Derivative Securities: Lecture 7 Further applications of Black-Scholes and Arbitrage Pricing Theory

Sources:

J. Hull

Avellaneda and Laurence

Black's Formula

- Sometimes its easier to think in terms of forward prices. Recalling that in Black-Scholes

$$d_{1,2} = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{S}{K}\right) + \frac{r-q}{\sigma\sqrt{T}} \pm \frac{\sigma\sqrt{T}}{2}$$
$$= \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{F}{K}\right) \pm \frac{\sigma\sqrt{T}}{2}, \quad F = Se^{(r-q)T}$$

$$BSCall(S,T,K,r,q,\sigma) = e^{-qT}SN(d_1) - e^{-rT}KN(d_2)$$
$$= e^{-rT}e^{(r-q)T}SN(d_1) - e^{-rT}KN(d_2)$$

$$\boxed{BSCall(S,T,K,r,q,\sigma) = e^{-rT}\cdot BSCall(F,T,K,0,0,\sigma)}$$

- Similarly,

$$\boxed{BSPut(S,T,K,r,q,\sigma) = e^{-rT}\cdot BSPut(F,T,K,0,0,\sigma)}$$

Symmetry between puts and calls in Black's formula

$$Call(F,T,K,r,\sigma) = e^{-rT}\left[F \cdot N\left(\frac{\ln(F/K)}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}\right) - K \cdot N\left(\frac{\ln(F/K)}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2}\right)\right]$$

$$Put(F,T,K,r,\sigma) = e^{-rT}\left[K \cdot N\left(\frac{\ln(K/F)}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}\right) - F \cdot N\left(\frac{\ln(K/F)}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2}\right)\right]$$

$$Call(F,T,K,r,\sigma) = Put(K,T,F,r,\sigma)$$

Call= ``exchanging cash for stock'', Put=``exchanging stock for cash''

Options on Foreign Exchange

- Example: Price a 90-day USD call/EUR put on 10,000,000 USD with strike 1.40. US rate=0.41%, EUR rate=1.50%. Current exchange rate: EUR= USD 1.3702, volatility: 16%.

$$F = (1.3702) \times \frac{1 + \left(\frac{90}{360}\right)0.0041}{1 + \left(\frac{90}{360}\right)0.0150} = 1.36648$$

Forward with USD as domestic currency

$$V(\text{per EUR notional}) = \frac{1}{1 + 0.25 \cdot 0.0041} BSPut(1.36648, 0.25, 1.40, 0, 0, 0.15)$$

$$= \frac{1}{1.001025} 0.062903 = \$0.062839$$

Put on 1 EUR notional amt. Use Black with USD rate

$$\text{Option Value} = \$0.062839 \times \frac{10{,}000{,}000}{1.40} = \$448{,}847.10 = EUR\ 327{,}577.8$$

Notional adjusted at strike price, option value in EUR converted at spot price

Options on Futures

- Consider an option on a futures contract. Typically, the option gives the right, but not the obligation to enter/exit a futures contract at a specified price and is cash-settled.

- Example: call option on a Eurodollar futures contract with strike 97 in 1 month. Futures settle in 3 months.

  -- in or before 1 month: can exercise and get long a futures. The payoff will be max(f-97,0) times the notional amount, cash-settled.

- Similar for a put option, with cash settlement of max(97-f,0).

Derivation (I)

- Assuming that the option fair value depends on futures price and time-to-maturity,

$$C = C(f, t), \quad t = \text{calendar time}, T = \text{maturity}$$

$$\Delta C = \frac{\partial C}{\partial t}\Delta t + \frac{\partial C}{\partial f}\Delta f + \frac{1}{2}\frac{\partial^2 C}{\partial f^2}(\Delta f)^2 + o(\Delta t)$$

- Exposure to futures : $\frac{\partial C}{\partial f} \cdot \Delta f$

- Assume that you `hedge' by taking a position in $-\frac{\partial C}{\partial f}$ contracts and in one option, the latter financed at rate *r*.
  The 1-day PNL is

$$PNL = -rC\Delta t + \Delta C - \frac{\partial C}{\partial f}\Delta f$$

Derivation (II)

