Inverse scattering for lossy medium material

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Abstract. We address the problem of the recovery
cients, the index of refraction $n(x)$ and the dissipat
$m(x)$, for the equation

$$\phi''(x) + k^2 n(x)\phi(x) + ik \cdot m(x)\phi(x) = 0, \quad x \in$$

and present an accurate, efficient, and stable numeri
the reconstruction of m(x) and n(x) from the scatte
ments.

Plan of the Talk

- Background and applications
- Introduction to 1-D inverse scattering problems
- Existing results and our new results
- Analytical tools for our inverse problem
- The inversion algorithm
- Numerical performance
- Discussion

Background and Applications

$$\phi''(x) + k^2 n(x)\phi(x) + ik \cdot m(x)\phi(x) = 0, \quad x \in$$

- Essentially a Helmholtz equation (1-D)

- Propagation of the time-harmonic waves through in
medium located inside $[a, b]$. Outside $[a, b]$, $n = 1$ and

- Reducible from Maxwell's equation in a 3-D layered

- Maxwell's case: $n(x)$ is permittivity, $m(x)$ is conduc

- Dissipative: $m(x) \geq 0$, solutions decay.

- Active material $m(x) \leq 0$, solutions grow. For neu
$m < 0$ could be quite large; ask Enrico Fermi and D
about their applications.

Introduction to 1-D Inverse Scattering Problems

- Inverse problem: Given solutions of differential eq
$[a, b]$, determine the coefficients of the equation.

- Inverse scattering is the inverse problem for wave e

- A general 1-D wave equation as a motivation:

$$u_{tt}(x,t) + \beta(x) u_t(x,t) = c^2(x)\rho(x) \left[\frac{1}{\rho(x)} u_x\right]_x, \ \ c(x) =$$

- By the time-harmonic substitution $u(x,t) = \phi(x)e^{-}$

$$\phi''(x) + \ell(x)\phi'(x) + k^2 n(x)\phi(x) + ik \cdot m(x)\phi($$

with three real coefficients

$$\ell(x) = -\frac{\rho'(x)}{\rho(x)}, \quad n(x) = \frac{c_0^2}{c^2(x)}, \quad m(x) = \frac{\beta}{c^2}$$

1-D Inverse Scattering Problems

- A simple Helmholtz equation (self-adjoint), already

$$\phi''(x) + k^2 n(x)\phi(x) = 0$$

- More complicated – two coefficients, and non-self-a

$$\phi''(x) + k^2 n(x)\phi(x) + ik \cdot m(x)\phi(x) = 0$$

↪ the subject of this talk

- Still more complicated – three coefficients, yet to b

$$\phi''(x) + \ell(x)\phi'(x) + k^2 n(x)\phi(x) + ik \cdot m(x)\phi(x$$

- A simple 1-D *forward* scattering problem

$$\phi''(x) + k^2 n(x)\phi(x) = 0, \quad x \in [a, b]$$

- k − wave number, a positive number in $(0, \infty)$
- n − index of refraction, $n(x) = 1 + q(x)$, $q = 0$ c
- ϕ − total wave field, $\phi(x) = \phi_0(x) + \psi(x)$,

- Only two possible incident wave fields: $\phi_0(x, k) =$
  $\hookrightarrow$ two corresponding scattered fields: $\psi_\pm(x, k)$, sa

$$\psi''(x) + k^2(1 + q(x))\psi(x) = -k^2 q(x)\phi_0(x), \quad x$$

subject to the outgoing radiation conditions

$\psi'(a) + ik\psi(a) = 0, \quad \psi'(b) - ik\psi(b) = 0 \quad$ (a third

Forward and Inverse Scattering Problems

$$\psi''(x,k) + k^2(1+q(x))\psi(x) = -k^2 q(x)\phi_0(x,k),$$
$$\psi'(a) + ik\psi(a) = 0, \quad \psi'(b) - ik\psi(b) = 0$$

- Forward problem: Given $k, q, \phi_0(x,k)$, determine $\psi($
 $\hookrightarrow$ the forward problem is well-posed.

