Weighted Monte-Carlo Methods for Multi-Asset Equity Derivatives:
Theory and Practice

Marco Avellaneda
Courant Institute of Mathematical Sciences
New York University

Summary

- Statement of the Calibration Problem for Multi-Asset Equity Derivatives
- Weighted Monte Carlo simulation (max-entropy)
- Application to Arbitrage Pricing of Basket Options
- Comparison between WMC and Steepest Descent Method
- Comments on Correlation Skew and the statistics of Implied and Historical Correlations

Calibration Problem for Multi-Asset Equity Derivatives

Given a group, or collection of stocks, build a stochastic model for the joint evolution of the stocks with the following properties:

- The associated probability measure on market scenarios is risk-neutral: all traded securities are correctly priced by discounting cash-flows
- The associated probability measure is such that stock prices, adjusted for interest and dividends, are martingales (local risk-neutrality)
- The model simulates the joint evolution of ~ 100 stocks
- All options (with reasonable OI), forward prices, on all stocks, must be fitted to the model. Number of constraints ~500 to ~1000 or more
- Efficient calibration, pricing and sensitivity analysis in real-time environment

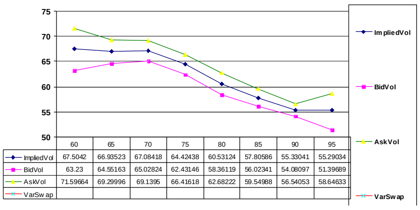
Example: Basket of 20 Biotechnology Stocks (Components of BBH)

Ticker	Price	ATM ImVol	Ticker	Price	ATM ImVol
ABI	17.85	55	GILD	30.05	46
AFFX	17.19	64	HGSI	16.99	84
ALKS	5.79	106	ICOS	23.62	64
AMGN	44.1	40	IDPH	43.31	72
BGEN	35.36	41	MEDI	27.75	82
CHIR	32.03	37	MLNM	11.8	92
CRA	10.2	55	QLTI	9.36	64
DNA	33.27	53.5	SEPR	6.51	84
ENZN	22.09	81	SHPGY	25.2	47
GENZ	21.66	56	BBH	81.5	32

Implied Volatility Skews

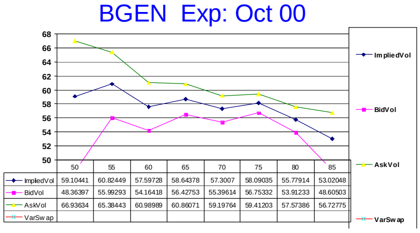
Multiple Names, Multiple Expirations

AMGN Exp: Oct 00



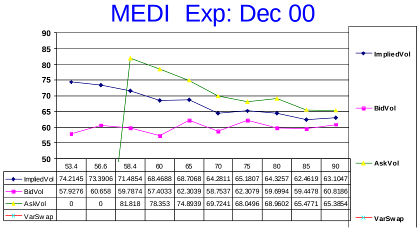
	60	65	70	75	80	85	90	95
ImpliedVol	67.5042	66.93523	67.08418	64.42438	60.53124	57.80586	55.33041	55.29034
BidVol	63.23	64.55163	65.02824	62.43146	58.36119	56.02341	54.08097	51.39689
AskVol	71.59664	69.29996	69.1395	66.41618	62.68222	59.54988	56.54053	58.64633
VarSwap								

BGEN Exp: Oct 00



	50	55	60	65	70	75	80	85
ImpliedVol	59.10441	60.82449	57.59728	58.64378	57.3007	58.09035	55.77914	53.02048
BidVol	48.36397	55.99293	54.16418	56.42753	55.39614	56.75332	53.91233	48.60503
AskVol	66.93634	65.38443	60.98989	60.86071	59.19764	59.41203	57.57386	56.72775
VarSwap								

MEDI Exp: Dec 00



	53.4	56.6	58.4	60	65	70	75	80	85	90
ImpliedVol	74.2145	73.3906	71.4854	68.4688	68.7068	64.2811	65.1807	64.3257	62.4619	63.1047
BidVol	57.9276	60.658	59.7874	57.4033	62.3039	58.7537	62.3079	59.6994	59.4478	60.8186
AskVol	0	0	81.818	78.353	74.8939	69.7241	68.0496	68.9602	65.4771	65.3854
VarSwap										

- 20-dimensional stochastic process
- fits option data (multiple expirations)
- martingale property

Multi-Dimensional Diffusion Model

$$\frac{dS_i}{S_i} = \sigma_i dZ_i + \mu_i dt \qquad \mu_i = r - d_i$$

ensures martingale property

$$dZ_i = \text{Brownian motion increment}$$

$$E\left(dZ_i dZ_j\right) = \rho_{ij} dt$$

1-Dimensional Problems

Dupire: local volatility as a function of stock price $\sigma = \sigma(S,t)$
Hull-White, Heston: more factors to model stochastic volatility
Rubinstein, Derman-Kani: implied ``trees''

These methods do not generalize to higher dimensions!