- Notice that the futures position requires no financing (except for margin, which we do not take into account). Hence,

$$PNL = -rC\Delta t + \frac{\partial C}{\partial t}\Delta t + \frac{\partial C}{\partial f}\Delta f + \frac{1}{2}\frac{\partial^2 C}{\partial f^2}(\Delta f)^2 - \frac{\partial C}{\partial f}\Delta f + o(\Delta t)$$

$$= -rC\Delta t + \frac{\partial C}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 C}{\partial f^2}(\Delta f)^2 + o(\Delta t)$$

$$= -rC\Delta t + \frac{\partial C}{\partial t}\Delta t + \frac{f^2\sigma^2}{2}\frac{\partial^2 C}{\partial f^2}\Delta t + \frac{f^2}{2}\frac{\partial^2 C}{\partial f^2}\left[\left(\frac{\Delta f}{f}\right)^2 - \sigma^2\Delta t\right] + o(\Delta t)$$

$$= \left(-rC + \frac{\partial C}{\partial t} + \frac{f^2\sigma^2}{2}\frac{\partial^2 C}{\partial f^2}\right)\Delta t + \frac{f^2}{2}\frac{\partial^2 C}{\partial f^2}\left[\left(\frac{\Delta f}{f}\right)^2 - \sigma^2\Delta t\right] + o(\Delta t)$$

Derivation (III)

- If we assume that

$$E\left[\left(\frac{\Delta f}{f}\right)^2\right] = \sigma^2 \Delta t + o(\Delta t)$$

the profit/loss from the hedged strategy would be <u>on average</u>

$$PNL = \left(\frac{\partial C}{\partial t} + \frac{\sigma^2 f^2}{2}\frac{\partial^2 C}{\partial f^2} - rC\right)\Delta t + o(\Delta t)$$

- We conclude that the fair value of the option should satisfy the PDE

$$\frac{\partial C(f,t)}{\partial t} + \frac{\sigma^2 f^2}{2}\frac{\partial^2 C(f,t)}{\partial f^2} - rC(f,t) = 0, \quad 0 < t < T$$

Black vs. Black Scholes PDE

- The PDE is a special case of the Black-Scholes PDE with *r=q* and therefore no first-order term.

- For European-style options, this leads to the boundary-value problem

$$\frac{\partial C(f,t)}{\partial t}+\frac{\sigma^2 f^2}{2}\frac{\partial^2 C(f,t)}{\partial f^2}-rC(f,t)=0, \quad 0<t<T$$

$$C(f,T)=F(f), \quad [F(f)=(f-K)^+ \ \text{(call)}, \ F(f)=(K-f)^+ \ \text{(put)}]$$

This has Black's formula as an exact solution.

- For American options, the PDE is solved numerically, e.g. with the trinomial scheme.

- If the interest rate is positive, American-style options which are deep-in-the-money should be exercised before expiration.

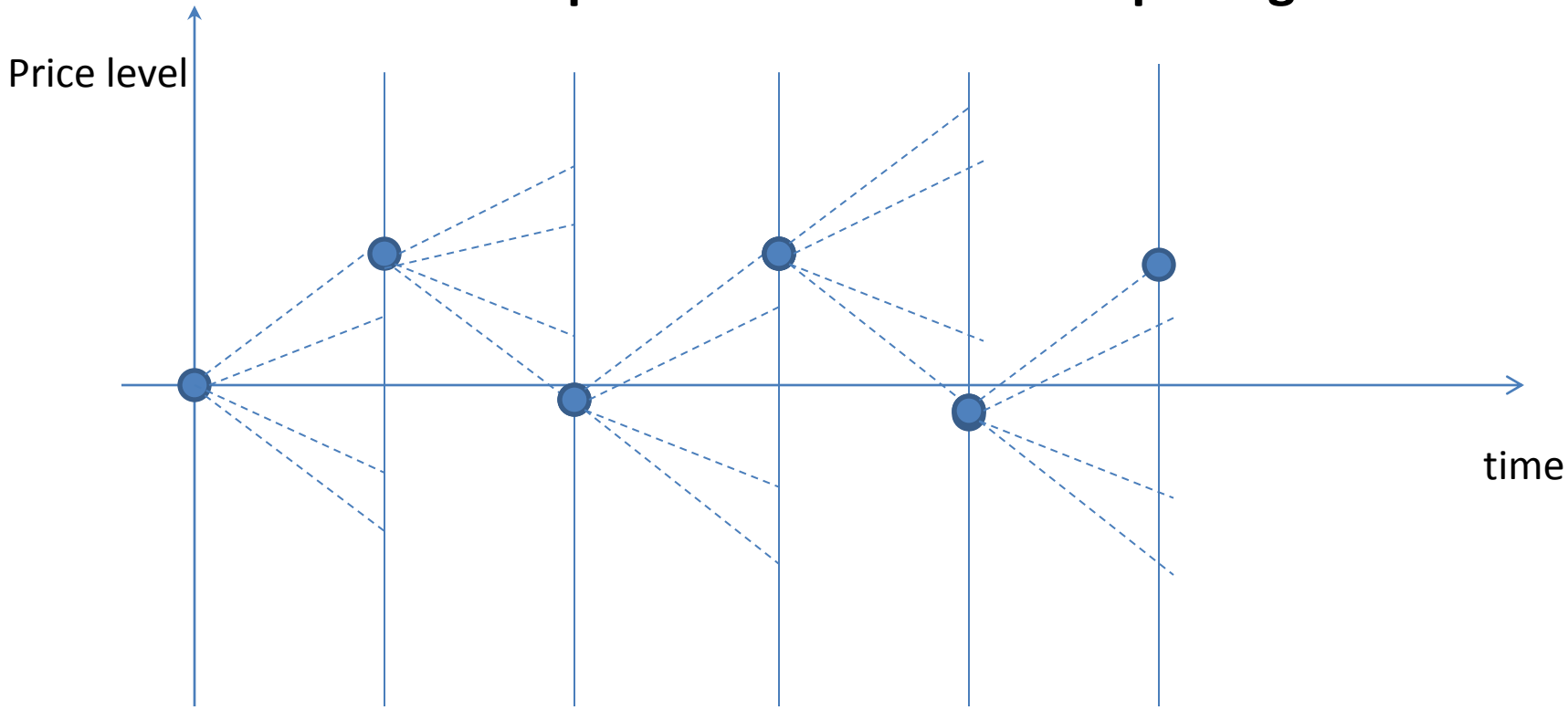
Multi-period models & beyond Black-Scholes