- Inverse problem: Given $\{\ \psi_\pm(a,k), \psi_\pm(b,k), k \in (0,\infty$
$\{\ q(x), x \in [a,b]\ \}$
 $\hookrightarrow$ the inverse problem is also well-posed (John Syl

Remark: Only one of the two functions–the two re
$\psi_+(a,k), \psi_-(b,k)$–is required to recover the scatterer

Generalization: The scatterer q may have imaginary

Scattering Data and Scattering Matrices

- For the general, three-coefficient, equation

$$\psi''(x,k) + \ell(x)\psi' + k^2(1 + q(x)\psi + ik \cdot m(x)\psi = -$$

all four measurements $\{ \ \psi_\pm(a,k), \psi_\pm(b,k), \quad k \in (0,$
used to recover the three coefficients $\{ \ \ell(x), q(x), m($

- Scattering matrix: organize the four functions an
algebraic and analytic properties for scattering proble

$$S(a,b,k) = \begin{bmatrix} \psi_+(a,k) & \psi_-(a,k) \\ \psi_+(b,k) & \psi_-(b,k) \end{bmatrix}$$

- A better definition for a proper scaling

$$S(a,b,k) = \begin{bmatrix} \psi_+(a,k)e^{ika} & \psi_-(a,k)e^{ika} \\ \psi_+(b,k)e^{-ikb} & \psi_-(b,k)e^{-ikb} \end{bmatrix} =: \Big[$$

- Inverse scattering: $\{S(a,b,k), \ x \in (0,\infty)\} \longrightarrow \{\ell, q,$

- Trace formula method for inversion: construct a sy

$$\begin{aligned} S'(x,b,k) &= F_0(S,k,\ell,q,m), \text{ for all } k \\ \ell'(x) &= F_1(\int S\ dk;\ \ell,q,m), \\ q'(x) &= F_2(\int S\ dk;\ \ell,q,m), \\ m'(x) &= F_3(\int S\ dk;\ \ell,q,m), \end{aligned}$$

and solve them from a to b with the initial values

$S(a,b,k)$ – scattering data, and $\ell(a) = q(a) = m$

- Frequency-global, space-local; ODEs amount to lin

Existing Results and Our New Results

- The simple inverse scattering problem

$$\psi''(x) + k^2(1 + q(x))\psi(x) = -k^2 q(x)\phi_0(x$$

Chen and Rokhlin (1991): Discovered a trace forn struct the scatterer $q \in C^l(\mathbb{R}^1)$ from scattering data { $(0, A]$ } with precision $O(1/A^l)$.

- The resulting algorithm is known as the most accu and stable scheme.

- Technique: Use symmetry and gain super-algebrai for smooth scatterer q.

Existing Results and Our New Results

- The two-coefficient equation

$$\psi''(x,k) + \ell(x)\psi' + k^2(1 + q(x)\psi + ik \cdot m(x)\psi = -$$

The only result (J. O. Powell, 1999) uses a first order
and the algorithm is unstable.

- Difficulty: the equation is no longer self-adjoint; s
longer present here.

- Our results: We obtained parallel results to those c

Analytical Tools: 1. Riccati Equations

- As is well-known

- A linear, scalar, and one-dimensional elliptic (diffe
  tion leads to a scalar Riccati equation.
- A linear, one-dimensional system of elliptic equat
  matrix Riccati equation.

