

CREDIT CONTAGION: PRICING CROSS-COUNTRY RISK IN BRADY DEBT MARKETS

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ABSTRACT. Credit contagion means that the credit deterioration of an entity causes the credit deterioration of other entities. In this paper, we build and test a continuous-time model for defaultable securities using a diffusive process for risk-free interest rate, and a finite-state continuous-time Markov process for the correlation of credit. The credit contagion, in particular, is established by relating transition rates of various credit states. Examples of derivative pricing with calibrated credit contagion model are provided. Initial empirical results with the benchmarks of Brady bonds show that our model is a viable new technique for the pricing and risk-managing of credit derivatives.

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1. INTRODUCTION

Credit contagion means that the credit deterioration of an entity causes the credit deterioration of other entities. One example of such contagion was the greater depreciation of many South American sovereignty debts in contrast to that of benchmark Brazilian *C*-bond during the summer financial crisis of 1998, when the former were considered being brought down by the benchmark bond. Credit contagion is of course one source of credit risk. In existing models, credit correlation is modeled with correlated diffusive processes for either spreads or firm values. As a result, the credit correlation built in in these models is symmetric. Yet credit contagion is often non-symmetric, in the sense that the default of entity A is likely to cause the simultaneous default of entity B, but not vice versa. Therefore, there are great interests in the markets for credit risk models which can cope with credit contagion.

The option pricing models (OPM) for defaultable securities have been following two different approaches. The first approach takes the firm's liability as contingent claims against the underlying asset. This approach was introduced by Merton(1974). In this approach, bankruptcy and bond non-payment occurs when the firm's assets drop below some pre-specified level, and the value of defaultable bonds are governed by diffusive-type partial differential equations with state variables to be the firm value and spot interest rate. The equations can be solved numerically or even analytically in some special cases. Important extensions and variants of Merton's model include those due to Black and Cox(1976), Shimko, Teejima and van Deventer(1993), Kim et al(1993), Hull and White(1991), and Longstaff and Schwartz(1992). These models take more tractable bankruptcy conditions, more general interest rate processes, or more general underlying firm value processes like jump-diffusion. The use of firm value is intuitively appealing, but it is also a major disadvantage for Merton's approach, because the firm value is not tradable and is only partially observable. Also, in actual applications one has to deal with the often complex priority structure of a firm's liabilities. The popular commercial "EDF" measure (Kealhofer, 1995) for default probability supplied by KMV Corporation was intended to estimate the firm value and offer prediction on credit migration.

The second approach specifies the credit spread as an exogenous stochastic process which does not explicitly depend on the firm's underlying assets. The advantage is that it allows exogenous assumptions to be imposed only on observable variables, and, as justified

by Duffie and Singleton(1994), Heath-Jarrow-Morton's(1992) framework for pricing risk-free rate derivatives extends naturally to that of risky rate derivatives. Such extension initiated a lot of studies on the term structure of credit spreads. Models of this approach include Artzner and Delbaen(1995), Duffie, Schroder and Skiadas(1996), Jarrow and Turnbull(1995), Lando(1994), Madan and Ural(1993), Nielsen and Ronn(1995) and others. Most of the models are developed for a single default event. Some of the models are extendable to multiple default events (Duffie and Singleton, 1999) like, for example, swaps with defaultable counterparties. On top of the risk-free short rate process, Jarrow, Lando and Turnbull(1997) introduced finite state Markov process to model the effect of credit rating (or migration) on defaultable bonds.

The new approach introduced in this paper is related to the model by Jarrow, Lando and Turnbull(1997), yet it is particularly targeted at credit contagion. We propose to treat correlated securities as a basket, identify possible states of single or multiple defaults, and, on top of a diffusion process for the spot interest rate, use a finite-state continuous-time Markov process to model defaults. We model the contagion of credit by some functional relation between various transition rates of the Markov process. The functional forms between the transition rates will be estimated with some methods of nonlinear regression. In formalism, our model bears some similarity to the model of Jarrow et al(1995) for corporate bonds with different credit ratings, but in their model the transition rates are conceptually independent of each other.

This model will be useful in debt markets where there is no or very little credit ratings, yet there exists strong price correlation. Such markets include some sector of corporate debts and some regional sovereign bond markets. In these markets, the liquidity is concentrated on some "benchmark" issues, yet the illiquid issues typically offer better return (or higher yields). A well-known trade is to long the high-yield illiquid bonds and hedge with low-yield liquid bonds. When the market is in turmoil and credit contagion occurs, hedging with existing models becomes unreliable and heavy losses can be generated.

In this paper, we limit ourselves to non-option instruments (which can be decomposed into a portfolio of zero-coupon defaultable bonds) and focus on a "reduced" model. Under the assumption that the interest-rate process and defaultable process are uncorrelated, the diffusion process for risk-free interest rate can be replaced by the spot forward rate curve. Consequently, the continuous-time model is reduced to a system of ordinary differential equations. With a prototype bond basket consisting of Argentina PAR bond and Brazilian

C-bond, we develop some technique to estimate the “contagion coefficient”. Finally, we will demonstrate the pricing and sensitivity calculation of default protection note and swaps using the calibrated model.

The paper is organized as follows. In § 2 we introduce the state-space representation of default states. Our continuous-time model is developed in § 3 in a general setting. The reduced model is put forward in § 4. In § 5 we discuss the estimation of contagion coefficients. In § 6 we develop the valuation techniques for prices and sensitivity parameters for default protection note and swaps. Finally in § 7 we conclude the paper.

