is termed resonant since it generates a relatively large, $O(\epsilon)$ response on the free surface. In this limit, the governing equation (17) is forced by the topography:

(23)
$$\Phi_{tt} - \Delta \Phi + \epsilon \left[-\frac{1}{3} \Delta^2 \Phi + \Phi_t \Delta \Phi + 2\nabla \Phi \nabla \Phi_t \right] = \epsilon U H_x (x + Ut, y).$$

The depth of the fluid layer is given by $1 + \epsilon^2 H(x, y)$. Equation (23) can be cast in the form of (19) and the forcing can be treated either exactly with the linear part of the equation or together with the nonlinear terms. In the results that follow we have used the latter method, for simplicity. Figure 4 presents typical results for a small localized bump $H(x, y) = -\operatorname{sech}^2((\alpha(x - x_0))^2 + (\beta(y - y_0))^2)$ and U = -1. In Figure 4(b)–(d) there are two bumps at x = 50, centered approximately under the large troughs of the free surface; the bumps are shown in Figure 4(a). In Figure 4(e), there is only one bump. Again, nonlinear effects are evident: in linear theory, there should be no disturbance ahead of the bumps but, nonlinearly, we see the generation of threedimensional nonlinear bow waves upstream. (For a discussion of related phenomena in the KdV and KP equations, see [5, 7].) It is also interesting to note the nonlinear interaction of the waves from one bump when they meet waves generated by the other bump in Figure 4(a)-(c). Nonlinearity inhibits the focusing and a large amplitude plane wave is generated, together with hexagonal-like patterns discussed in [10] (see Figure 4(c), $x \approx 60, y \approx 100$). In the wake of the topography one sees a wave pattern similar to that of the Kelvin wake.

4. Conclusions. A general procedure has been presented for the numerical solution of a large class of nonlinear wave equations. This procedure uses a symmetric factorization to reduce a second order real equation to a first order complex one and uses an integrating factor in Fourier space to solve the linear part of the equation exactly. The resulting system of ordinary differential equations for the Fourier modes of the solution is not stiff and can be solved by any standard pseudospectral method.

Applications have been presented to the study of long surface waves, their diffraction and their interaction with a bottom topography. Further applications that are currently being developed include the effects of rotation and stratification on geophysical flows and the interaction of waves with topographic features of a magnitude comparable to the waves themselves.

Some of these applications require slight generalizations of the procedure developed here. In particular, the Fourier modes have to be replaced by more general eigenfunctions of the linear operator either when this does not have constant coefficients, as is the case of the latitude-dependent Coriolis force, or when boundary conditions

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FIG. 4. Generation of surface waves by a flow over topography. We solve 23 with $H(x,y) = -\operatorname{sech}^2((0.3x)^2 + (0.15y)^2)$ and U = -1 on a 128×128 grid for (b)-(d) (two copies of the domain are shown to highlight the interaction of the waves from two adjacent bumps centered at y = 50, 150), and on a 128×256 grid for e, with a domain twice as large to minimize the wave interactions from the periodic array of bumps centered at y = 100). (a) Bottom topography for figures (b)-(e). (b) t = 100.5. (c) t = 201.1. (d) Grayscale plot for t = 301.6. (e) Grayscale plot for t = 301.6.

other than periodic are required, such as decay at infinity or no-flow through rigid boundaries. Another straightforward generalization involves working with systems instead of a single wave equation. This latter case is important in the study of stably stratified flows and of interactions between rotational and irrotational components of free surface fluid flows.

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