- It turns out that the scattering matrices $S^l(x) =$
$S^r(x) = S(x, b, k)$ satisfy the matrix Riccati equatio
$q - m/ik)$

$$\frac{dS^l}{dx} = \frac{ik}{2}\Big\{q(x)(E_2^l + S^l J_1)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}(J_1^* S^l + E_2^l) + \Big[$$

$$\frac{dS^r}{dx} = -\frac{ik}{2}\Big\{q(x)(E_1^r + S^r J_1^*)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}(J_1 S^r + E_1^r) +$$

- Entry by entry, the equations are (with $q \leftarrow q - m/i$

$$
\begin{aligned}
\frac{dS^l_{22}}{dx} &= \frac{ik}{2}[q(x)(1+S^l_{22})^2 + 4S^l_{22}], \\
\frac{dS^l_{12}}{dx} &= \frac{ik}{2}[q(x)(1+S^l_{22})(e^{ik(x-a)} + S^l_{12}) + \\
\frac{dS^l_{21}}{dx} &= \frac{ik}{2}[q(x)(1+S^l_{22})(e^{ik(x-a)} + S^l_{21}) + \\
\frac{dS^l_{11}}{dx} &= \frac{ik}{2}q(x)(e^{ik(x-a)} + S^l_{12})(e^{ik(x-a)} + S^l_{2}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{dS^r_{11}}{dx} &= -\frac{ik}{2}[q(x)(1+S^r_{11})^2 + 4S^r_{11}], \\
\frac{dS^r_{12}}{dx} &= -\frac{ik}{2}[q(x)(1+S^r_{11})(e^{ik(b-x)} + S^r_{12}) + \\
\frac{dS^r_{21}}{dx} &= -\frac{ik}{2}[q(x)(1+S^r_{11})(e^{ik(b-x)} + S^r_{21}) + \\
\frac{dS^r_{22}}{dx} &= -\frac{ik}{2}q(x)(e^{ik(b-x)} + S^r_{12})(e^{ik(b-x)} + S
\end{aligned}
$$

Analytical Tools: 2. WKBJ Expansions

- The well-posedness of the Riccati equations were es
therefore can write down the asymptotics for large fr

$$S^l_{22}(x,k) = \frac{1-\sqrt{n}}{1+\sqrt{n}} + \frac{1}{2n(1+\sqrt{n})^2}\left[-q' + 2m\sqrt{n}\right]\frac{1}{ik}$$

$$S^r_{11}(x,k) = \frac{1-\sqrt{n}}{1+\sqrt{n}} + \frac{1}{2n(1+\sqrt{n})^2}\left[+q' + 2m\sqrt{n}\right]\frac{1}{ik}$$

where,

$$n(x) = 1 + q(x)$$

$$S^{2,l}_{22} = \frac{\frac{dS^{1,l}_{22}}{dx} + m\left(1+S^{0,l}_{22}\right)S^{1,l}_{22} - \frac{i}{2}q\left(S^{1,l}_{22}\right.}{i\left[q\left(1+S^{0,l}_{22}\right)+2\right]}$$

$$S^{2,r}_{11} = \frac{\frac{dS^{1,r}_{11}}{dx} - m\left(1+S^{0,r}_{11}\right)S^{1,r}_{11} + \frac{i}{2}q\left(S^{1,r}_{11}\right.}{-i\left[q\left(1+S^{0,r}_{11}\right)+2\right]}$$

Analytical Tools: 3. Trace Formulae

- For $q(x)$:

$$q' = \frac{1}{\pi}(1+q)(1+\sqrt{1+q})^2 \times \lim_{A\to+\infty}\int_{-A}^{A}(S_{22}^{l}(x,k) -$$

- For $m(x)$:

$$m = \frac{1}{2}\sqrt{1+q}\,(1+\sqrt{1+q})^2 \times \lim_{A\to+\infty}\frac{1}{2A}\int_{-A}^{A} ik(S_{22}^{l}(x$$

- Neither stable nor of high order for finite A.