2. STATE-SPACE REPRESENTATION

We consider a basket of N defaultable securities(henceforth refereed to as “bonds”). The **default states** associated with the default of one or several bonds in the basket will be identified with the subsets of $\{\phi, 1, 2, 3, \dots, N-1\}$. Here, $\{\phi\}$ is an empty set, corresponding to the state of no-default. Accordingly, when the system is in state S , the bonds with indices $i \in S$ have defaulted and the bonds with i not belonging to S have not. Default states can also be visualized in the “occupation” representation, i.e, as arrays of zeros and ones. We then write

$$S = (s_1, s_2, \dots, s_N)$$

where $s_j = 1$ iff $j \in S$. Lets take the basket of an Argentinean bond(“1”) and a Brazilian bond(“2”) for example. The default states are

$$\{\phi\}, \{1\}, \{2\}, \text{ and } \{1, 2\}. \tag{1}$$

The corresponding “occupation” representation of these states are

$$\{00\}, \{10\}, \{01\}, \text{ and } \{11\}.$$

Later we will refer the above states as state 1, 2, 3 and 4 accordingly.

We describe the underlying dynamics of default events as a finite-state Markov process. The dynamics is completely determined once we specify the transition rates between different states. Transition rates are calibrated to fit the market prices, spreads and our expectations about the correlation between default events for different bonds.

Since default events are “irreversible”, a transition from state S_1 to S_2 can happen only if $S_1 \leq S_2$. This gives the set of all states a partially ordered structure from which it is easy to show that the maximum possible number of non-zero transition rates is $3^N - 2^N$.

In practice, we can selectively rule out some states that we believe are not reachable in the time horizon of interest, in order to reduce the number of free parameters and make the implementation more efficient.

Notice that the extreme case of 1) independence of default events and 2) extreme correlation of default events (the default of bond i implies the default of bond j) leads to simplification of the general framework. In the next section, we provide the basic premises for multiple-default models.

3. MODEL WITH STOCHASTIC INTEREST RATE PROCESS

In the general setting the random dynamics driving the prices of defaultable debt-instruments consists of the interest rate dynamics and default dynamics. We assume that the interest-rate component of the model is described by a diffusion process with infinitesimal generator

$$L = \sum_{ij} a_{ij} D_{ij} + \sum_i \mu_i D_i \quad (2)$$

and that interest rates are functions of the state-variables (in practice, the model will be exponential-affine). The short rate is denoted by r .

The finite-state Markov process is specified by its generator matrix, namely, Λ , of transition rates between states. The (i, j) entry of Λ , i.e. λ_{ij} , has the meaning given by the equation

$$Prob\{\text{State } i \text{ at time } t + dt \mid \text{State } j \text{ at time } t\} = \lambda_{ij} dt + o(dt) \quad (3)$$

Note that λ_{ij} must be non-negative for all i, j and

$$\lambda_{ii} = - \sum_{j \neq i} \lambda_{ij} \quad \text{for } i = 0, \dots, N \quad (4)$$

In Figure 1 we display the default diagram for a basket with 2 bonds. Notice that we must specify 5 = 3² - 2² transition rates.

Given that the system is in state i , we label the value of a defaultable security by v_i . The values of the security for all states therefore are represented by a vector $V = (v_0, v_1, \dots, v_N)^T$.

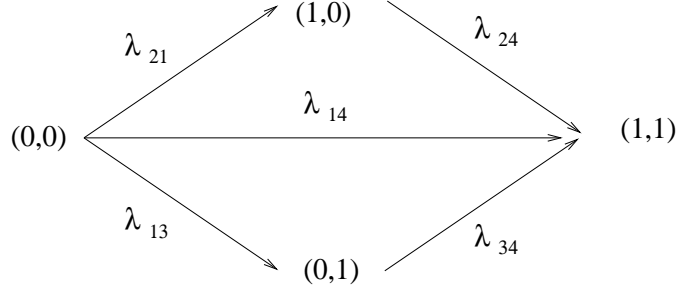


FIGURE 1. Default diagram

The evolution equations for V have the general form

$$V_i + LV + \Lambda V - rV = 0. \quad (5)$$

Individually the equations read

$$\frac{\partial v_i}{\partial t} + \sum_{ij} a_{ij} D_{ij} v_i + \sum_i \mu_i D_i v_i + \sum_j \lambda_{ij} v_j - r v_i = 0, \quad (6)$$

$$i = 0, 1, \dots, N.$$

The main feature of the above approach is that it provides a framework for us to incorporate the correlation of defaults into modeling. Such correlation can be crucial for us to understand the relative performance of bond prices. From the view point of calibration, this feature leads to the reduction of the number of free parameters, and makes calibration amenable. Let us illustrate such feature with the Argentina/Brazil bond basket, which has four states listed in (1). In this case,

$$\lambda_{12} dt = Prob\{\text{Argentina defaults at time } t + dt \mid \text{Brazil has not defaulted at time } t\}, \quad (7)$$

and

$$\lambda_{34} dt = Prob\{\text{Argentina defaults at time } t + dt \mid \text{Brazil has defaulted at time } t\} \quad (8)$$

If one believes that the probability of Argentinean default becomes much higher in case of that Brazil has already defaulted, then, instead of specifying λ_{12} and λ_{24} individually, one can relate λ_{34} to λ_{12} by

$$\lambda_{34} = \alpha \lambda_{12}, \quad (9)$$

for some appropriately chosen α which is bigger than 1. Similarly, one can relate λ_{13} and λ_{24} by

$$\lambda_{24} = \beta \lambda_{13}, \quad \text{with some } \beta \quad (10)$$

As the result, λ_{24} and λ_{34} are no longer free parameters. Here, α and β are named as *contagion coefficients* that are defined exogeneously. Furthermore, without a great loss of generality we can put $\lambda_{14} = 0$, i.e., zero possibility for simultaneous default. In fact, the simultaneous default can be modeled by using large α or β . If we take $\alpha = \infty$, then we mean that Argentinean default becomes immediate with probability 1 once Brazil has defaulted. On the other hand, if there is no correlation between the default events of individual securities, we simply put $\alpha = \beta = 1$. Note that α and β do not have to be the same. That is, the impact of one default on the default probability of the other needs not be symmetry.