- In a previous lecture, we presented option pricing as an expected value over a **probability measure**.

- Using a hedging argument, we also presented a **PDE approach** for pricing options and showed (numerically) that this lead to the same prices in the case of European options.

- To connect both methods and increase our understanding of pricing of derivatives, we need to consider **multi-period models** and a more general setting.

A multi-period model for asset-pricing



Assumptions

- N trading dates
- On each trading date, we know the interest rate for the next period, the dividend yield for the next period and the price level, i.e.

$$(S_t, q_t, r_t),\ t = t_1, t_2, \ldots, t_N$$

- S and q can represent vectors, i.e., multiple securities

Arbitrage opportunities

- An arbitrage opportunity consists of a trading strategy which has zero initial cost and which gives rise to non-negative cash-flows in all future states of the market with at least one state with positive cash flows.

- In a 1-period model, we saw that the absence of arbitrage opportunities implies the existence of a probability measure in the set of states at time T such that

all traded securities satisfy

$$V_0 = \frac{1}{1+r\Delta t} E^p\{V_T\}$$

In particular,

$$F_{0,T} = E^p\{S_T\}$$

The Hahn-Banach Theorem

- Let X be a finite space representing the future states of a market, and let $\Pi(F)$ be a linear functional defined on functions of future states:

$$\Pi(\lambda F + \mu G) = \lambda \Pi(F) + \mu \Pi(G)$$

Here, the functional represents the price of a derivative with payoff *F(x)* outcome *x* is realized.

- If $\Pi(\ )$ has the property that $F(x) > 0 \quad \forall x \in X \quad \Rightarrow \quad \Pi(F) > 0,$ then

$$\Pi(F) = \sum_{x \in X} w(x) F(x) \qquad \forall F$$

for some positive weights $\{w(x),\ x \in X\}$

Absence of arbitrage opportunities: the formalization of ``fair value''

- In the one-period model, we found that the absence of arbitrage opportunities implied that there exists a probability measure on future states such that any derivative security with value V_F at time 0 and value F(S) at time T should have a price V_F satisfying

$$V_F(S_0) = \frac{1}{1+rT} E^P \{F(S_T)\}$$

- In a multi-period model, the same equation should hold conditionally on the current state of the market, i.e. if a security pays a cash-flow F(S) at time t_{n+1}, its value at time t_n should satisfy