Analytical Tools: 4. Active Material

- Want to create symmetry by using some other equ

$$\psi''(x,k) + \ell(x)\psi' + k^2(1 + q(x)\psi + ik \cdot m(x)\psi \;=\; -$$
$$\psi''(x,k) + \ell(x)\psi' + k^2(1 + q(x)\psi - ik \cdot m(x)\psi \;=\; -$$

- There were two scattering matrices $S^l(x) = S(a,x,k$
$S(x,b,k)$

- There are now four scattering matrices $S^{\pm l}(x)$, $S^{\pm r}$
tains a WKBJ expansion:

$$S^{\pm l}_{22}(x,k) \;=\; \frac{1-\sqrt{n}}{1+\sqrt{n}} + \frac{1}{2n(1+\sqrt{n})^2}\left[-q' \pm 2m\sqrt{n}\right]\frac{1}{ik}$$
$$S^{\pm r}_{11}(x,k) \;=\; \frac{1-\sqrt{n}}{1+\sqrt{n}} + \frac{1}{2n(1+\sqrt{n})^2}\left[+q' \pm 2m\sqrt{n}\right]\frac{1}{ik}$$

where,

$$n(x) = 1 + q(x)$$

$$S_{22}^{2,l} = \frac{\frac{dS_{22}^{1,l}}{dx} \pm m\left(1 + S_{22}^{0,l}\right) S_{22}^{1,l} - \frac{i}{2} q \left(S_{22}^{1,l}\right.}{i\left[q\left(1 + S_{22}^{0,l}\right) + 2\right]}$$

$$S_{11}^{2,r} = \frac{\frac{dS_{11}^{1,r}}{dx} \mp m\left(1 + S_{11}^{0,r}\right) S_{11}^{1,r} + \frac{i}{2} q \left(S_{11}^{1,\right.}{-i\left[q\left(1 + S_{11}^{0,r}\right) + 2\right]}$$

- Symmetry: It is easy to show that $S_{22}^{+l} + S_{22}^{-l}$ and the jugate of $S_{11}^{+r} - S_{11}^{-r}$ have identical WKBJ expansio n and m on the line $\mathbb{R}^1$, just as in the classical $S_{22}^{+l} = S_{22}^{-l} = S_{22}^{l}$, and $S_{11}^{+r} = S_{11}^{-r} = S_{11}^{r}$) where t tion coefficients satisfy $S_{22}^{l} = \overline{S_{11}^{r}}$.

Analytical Tools: 4. New Trace Formulae

The newly established symmetry can be used in a s
way to yield trace formulae:

- For $q(x)$,

$$q' = \frac{1}{\pi}(1+q)(1+\sqrt{1+q})^2 \times \int_{-\infty}^{\infty}(S_{22}^{+l} + S_{22}^{-l} - S_{11}^{+r} -$$

- For $m(x)$,

$$m = \frac{1}{2}\sqrt{1+q}\,(1+\sqrt{1+q})^2 \times \int_{-\infty}^{\infty}(S_{22}^{+l} + S_{11}^{+r} - S_{22}^{-l} -$$

- Stable and super-algebraic convergent

Analytical Tools: 5. Merging and Conjugate Operati

- Remember only the scattering matrix $S^{+r}(a,k)$ is
scattering data, which is to be used as the inital value
equaiton for $S^{+r}(x,k)$.

- Fortunately, the other three scattering matrices S^{+l}
can be obtained with the merging and conjugate ope

- Merging is useful to calculate the scattering matr
chunk $[a,x]$ from that for the right chunk $[x,b]$; name
S^{+l} from S^{+r}, and S^{-l} from S^{-r}.