4. MODEL WITH STOCHASTIC INTEREST RATE PROCESS

If the following conditions hold our model can be simplified substantially: 1) the interest-rate process and defaultable process are uncorrelated; and 2) the securities for calibration and the security to be priced are all straight debt instruments(which can be decomposed into a portfolio of zero-coupon defaultable bonds). Under the two conditions, one can prove that the infinitesimal generator of the interest rate process together with short-rate discounting term can be replaced by a linear zero-order term which represents the discounting with the forward rate of U.S. treasury, and equation (6) becomes

$$V_t - fV + \Lambda V = 0, \quad (11)$$

where the scalar term $f = f(0; t)$ is the spot forward rate of U.S. treasury. Once the state-dependent cash flows of a security are specified, we can obtain a price by solving the above ordinary differential equations. Clearly, the calibration for and pricing with the equation (11) both become simple mathematical exercises. We want to comment here that Condition 1 is nothing unusual. It appears in a market where the prices of defaultable debts are dominated by the probability of default. Such markets were seen in Asia over last year and sovereignty debt market in East Europe and South America around and after the July of 1998.

Equation (11) can be solved analytically provided that Λ commutes with $d\Lambda/dt$. The details are given in Appendix A. When Λ and $d\Lambda/dt$ don't commute we employ numerical ODE methods to solve equations (11).

5. DETERMINATION OF THE CONTAGION COEFFICIENTS

In our credit contagion model for two-bond basket, there are four parameters, namely, hazard rates $\lambda_{12}, \lambda_{13}$ for the first default, and contagion coefficients α and β , needed to be

”calibrated” to the market, before the model can be used to price other derivatives. In practice the calibration is achieved by minimizing some target function. To calibrate to a two-bond basket, a popular choice of the target function is

$$\sum_i (P_{model}^a(T_i) - P_{market}^a(T_i))^2 + (P_{model}^b(T_i) - P_{market}^b(T_i))^2, \quad (12)$$

where the super index variables a and b refer to the two bonds, T_i 's correspond to certain trading dates of interest, P_{model} and P_{market} refer to model price and market price, respectively. We want to find $\lambda_{12}, \lambda_{13}, \alpha$ and β that minimize the target function. The technical details for calculating model prices are given in Appendix B.

To solve minimization problem (12) directly can be costly but not necessarily helpful. In applications, we want to specify α and β a priori, and leave λ_{12} and λ_{13} for calibration (For clarity we rename λ_{12} and λ_{13} to be λ_a and λ_b from now on). Our interest in this section is to develop a technique to extract α and β from the available price history. For this purpose, we make simplification and impose an assumption to reduce number of variables into two. We assume that the contagion occurs only in one way, and postulate a linear relation between the first-to-default rates λ_a and λ_b :

$$\begin{aligned} \beta &= 1, \\ \lambda_a &= c_1 \lambda_b + c_2. \end{aligned} \quad (13)$$

The coefficients c_1 and c_2 , meanwhile, are obtained from solving the linear regression problem

$$\min_{c_1, c_2} \sum_i (s_a(T_i) - (c_1 s_b(T_i) + c_2))^2 \quad (14)$$

where s_a and s_b are spreads of the two bonds over US 30-year treasury rate. Using the closing prices Argentina PAR-bond and Brazilian C-bond since July 30, 1993 when C-bond was launched, we obtain $c_1 = 0.4072$ and $c_2 = 0.0072$. Because $\beta = 1$, λ_b is naturally equal to the spread of the C-bond over the US 30-year treasury yield, i.e., $\lambda_b = s_b$. The relation between λ_a and λ_b , and the relation between s_a and s_b are illustrated in Figure 2, where the line plot is for (λ_a, λ_b) , and the dot-plot is for (s_a, s_b) . To some extent, we attribute the deviation of (s_a, s_b) from the λ_a - λ_b line to the credit contagion coefficient α . After the specification of β, λ_a and λ_b , we end up with a minimization problem with a single variable α .

The minimization problem is solved with the following patterns of price intakes:

- a. Closing prices since initiation;
- b. Monthly closing prices since initiation;

- c. Closing prices in the last year;
- d. Closing prices in the last six month;
- e. Closing prices in the last three month;
- f. Closing prices in the last month.

In calculating the model prices, precise features of the bonds must be taken into account, which include interest payment capitalization, principle amortization, principle collatorization and 12-month rolling interest guarantee. The terms of the two benchmark bonds are given by Table D.1 and Table D.2 in Appendix D.

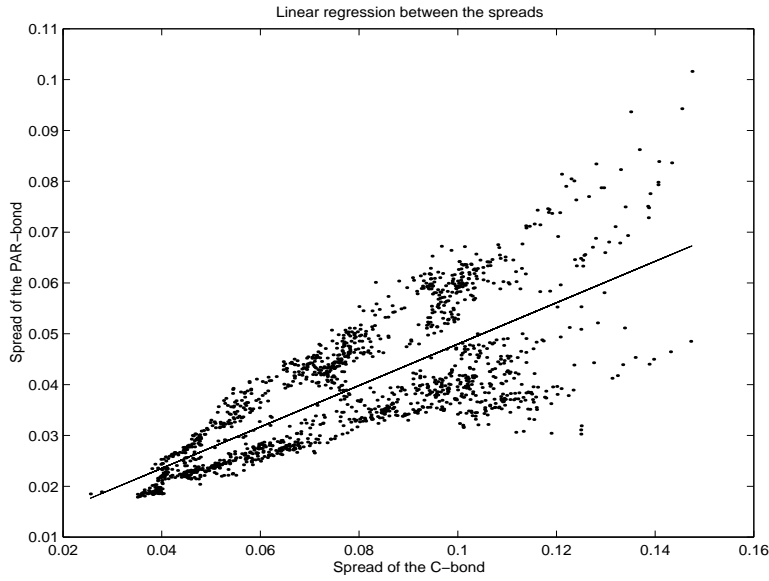


FIGURE 2. Spreads and first-to-default rates

The results of calibrations are posted in Table 1, where R stands for recovery rate.