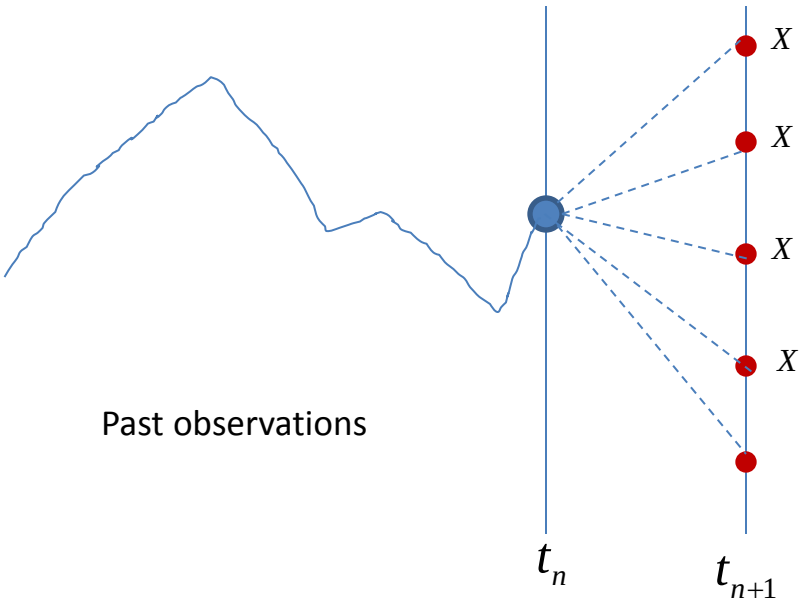
$$V(S_{t_1}, S_{t_2}, ..., S_{t_n}; t_n) = \frac{1}{1+r_{t_n}\Delta t} E^{p_{n,n+1}} \{F(S_{t_{n+1}})\}$$

The probability $p_{n,n+1}$

- The probability can be seen as a function of the observations up to time t_n forecasting the price level at time t_{n+1}:

$$p_{n,n+1}(X) = p_{n,n+1}\left\{S_{t_{n+1}} = X \mid \left(S_{t_i}, d_{t_i}, r_{t_i}\right), i \le n\right\}$$

Observations up to time n



- Main idea:
  The pricing probability from date n to date n+1 must satisfy a constraint, associated with forward pricing.

Forwards and probability constraints

- If interest and dividend are known at date n, then the forward price is known as well, and we must have

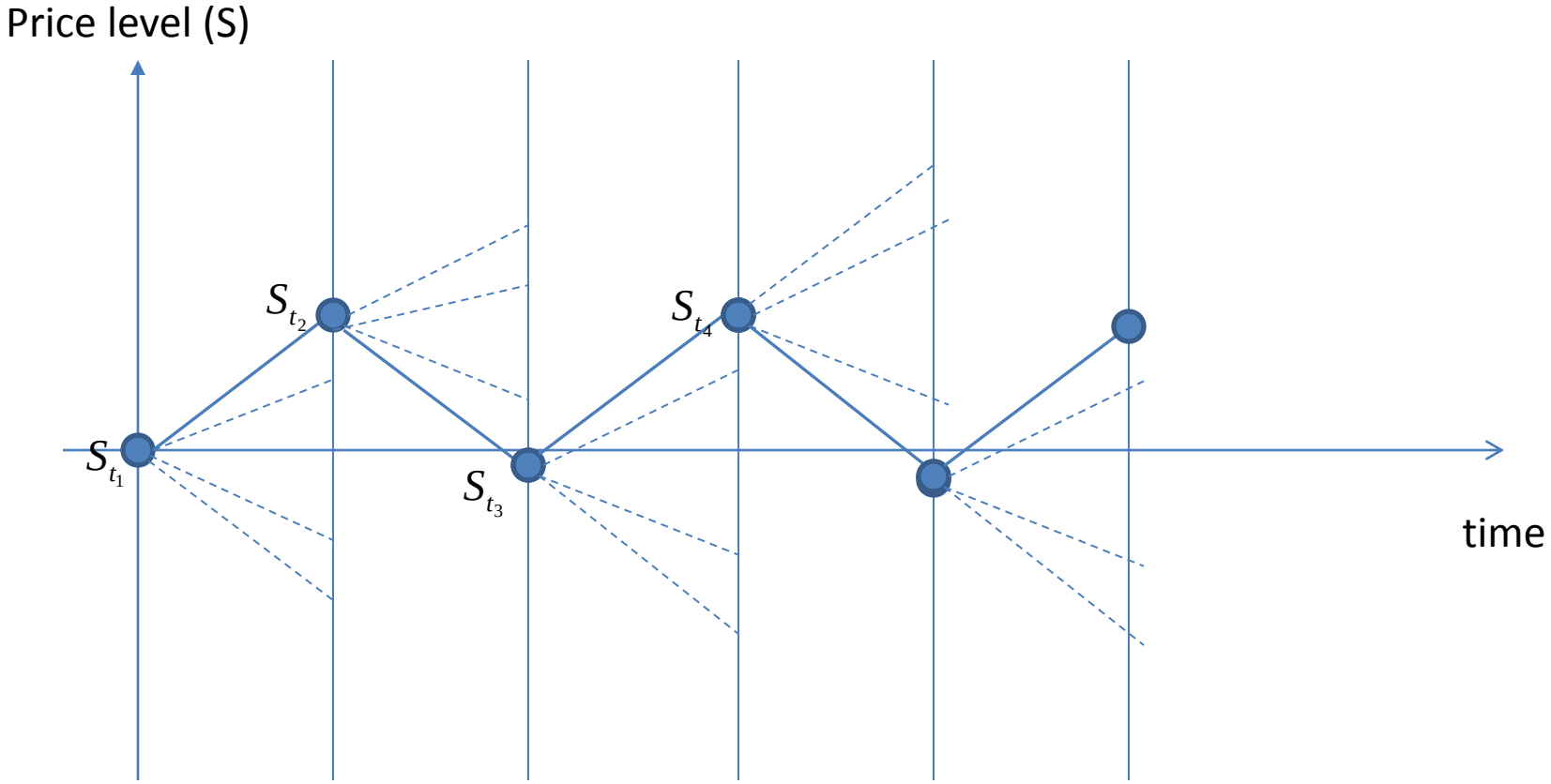
$$0 = \frac{1}{1+r_{t_n}\Delta t} E^{p_{n,n+1}}\left\{S_{t_{n+1}} - F_{t_n,t_{n+1}}\right\} \quad \therefore \quad E^{p_{n,n+1}}\left\{S_{t_{n+1}}\right\} = F_{t_n,t_{n+1}}$$

