Gauss-Markov processes

We shall investigate conditions under which the diffusion process defined by

$$dX \;=\; \sigma(X,\,t)\;dZ \;+\; \mu(X,\,t)\,dt \tag{1}$$

is *Gaussian*. (Recall that, by definition, a process is Gaussian if and only if the probability distribution of any vector

$$(X(t_1),\;X(t_2),\;...\;X(t_M)\;)\;,\quad t_1\;<\;t_2\;<\;...\;<\;T_M$$

(where $t_1.\;...\,t_M$ are arbitrary times), is multivariate Gaussian).

(i) Show that if Y is a Gaussian process, we have

$$\mathbf{E}\{Y(T)\mid Y(t)\;\}\;=\;a(t,\,T)\,Y(t)\;+\;b(t,\,T)$$

where a and b are deterministic functions.

(ii) Since uncorrelation is equivalent to independence in "Gaussland", conclude that

$$Y(T)\;=\;a(t,\,T)\,Y(t)\;+\;b(t,\,T)\;+\;\xi(t,\,T)$$

where $\xi(t,\,T)$ is a Gaussian r.v. with mean zero which is independent of $Y(t)$, and therefore that

$$\mathbf{E}\left\{\;e^{\theta\,Y(T)}\mid Y(t)\;\right\}\;=\;\exp\left\{\frac{\theta^2}{2}\,\sigma^2(t,\,T)\;+\;\theta\,a(t,\,T)\,Y(t)\;+\;\theta\,b(t\,T)\;\right\}\;. \tag{2}$$

(iii) Show from (1) that the following identities hold

$$\mathbf{E}\{X(t+\;\Delta t)\mid X(t)\}\;=\;X(t)\;+\;\mu(X(t),t)\,\Delta t\;+\;o(\Delta t)$$

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$$\mathbf{E}\{(X(t+\ \Delta t))^2 \mid X(t)\} \;=\; \sigma^2(X(t),t)\,\Delta t \;+\; o(\Delta t)$$

Conclude that if $X(t)$ is Gaussian, then

$$\sigma(X,\,t) \;=\; \sigma(t) \quad \text{and} \quad \mu(X,\,t) \;=\; \alpha(t)\,X \;+\; \beta(t) \tag{3}$$

(iv) Show that the process X which satisfies equation (1) with parameters given in (3) is given explicitly by the formula

$$X(t) \;=\; \psi(0,\,t)\,X(0) \;+\; \int_0^t \psi(s,\,t)\,\alpha(s)\,ds \;+\; \int_0^t \psi(s,\,t)\,\sigma(s)\,dZ(s)$$

where

$$\psi(s,\,t) \;=\; \exp\left(\int_s^t \alpha(u)\,du\right)$$

(v) **The case of constant α, β and σ.** We pass to more standard notation, setting

$$\alpha \;=\; -\kappa\,, \quad \beta \;=\; \kappa\,\theta\,.$$

Show that $X(t)$ is normal with mean

$$m(t) \;=\; e^{-\kappa\,t}\,X(0) \;+\; \left(1 \;-\; e^{-\kappa\,t}\right)\,\theta$$

and variance

$$v(t) \;=\; \frac{\sigma^2}{2\kappa}\,\left(1 \;-\; e^{-\,2\kappa\,t}\right)$$

Conclude that the process $X(t)$ has an asymptotic distribution if $\kappa \;>\; 0$. This constant is usually known as the **rate of mean-reversion** of the process.

(vi) Last but not least, write a program that simulates the process with $\kappa \;=\; 0.03,\ \theta\ 0.05,\ \sigma \;=\; 0.01$. Using Monte Carlo simulation, compute

$$\mathbf{P}\left\{ \max_{0<t<2} X(t) \;>0.075\;|X(0)\;=\;0.057\right\}$$

(Interpretation: assuming that $X(t)$ is the short-term interest rate, and the given values for the parameters, this is the probability that the short-term interest rate will rise from 5.70% to 7.50% in less than two yrs.)