	$R = 0.0$	$R = 0.1$
a.	$\alpha = 5.4380$	$\alpha = 10.8608$
b.	$\alpha = 5.4925$	$\alpha = 10.7996$
c.	$\alpha = 4.7081$	$\alpha = 9.5246$
d.	$\alpha = 5.2468$	$\alpha = 10.5149$
e.	$\alpha = 6.8674$	$\alpha = 18.7549$
f.	$\alpha = 6.3028$	$\alpha = 16.7522$

Table 1. Contagion coefficients for different duration and recovery rates

It is interesting to note that α is very stable for data of long period. In Figure 3, we offer a comparison between the actual *yield to maturity* and the *yield to maturity* produced by our

model(using the prices of C-bond, $R = 0.0$ and $\alpha = 5.438$). It can be seen the YTM curves are very close, which suggests that our model has high degree of predicting power.

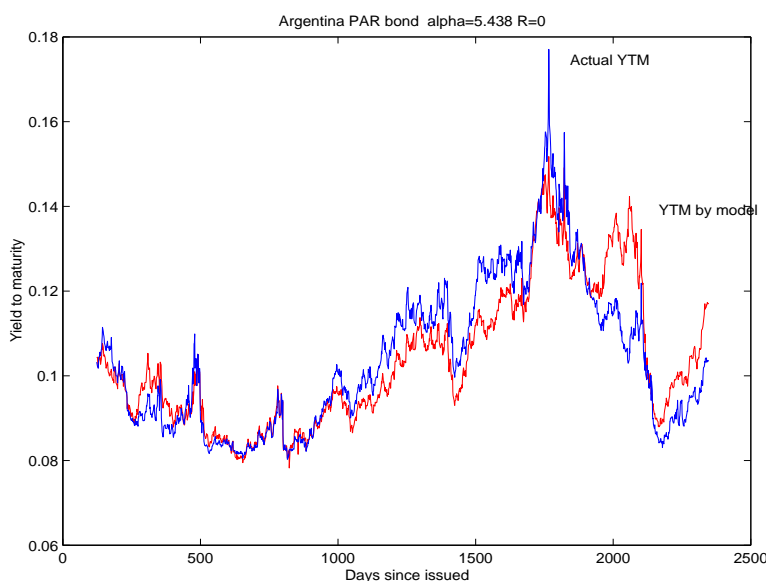


FIGURE 3. Spreads and first-to-default rates

6. DERIVATIVE PRICING WITH CALIBRATED DEFAULT PROCESS

In this section we demonstrate the pricing of derivatives with calibrated default process. In specific, we take a default protection note and a serious swaps as examples. For given α and β , we first calibrate the model to the prices of Argentina PAR bond and Brazilian C-bond of a specific date, and then use the calibrated model to price two kinds of derivatives. The date is taken to be September 15, 1998, and the closing stripped prices were $B_{A,m} = \$29.67$ and $B_{B,m} = \$66.17$, respectively. The sensitivity of the prices of the derivatives with respect to various parameters are also calculated and presented. The details are listed in Appendix C.

Example 1: *Argentinean default protection note.* The maturity of the note is two years, and the payoff of the note is \$100 if Argentina does not default within the two years or otherwise nothing.

Our interests in this problem is on the effect of different contagion coefficient α on the price and various sensitivity quantities. For α increasing continuously from 1 to 100, and the rest of the inputs fixed ($\beta = 1, R = 0.1, B_{A,m}$ and $B_{B,m}$), we compute the price, change in price of the note for a 1 basis point change in interest rate(BPV Tsy), change in price of the note for a 1 basis point change in stripped spread of Argetina(BPV spread), change in price

of the note for a 1 basis point change in Argetina/Brazil cross country spread. The results are presented as plots of functions of α , given in Figure 7. The main point of interest is that the value of the default note and its sensitivity parameters are not monotonic functions of α . The value and the parameters are most sensible to changes of α within $\alpha \in [1, 10]$, where maximum or minimum of the value and sensitivity parameters are achieved. For α increases beyond 10, the value and sensitivity parameters approach some asymptotes. It is interesting to note that the sensitivity to cross-country spread converges to zero, meaning that the value will no longer be sensitive to the spread if the Brazilian default will almost certainly bring down Argentina.