$$E^{p_{n,n+1}}\left\{S_{t_{n+1}}\right\} = \frac{1+r_{t_n}\Delta t}{1+q_{t_n}\Delta t} S_{t_n}$$

forward price= expectation of future spot

``currency-type model for the dividend'' (for simplicity)

Probabilities for prices defined over paths



$$P\left(S_{t_1}, S_{t_2}, S_{t_3}, \ldots, S_{t_N}\right) \equiv p_{0,1}\left(S_{t_1}\right) \cdot p_{1,2}\left(S_{t_2}\right) \cdot p_{2,3}\left(S_{t_3}\right) \ldots p_{N-1,N}\left(S_{t_N}\right)$$

- Given the 1-step probabilities $p_{n,n+1}$, this formula defines a probability over paths

Characterization of *P( )*

- Define the <u>discounted total return process T by</u>

$$T_{t_n} \equiv \left(\frac{\prod_{j=0}^{n-1} \left(1 + q_{t_j} \Delta t\right)}{\prod_{j=0}^{n-1} \left(1 + r_{t_j} \Delta t\right)} \right) \cdot S_{t_n}, \quad T_{t_0} = S_{t_0}$$

- Then,

$$E^P \left\{ T_{t_{n+1}} \mid T_{t_n} \right\} = E^{p_{n,n+1}} \left\{ T_{t_{n+1}} \right\}$$

$$= \left(\frac{\prod_{j=0}^{n} \left(1 + q_{t_j} \Delta t\right)}{\prod_{j=0}^{n} \left(1 + r_{t_j} \Delta t\right)} \right) E^{p_{n,n+1}} \left\{ S_{t_{n+1}} \right\} = \left(\frac{\prod_{j=0}^{n} \left(1 + q_{t_j} \Delta t\right)}{\prod_{j=0}^{n} \left(1 + r_{t_j} \Delta t\right)} \right) \frac{1 + r_{t_n} \Delta t}{1 + q_{t_n} \Delta t} S_{t_n}$$

$$= \left(\frac{\prod_{j=0}^{n-1} \left(1 + q_{t_j} \Delta t\right)}{\prod_{j=0}^{n-1} \left(1 + r_{t_j} \Delta t\right)} \right) S_{t_n} = T_{t_n}$$

$$E^P \left\{ T_{t_{n+1}} \mid T_{t_n} \right\} = T_{t_n}$$

No-arbitrage theorem for multi-period asset pricing

- If there are no arbitrage opportunities, there exists a probability measure over paths of traded securities, *P(),* such that the discounted total-return process of any traded security is a **martingale** under P, i.e.,

setting

$$T_{t_n} \equiv \left(\frac{\prod_{j=0}^{n-1} \left(1 + q_{t_j} \Delta t\right)}{\prod_{j=0}^{n-1} \left(1 + r_{t_j} \Delta t\right)} \right) \cdot S_{t_n}, \quad T_{t_0} = S_{t_0}$$

we have

$$E^P \left\{ T_{t_{n+1}} \mid T_{t_n} \right\} = T_{t_n} \quad .$$