Main Challenges in Multi-Asset Models

- Modeling correlation, or co-movement of many assets
- Correlation may have to match market prices if index options are used as price inputs (time-dependence)
- Fitting single-asset implied volatilities which are time- and strike-dependent
- Large body of literature on 1-D models, but much less is known on intertemporal multi-asset pricing models

Beware of ``magic fixes'', e.g. Copulas

Weighted Monte Carlo

Avellaneda, Buff, Friedman, Grandchamp, Kruk: IJTAF 1999

- Build a discrete-time, multidimensional process for the asset price
- Generate many scenarios for the process by Monte Carlo Simulation
- Fit all price constraints using a Maximum-Entropy algorithm

Example 1: Discrete-Time Multidimensional Markov Process

Modeled after a diffusion

$$S_{n+1}^{(i)} = S_n^{(i)} \cdot \left[1 + \sigma_n^{(i)} \left(\sum_{j=1}^{N} \alpha_{ij} \xi_{n,j} \right) \sqrt{\Delta t} + \mu_n^{(i)} \Delta t \right]$$

$$\xi_{n,j} = \text{i.i.d. normals}$$

- Correlations estimated from econometric analysis
- Vols are ATM implied or estimated from data
- Time-dependence, seasonality effects, can be incorporated

Example 2: Multidimensional Resampling

Bootstrap: B. Efron

$$S_{ni} = \text{historical data matrix} \qquad n \le \nu \ (\text{sample size})$$

$$X_{ni} = \frac{S_{ni} - S_{(n-1)i}}{S_{(n-1)i}} \qquad Y_{ni} = \frac{X_{ni}}{\sqrt{\sum_{m=1}^{\nu} \left(X_{mi} - \overline{X_i} \right)^2}}$$

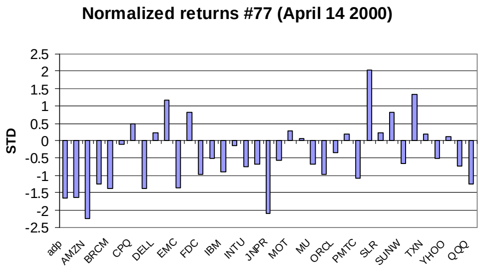
Use resampled standardized moves to generate scenarios

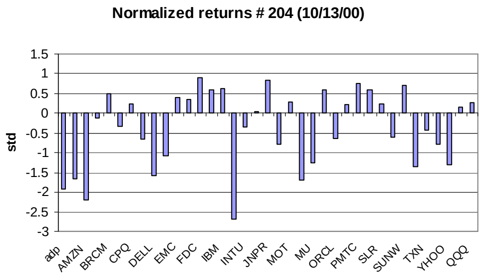
$$S_{n+1}^{(i)} = S_n^{(i)} \cdot \left[1 + \sigma_n^{(i)} Y_{R(n),i} \sqrt{\Delta t} + \mu_n^{(i)} \Delta t \right]$$

$$R(n) = \text{ random number between 1 and } \nu$$

R(n) can be uniform or have temporal correlation

Two draws from the empirical distribution (12/99-12/00)





Simulation consists of sequence of random draws from standardized empirical distribution

Calibration to Option and Forward Prices

- Evaluate Discounted Payoffs of reference instruments along different paths

$$g_{ij} = e^{-rT_j} \max\left(S_{i,T_j}^{a_j} - K_j, 0\right)$$

$$i = 1,\ldots,N \quad \text{(number of simulated paths)}$$

$$j = 1,\ldots,M \quad \text{(number of reference instruments)}$$

$$C_j = \text{midmarket price of } j^{th} \text{ reference instrument}$$

- Solve

$$\begin{pmatrix} C_1 \\ \cdots \\ C_M \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} & \cdots & \cdots & g_{1N} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ g_{M1} & \cdots & \cdots & \cdots & g_{MN} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ \cdots \\ p_N \end{pmatrix}$$