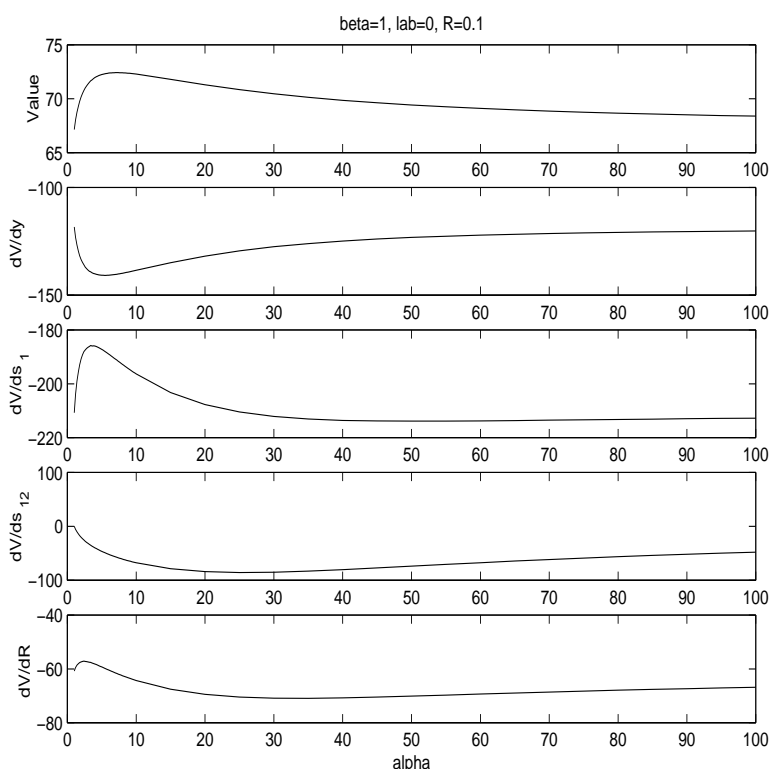


FIGURE 7. Value and sensitivities of default note

Example 2: Swaps of different tenors. We consider a sequence of swaps such that, if Argentina defaults, the payment from the fixed leg will stop. The terms of the swaps are

- Tenor: 5-15 years
- Principle: \$100
- Interest rate for floating leg: LIBOR
- Interest rate for fixed leg: 15%
- Payment schedule: semiannually

Yet again, we calibrate the model for alternatively fixed α and tenor, and then calculate the price and sensitivity parameters as functions of tenor and α , respectively. For the 5-year swap, the price and sensitivity parameters as functions of α , which varies from 1 to 100, are given in Figure 8. As one can see that the patterns of changes in price and sensitivity parameters are similar to those of the default note. Drastic changes are only seen for $\alpha \in (1,10)$. The plots for value and sensitivity parameters for fixed $\alpha(=5)$ yet varying tenor(from 5 year to 15 years) are given in Figure 9. It is seen that the absolute values of the outcomes essentially increase with the maturity, and the changes are gradual and mild.