- The conjugate operation is useful to obtain the scatt
for the active material from the those of the passive

Analytical Tools: 5.1. Merging Operation

- The vector form of the formula:

$$S = E_2 S^r E_2 +$$
$$(E_1 + E_2 S^r J_2)\left\{S^l + \frac{S^r_{11}}{1 - S^r_{11} S^l_{22}} \begin{bmatrix} S^l_{12} \\ S^l_{22} \end{bmatrix} \begin{bmatrix} S^l_{12} & S^l_{22} \end{bmatrix}\right\}(J$$

where $E_1,\ E_2,\ J_2$ are some 2-by-2 constant matrices

- The explicit, scalar form of the formula:

$$S^l_{22} = \frac{S^r_{22} - S_{22}}{S^r_{11}(S^r_{22} - S_{22}) - (S^r_{12} + e^{ik(b-x)})^2},$$

$$S^l_{12} = \frac{1 - S^r_{11} S^l_{22}}{(S^r_{12} + e^{ik(b-x)})^2} \times$$
$$[(S^r_{12} + e^{ik(b-x)})(S_{12} - e^{ik(x-a)} S^r_{12}) - S^r_{11}(S_{22} - S^r_{2}$$
$$= \frac{(S^r_{12} + e^{ik(b-x)})(e^{ik(x-a)} S^r_{12} - S_{12}) - S^r_{11}(S^r_{22} - S_{22}) e^{ik}}{S^r_{11}(S^r_{22} - S_{22}) - (S^r_{12} + e^{ik(b-x)})^2}$$

$$S^l_{11} = S_{11} - (S^l_{12} + e^{ik(x-a)})^2 \frac{S^r_{11}}{1 - S^r_{11} S^l_{22}}.$$

Analytical Tools: 5.2. Conjugate Operation

The conjugate operation is quite simple, and is a dire
an examination of the Wronskians of the two equ
passive and active media.

$$S^{+l}(x,k) \cdot (S^{-l}(x,k))^* = I; \quad S^{+r}(x,k) \cdot (S^{-r}(x,$$

provided that $S^{-l}(x,k)$ and $S^{-r}(x,k)$ exist.

- The well-posedness of the active medium problem
only for small m.

The Inversion Algorithm

Solve the inital value problem for q and $S^{+r}(x,k)$ for
and for each of the positive frequencies $k \in (0,\infty)$

$$\frac{dS^{+r}}{dx} = -\frac{ik}{2}\Big\{(q - m/ik)(E_1^r + S^{+r}J_1^*)\left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right](J_1S$$

$$+\left[\begin{array}{cc} 4S_{22}^{+r} & 2S_{12}^{+r} \\ 2S_{21}^{+r} & 0 \end{array}\right]\Big\},$$

$$q' = \frac{1}{\pi}(1+q)(1+\sqrt{1+q})^2 \times \int_{-\infty}^{\infty}(S_{22}^{+l} + S_{22}^{-l} - S_{11}^{+r}$$

$$m = \frac{1}{2}\sqrt{1+q}\,(1+\sqrt{1+q})^2 \times \int_{-\infty}^{\infty}(S_{22}^{+l} + S_{11}^{+r} - S_{22}^{-l}$$

with the initial values $S^{+r}(a,k),\ q(a) = m(a) = 0$.

- The required entries $S_{22}^{+l},\ S_{22}^{-l},\ S_{11}^{-r}$ can be obtained
the merging and conjugate opertions.

Discussions and Conclusions

- It seems that no one has been able to contructed t
of some intermediate order: from second order and
have first order or super-algebraic convergence.

- Owing to the use of the active material, there is a
the magnitude of the dissipative term m. The trace r
work for large m.

- Trace method is a special case of the so-called space-
global approach. It is not a flexible method in the
whelming analytical and algebraic requirements of fun
in the formulae.

- Space-local, frequency-global approaches v.s. Space
local
  $\hookrightarrow$ different ways to linearize

- SLFG is always well-posed, easier to analyze, but c
cretize x in high order; inefficient in utilizing the sc
unable to recover discontinuous coefficients withou
shooting.

- SGFL is ill-posed, difficult to analyze, but there is n
instability in actual computation; efficient in utilizing
data; easy to construct high order schemes; convenier
discontinuous coefficients

- SGFL is the choice for accurate and reliable algo
inversion

- SGFL is expected to work for the inversion of n a
m.