- Repricing condition

$$\boxed{C_j = E^P\left(g_j(S)\right), \qquad j = 1,2,\ldots,M}$$

Maximum-Entropy Algorithm

$$H(p) = -\sum_{i=1}^{N} p_i \log p_i = -D(p \parallel u) \quad u = \left(\frac{1}{N}, \ldots, \frac{1}{N}\right)$$

Algorithm

$$\max_p H(p) \quad \text{subject to price constraints}$$
$$\min_p D(p \parallel u) \qquad \qquad \text{"}$$

Stutzer, 1996; Buchen and Kelly, 1997;
Avellaneda, Friedman, Holmes, Samperi, 1997; Avellaneda 1998
Cont and Tankov, 2002, Laurent and Leisen, 2002,
Follmer and Schweitzer, 1991; Marco Frittelli MEM

Calibrated Probabilities are Gibbs Measures

Lagrange multiplier approach for solving constrained optimization gives rise to M-parameter family of Gibbs-type probabilities

$$p_i = p_i = \frac{1}{Z(\lambda)} \exp\left[\sum_{j=1}^{M} \lambda_j g_{ij}\right], \qquad i = 1,2,\ldots,N$$

Unknown parameters

$$Z(\lambda) = \sum_{i=1}^{N} \exp\left[\sum_{j=1}^{M} \lambda_j g_{ij}\right]$$

Boltzmann-Gibbs partition function

Calibration Algorithm
How do we find the lambdas?

- Minimize in lambda

$$W(\lambda) = \log Z(\lambda) - \sum_{j=1}^{M} \lambda_j C_j$$

- W is a convex function
- The minimum is unique, if it exists
- W is differentiable in C, lambda
- Use L-BFGS Quasi-Newton gradient-based optimization routine

Boltzmann-Gibbs formalism

$$\frac{\partial W(\lambda)}{\partial \lambda_j} = E^{P_\lambda}\left(G_j(X)\right) - C_j$$

Gradient=difference between market px and model px

$$\frac{\partial^2 W(\lambda)}{\partial \lambda_j \partial \lambda_k} = E^{P_\lambda}\left(G_j(X) G_k(X)\right) - C_j C_k = Cov^{P_\lambda}\left(G_j(X), G_k(X)\right)$$

Hessian=covariance of cash-flows under pricing measure

Numerical optimization with known gradient & Hessian

Least-Squares Variant

$$\chi^2 = \sum_{j=1}^{M}\left(\sum_{i=1}^{N} g_{ij} p_i - C_j\right)^2 = \sum_{j=1}^{M}\left(E^P\left(g_j(S)\right) - C_j\right)^2$$

$$\min_p \left\{ -H(p) + \frac{\chi^2}{2\varepsilon^2} \right\}$$

Max entropy with least-squares constraint

$$\min_\lambda \left\{ \ln Z(\lambda) + \sum_{j=1}^{M} \lambda_j C_j + \frac{\varepsilon^2}{2} \sum_{j=1}^{M} \lambda_j^2 \right\}$$

Equivalent to adding quadratic term to objective function

Sensitivity Analysis

$h(X) = \text{payoff function of ``target security''}$

$E^{P_\lambda}(h(X)) = \text{model value of} \quad ''$

$$\frac{\partial E^{P_\lambda}(h(X))}{\partial C_j} = \frac{\partial E^{P_\lambda}(h(X))}{\partial \lambda_k} \frac{\partial \lambda_k}{\partial C_j}$$

$$= Cov^{P_\lambda}\left(h(X), g_k(X)\right) \cdot \left(\frac{\partial C}{\partial \lambda_k}\right)^{-1}_{kj}$$

$$= Cov^{P_\lambda}\left(h(X), g_k(X)\right) \cdot \left(Cov^{P_\lambda}\left(g_\bullet(X), g_\bullet(X)\right)\right)^{-1}_{jk}$$

Price-Sensitivities=``Betas”

Solve LS problem:

$$\min_{\beta,\alpha} \sum_{i=1}^{\nu} p_i \left(h(X_i) - \alpha - \sum_{j=1}^{M} \beta_j G_j(X_i) \right)^2$$

Uncorrelated to gj(X)

$$h(X) = \alpha + \sum_{j=1}^{M} \beta_j g_j(X) + \varepsilon(X)$$

Minimal Martingale Measure?