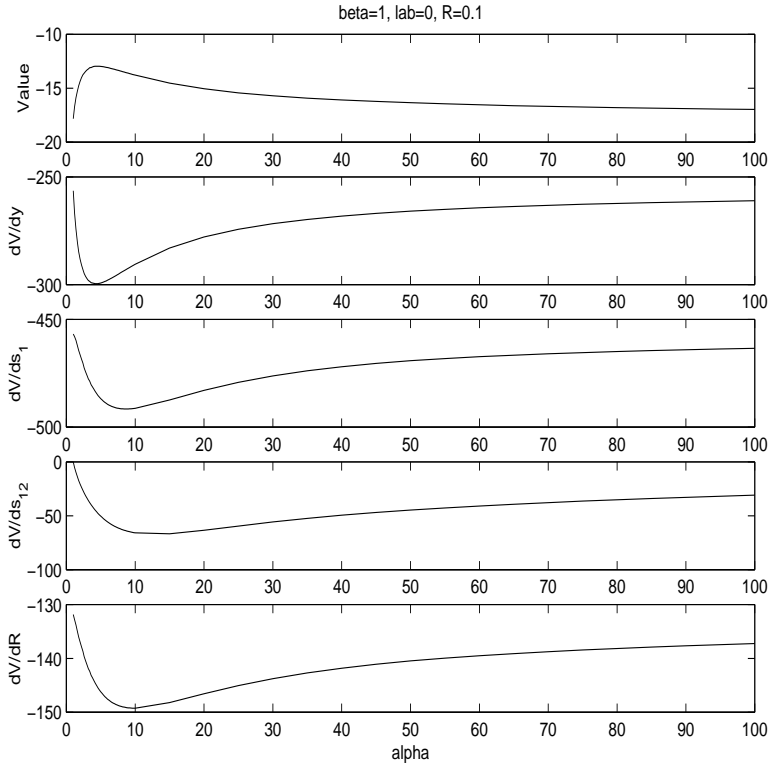


FIGURE 8. Value and sensitivities of swap

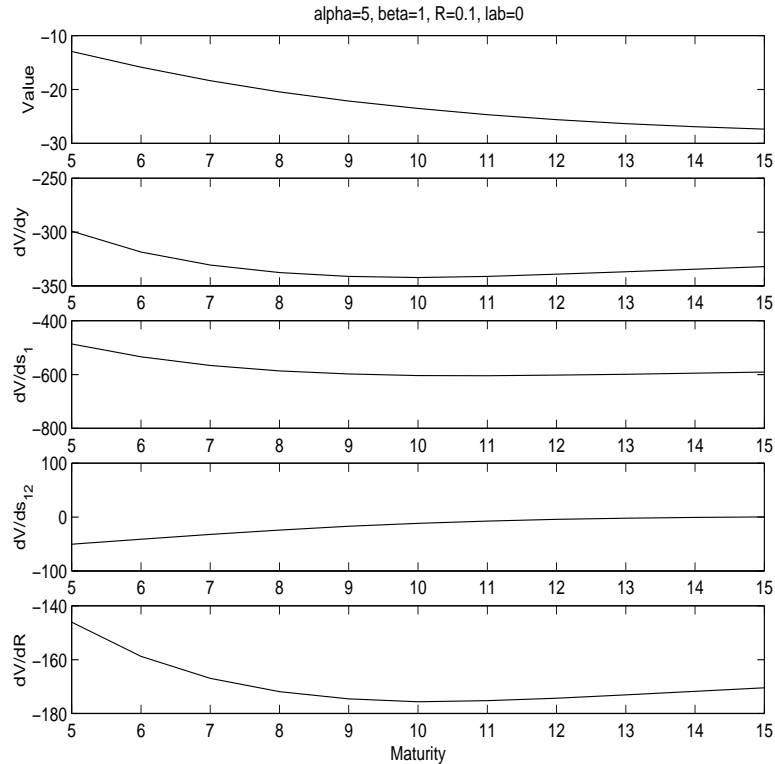


FIGURE 9. Swap values vs.. maturity

7. CONCLUSION

We have proposed a new model for credit contagion based on the joint processes of diffusion and finite-state transition for risk-free interest rate and default. Studies for a reduced case when the interest rate is deterministic is carried out in the paper. For the reduced case, the "contagion coefficient" is estimated with Argentinean and Brazilian benchmark bonds, under the assumption of one-way credit contagion from Brazil to Argentina. The reduced model can be used to price any straight sovereignty debts of Argentina. A default protection note and a series of swaps of Argentina are used as two examples in the paper. We have also developed techniques to calculate the sensitivity to an infinitesimal plain shift of yield, to the changes of spread, cross spread, and to the recovery rate. For options instead of straight-debt instruments, we will have to use and calibrate the diffusive process for the risk-free interest rates. This will be the subject of future studies.

REFERENCES

- [1] Artzner, P. and F. Delbaen, 1995, Default Risk Insurance and Incomplete Markets, *Mathematical Finance* 5: 187-195.
- [2] Avellaneda, M, 1998, The minimum-Entropy algorithm and Related Methods for Calibrating Asset-Pricing models, *Journee Annuelle*, 40-51.
- [3] Black, F., and J. Cox, 1976, Valuing Corporate Securities: Some Effects of Bond Indenture Provisions, *Journal of Finance*, 31, 351-367.
- [4] Duffie, D., M. Schrod, and C. Skiadas, 1996, Recursive Valuation of Defaultable securities and the Timing of Resolution of Uncertainty, *Annals of Applied Probability*, 6,1075-1090.
- [5] Duffie, D. and K.J. Singleton, 1994, Modeling term structures of defaultable bonds, forthcoming: *Review of Financial Studies*
- [6] Duffie, D. and K.J. Singleton, 1999, Simulating Correlated Defaults, working paper, Stanford University.
- [7] Heath, D.R., R. Jarrow and A. Merton, 1992, Bond Pricing and the Term S tructure of Interest Rates: A New Methodology for Contingent Claim Valuation, *Econometrica*, 60, 77-105.
- [8] Hull, J. and A. White, 1991, The Impact of Default Risk on Option Prices, working paper, University of Toronto.
- [9] Jarrow, R., D. Lando and S.M. Turnbull, 1997, A Markov model for the term structure of credit risk spread, *The Review of Financial Studies*, 10, 481-523.
- [10] Jarrow, R.D. and S.M. Turnbull, 1995, Pricing Derivatives on Financial Securities Subject to Credit Risk, *The Journal of Finance*, 50, 53-86.
- [11] Kealhofer, S., 1995, Managing Default Risk in Portfolios of Derivatives, in E. Van Hertsen and P. Fields, editors, *Derivative Credit Risk, Advances in Measurement and Management*, London: Risk Publications.
- [12] Kim, J., K. Ramaswany, and S. Sundaresan, 1993, Does default risk in coupons affect the Valuation of corporate bonds?: A contingent claims model, *Financial Management* 117-131
- [13] Lando, D., 1994, Three Essays on Contingent Claims Pricing, Ph.D. thesis, Cornell University.
- [14] Longstaff, F. and E. Schwartz, 1992, Valuing Risky debt: a new approach, working paper, UCLA.
- [15] Madan, D. and H. Unal, 1993, Pricing the Risks of Default, working paper, College of Business, University of Maryland.
- [16] Merton, R., 1974, On the pricing of corporate debt: the risk structure of interest rates, *The Journal of Finance* 29, 449-470.
- [17] Nielsen, S. and E. Ronn, 1995, The Valuation of Default Risk in Corporate Bonds and Interest Rates, *The Journal of Finance*, 29, 449-470.

APPENDIX A. ANALYTICAL SOLUTION TO SYSTEM OF ODE'S

Consider a risky bond which will make a single payment V at maturity T if default does not happen up to the maturity. According to the states we can specify the vector of payoff $V(T)$. If λ and $d\Lambda/dt$ commute, we can solve (11) directly and obtain

$$\begin{aligned} V(0) &= \exp\left(-\int_0^T (f(\tau)I - \Lambda(\tau))d\tau\right) V(T) \\ &= P(0, T) \cdot \exp\left(\int_0^T \Lambda(\tau)d\tau\right) V(T) \end{aligned} \quad (\text{A.1})$$

where

$$P(0, T) = \exp\left(\int_0^T f(\tau)d\tau\right)$$

is the value of the US zero-coupon bond with maturity T . Let

$$\Theta = \frac{1}{T} \int_0^T \Lambda(\tau)d\tau.$$

Then $e^{\Theta T}$ is the T -period probability transition matrix. Using the Jordan canonical form of Θ : $\Theta = XJX^{-1}$, where J is a (block) diagonal matrix, we can evaluate $e^{\Theta T}$ by

$$e^{\Theta T} = Xe^{JT}X^{-1} \quad (\text{A.2})$$

If the states are partially ordered, i.e., we will never have $S_i \geq S_{i+1}$, then $\Lambda(\tau)$ and consequently Θ are both upper triangular matrices, since default events are ‘‘irreversible’’. In such case, the factorization of Θ into its Jordan canonical form can be achieved by finite number of elementary arithmetic operations. The computations can be reduced further if Λ is assumed to be τ -independent (or time homogeneous), when there is $\Theta = \Lambda$. Take two-bond basket and use the order given in (1), for example, we have the factor matrices as

$$J = \text{diag}(\lambda_{11}, \lambda_{22}, \lambda_{33}, 0), \quad (\text{A.3})$$

$$X = (X_1, X_2, X_3, X_4), \quad (\text{A.4})$$

with

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} \frac{\lambda_{12}}{\lambda_{33}-\lambda_{11}} \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} \frac{\lambda_{12}x_{23}+\lambda_{13}}{\lambda_{33}-\lambda_{11}} \\ \frac{\lambda_{23}}{\lambda_{33}-\lambda_{22}} \\ 1 \\ 0 \end{pmatrix}, \quad X_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \quad (\text{A.5})$$

Here, in particular, x_{23} refers to the second component of X_3 .

APPENDIX B. VALUATION OF DEFAULTABLE BONDS

The model prices of two correlated defaultable bonds can be calculated as

$$\begin{aligned} P_{model}^a(T_i) &= \sum_{T_j^a \geq T_i} C_{T_j^a}^a P_a(T_j^a), \\ P_{model}^b(T_i) &= \sum_{T_j^b \geq T_i} C_{T_j^b}^b P_b(T_j^b), \end{aligned} \tag{B.1}$$

where $C_{T_j^a}^a$ and $C_{T_j^b}^b$ are the cash flows of the two benchmarks, $P_a(T)$ and $P_b(T)$ are the model values of zero-coupon risky bonds of Argentina and Brazil with maturity T , given by

$$\begin{aligned} P_a(T) &= P(0, T) * [(1 - R) \cdot Prob(\tau_a > T) + R], \\ P_b(T) &= P(0, T) * [(1 - R) \cdot Prob(\tau_b > T) + R], \end{aligned} \tag{B.2}$$

where R is the recovery rate (assuming the same for both country), and τ_a and τ_b are the default arrival times. $Prob(\tau_a > T)$ and $Prob(\tau_b > T)$ thus are the survival probabilities of the two bonds up to time T , respectively. Through standard probabilistic arguments (or by solving the ODE (11) we can obtain

$$\begin{aligned} Prob(\tau_a > T) &= e^{-(\lambda_{12} + \lambda_{13})T} + \int_0^T e^{-(\lambda_{12} + \lambda_{13})s} \lambda_{13} e^{-\alpha \lambda_{12}(T-s)} ds \\ &= \frac{\lambda_{13}}{\lambda_{13} + (1 - \alpha)\lambda_{12}} e^{-\alpha \lambda_{12}T} + \frac{(1 - \alpha)\lambda_{12}}{\lambda_{13} + (1 - \alpha)\lambda_{12}} e^{-(\lambda_{12} + \lambda_{13})T} \end{aligned} \tag{B.3}$$

By symmetry we also obtain

$$Prob(\tau_b > T) = \frac{\lambda_{12}}{\lambda_{12} + (1 - \beta)\lambda_{13}} e^{-\beta \lambda_{13}T} + \frac{(1 - \beta)\lambda_{13}}{\lambda_{12} + (1 - \beta)\lambda_{13}} e^{-(\lambda_{12} + \lambda_{13})T}. \tag{B.4}$$

APPENDIX C. CALCULATION OF SENSITIVITIES

If we price a portfolio of defaultable bonds with a model calibrated to bond prices B_a and B_b , then the portfolio value can be considered as a function of US treasury yields, input bond prices, and recovery rate R , i.e.,

$$V = V(y, R, B_a, B_b).$$

Alternatively, we can consider the value as a function of treasury yield, the recovery rate, and the first-to-default rates λ_a and λ_b ,

$$V = \tilde{V}(y, R, \lambda_a, \lambda_b),$$

while λ_a and λ_b are defined implicitly by calibration conditions:

$$\begin{aligned} B_a(y, \lambda_a, \lambda_b, R) &= B_{a,mkt}, \\ B_b(y, \lambda_a, \lambda_b, R) &= B_{b,mkt}. \end{aligned} \tag{C.1}$$

Here, $B_{a,mkt}$ and $B_{b,mkt}$ are the market prices of the benchmark bonds. For the purpose of hedging, we need in the sensitivities with respect to the changes in yield, recovery rate, spread and cross-spread. The calculations can be proceeded in two steps. First, we compute sensitivities with respect to the changes in yield, recovery rate and input bonds prices. Then, we convert them to the sensitivities needed by applying a Jacobi matrix.

a. Sensitivity to treasury yield, recovery rate and input bonds

By chain rule we have

$$\begin{pmatrix} \frac{\partial V}{\partial y} [5pt] \\ \frac{\partial V}{\partial R} [5pt] \\ \frac{\partial V}{\partial B_a} \\ \frac{\partial V}{\partial B_b} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{\partial \lambda_a}{\partial y} & \frac{\partial \lambda_b}{\partial y} \\ 0 & 1 & \frac{\partial \lambda_a}{\partial R} & \frac{\partial \lambda_b}{\partial R} \\ 0 & 0 & \frac{\partial \lambda_a}{\partial B_a} & \frac{\partial \lambda_b}{\partial B_a} \\ 0 & 0 & \frac{\partial \lambda_a}{\partial B_b} & \frac{\partial \lambda_b}{\partial B_b} \end{pmatrix} \begin{pmatrix} \frac{\partial \tilde{V}}{\partial y} [5pt] \\ \frac{\partial \tilde{V}}{\partial R} [5pt] \\ \frac{\partial \tilde{V}}{\partial \lambda_a} [5pt] \\ \frac{\partial \tilde{V}}{\partial \lambda_b} \end{pmatrix} \tag{C.2}$$

To compute the coefficient matrix, we define

$$J = \begin{pmatrix} \frac{\partial B_a}{\partial \lambda_a} & \frac{\partial B_a}{\partial \lambda_b} \\ \frac{\partial B_b}{\partial \lambda_a} & \frac{\partial B_b}{\partial \lambda_b} \end{pmatrix} \tag{C.3}$$

which is the Jacobi matrix for transformation $(\lambda_a, \lambda_b) \rightarrow (B_a, B_b)$. Clearly we have

$$\begin{pmatrix} \frac{\partial \lambda_a}{\partial B_a} & \frac{\partial \lambda_b}{\partial B_a} \\ \frac{\partial \lambda_a}{\partial B_b} & \frac{\partial \lambda_b}{\partial B_b} \end{pmatrix} = J^{-1}. \tag{C.4}$$

To calculate $(\frac{\partial \lambda_a}{\partial y}, \frac{\partial \lambda_b}{\partial y})$, we need equations (C.1) for fixed R , $B_{a,mkt}$ and $B_{b,mkt}$:

$$\begin{aligned} B_a(y, \lambda_a(y), \lambda_b(y)) &= B_{a,mkt}, \\ B_b(y, \lambda_a(y), \lambda_b(y)) &= B_{b,mkt}, \end{aligned} \tag{C.5}$$

Differentiating (C.5) with respect to y we have

$$\begin{aligned}\frac{\partial B_a}{\partial y} + \frac{\partial B_a}{\partial \lambda_a} \frac{\partial \lambda_a}{\partial y} + \frac{\partial B_a}{\partial \lambda_b} \frac{\partial \lambda_b}{\partial y} &= 0, \\ \frac{\partial B_b}{\partial y} + \frac{\partial B_b}{\partial \lambda_a} \frac{\partial \lambda_a}{\partial y} + \frac{\partial B_b}{\partial \lambda_b} \frac{\partial \lambda_b}{\partial y} &= 0,\end{aligned}\tag{C.6}$$

Hence,

$$\begin{pmatrix} \frac{\partial \lambda_a}{\partial y} & \frac{\partial \lambda_b}{\partial y} \end{pmatrix} = - \begin{pmatrix} \frac{\partial B_a}{\partial y} & \frac{\partial B_b}{\partial y} \end{pmatrix} J^{-T}.\tag{C.7}$$

Through similarly derivations we have

$$\begin{pmatrix} \frac{\partial \lambda_a}{\partial R} & \frac{\partial \lambda_b}{\partial R} \end{pmatrix} = - \begin{pmatrix} \frac{\partial B_a}{\partial R} & \frac{\partial B_b}{\partial R} \end{pmatrix} J^{-T}.\tag{C.8}$$

b. Sensitivities to spread and cross spread

The sensitivities with respect to spread s_1 and cross-country spread s_{12} are related to sensitivities with respect to bonds B_a and B_b through the relation

$$\begin{aligned}\begin{pmatrix} \frac{\partial V}{\partial s_1} \\ \frac{\partial V}{\partial s_{12}} \end{pmatrix} &= \begin{pmatrix} \frac{\partial B_a}{\partial s_1} & \frac{\partial B_b}{\partial s_1} \\ \frac{\partial B_a}{\partial s_{12}} & \frac{\partial B_b}{\partial s_{12}} \end{pmatrix} \begin{pmatrix} \frac{\partial V}{\partial B_a} \\ \frac{\partial V}{\partial B_b} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial B_a}{\partial y_a} & \frac{\partial B_b}{\partial y_b} \\ 0 & -\frac{\partial B_b}{\partial y_b} \end{pmatrix} \begin{pmatrix} \frac{\partial V}{\partial B_a} \\ \frac{\partial V}{\partial B_b} \end{pmatrix} \\ &= \begin{pmatrix} -D_a B_a & -D_b B_b \\ 0 & D_b B_b \end{pmatrix} \begin{pmatrix} \frac{\partial V}{\partial B_a} \\ \frac{\partial V}{\partial B_b} \end{pmatrix}\end{aligned}\tag{C.9}$$

where y_a and y_b are the yields, and D_a and D_b are the McCauley duration of bonds B_a and B_b , respectively.

APPENDIX D. TERMS OF THE BENCHMARK BONDS

Terms of the Argentinean PAR Bond:

ISSUER:	Republic of Argentina	
CURRENCY:	U.S. Dollars	
AMOUNT ISSUED:	US\$12,488.9 million	
AMOUNT OUTSTANDING:	US\$11,269 million	
DATE ISSUED:	March 31, 1993	
MATURITY DATE:	March 31, 2023	
TENOR:	30 years	
AMORTIZATION:	Bullet	
COUPON:	Fixed rate, Step on coupon	
	Year 1	4.00%
	Year 2	4.25%
	Year 3	5.00%
	Year 4	5.25%
	Year 5	5.50%
	Year 6	5.75%
	Year 7-30	6.00%
GUARANTOR:	US Treasury zero-coupon bonds due 2023 to collateralize principle 12 month of rollin ginterest guarantee at 6%. The remainder of interest is not. collateralized.	
ORIGIN:	Refinancing of foreign debt to commercial banks, under the Brady agreement signed in November 1992	

Terms of the Brazilian C-Bond:

ISSUER:	Federal Republic of Brazil	
CURRENCY:	U.S. Dollars	
AMOUNT ISSUED:	US\$7.39 billion	
AMOUNT OUTSTANDING:	US\$8.14 billion	
DATE ISSUED:	April 15, 1994	
MATURITY DATE:	April 15, 2014	
TENOR:	10 years	
AMORTIZATION:	21 semiannual payments beginning April 15, 2004	
COUPON:	8% fixed, with a portion payable in cash and the remainder capitalized(added to principle) in years 1-6, according to the following schedule.	
	Year 1-2	4.00%
	Year 3-4	4.50%
	Year 5-6	5.00%
	Year 7-20	8.00%
	Interest payable semiannually.	
GUARANTOR:	Federal Republic of Brazil	
ORIGIN:	Brady Plan(April 1994)	