Avellaneda and Fisher, forthcoming 2003

- Boltzmann-Gibbs posterior measure with price constraints is not a local martingale
- Remedy: include additional constraints:

$$g(S) = \psi\left(S_{t_1},\dots,S_{t_N}\right)\left(S_{t_{N+1}} - S_{t_N}\right) \qquad \psi\left(S_{t_1},\dots,S_{t_N}\right) = \text{polynomial function}$$

$$\text{Martingale constraint:} \qquad E^P\left(g(S)\right) = 0 \quad \text{for all } \psi$$

- Constrained Max-Entropy problem with martingale constraints: Follmer-Schweitzer MEM under constraints
- In practice, use only low-degree polynomials (deg=0 or deg=1)

Performance of the algorithm for equity derivatives

- 100+ Stocks easily implemented
- 6 months to one year horizon (~ 5 expirations)
- 1000+ options and forwards
- Calibration time: < 5 minutes on single-processor PIII with 900 MHZ
- Scales almost linearly with number of processors.

Allows for real-time implementation and for exploring new groups of stocks with a relatively small computational effort

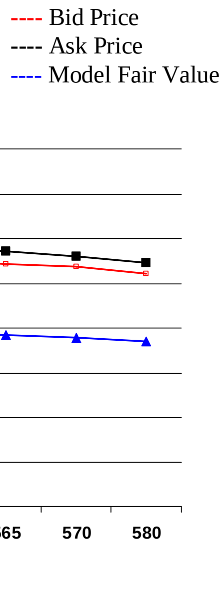
Application: “Relative Value’’ Pricing of Index Options

- Observe current prices of index options
- Observe current prices of options on index components
- Build model to determine whether index options are ``rich’’ or ``cheap’’ in relation to the components

Crucial points:

(I) incorporate volatility skew for each component
(II) fit the information together (across components)

Broad Market Index Options
Pricing Date: Oct 9, 2001



Some Exchange-Traded Funds and Indices with Options

QQQ : NASDAQ 100 Trust (AMEX)

MSH: Morgan Stanley 35 High-Technology Index (AMEX)

SOX: Semiconductor Index (PHLX)

BTK: Biotechnology Index (AMEX)

XNG: Natural Gas Index (AMEX)

XAU: Gold Index (AMEX)

COMS	CMGI	LGTO	PSFT
ADPT	CNET	LVLT	PMCS
ADCT	CMCSK	LLTC	QLGC
ADLAC	CPWR	ERICY	QCOM
ADBE	CMVT	LCOS	QTRN
ALTR	CEFT	MXIM	RNWK
AMZN	CNXT	MCLD	RFMD
APCC	COST	MEDI	SANM
AMGN	DELL	MFNX	SDLI
APOL	DLTR	MCHP	SEBL
AAPL	EBAY	MSFT	SIAL
AMAT	DISH	MOLX	SSCC
AMCC	ERTS	NTAP	SPLS
ATHM	FISV	NETA	SBUX
ATML	GMST	NXTL	SUNW
BBBY	GENZ	NXLK	SNPS
BGEN	GBLX	NWAC	TLAB
BMET	MLHR	NOVL	USAI
BMCS	ITWO	NTLI	VRSN
BVSN	IMNX	ORCL	VRTS
CHIR	INTC	PCAR	VTSS
CIEN	INTU	PHSY	VSTR
CTAS	JDSU	SPOT	WCOM
CSCO	JNPR	PMTC	XLNX
CTXS	KLAC	PAYX	YHOO

NASDAQ 100 Index (NDX) and ETF (QQQ)

- Capitalization-weighted average of 100 largest stocks in NASDAQ
- QQQ trades as a stock
- QQQ index option is the most actively traded contract in AMEX

Morgan Stanley 35 (MSH)

ADP	JDSU
AMAT	JNPR
AMZN	LU
AOL	MOT
BRCM	MSFT
CA	MU
CPQ	NT
CSCO	ORCL
DELL	PALM
EDS	PMTC
EMC	PSFT
ERTS	SLR
FDC	STM
HWP	SUNW
IBM	TLAB
INTC	TXN
INTU	XLNX
	YHOO

- 35 Underlying Stocks
- Equal-dollar weighted index, adjusted annually
- Each stock has typically O(30) options over a 1yr horizon

Test problem: 35 tech stocks

Price options on basket of 35 stocks underlying the MSH index

Number of constraints: 876

Number of paths: 10,000 to 30,000 paths

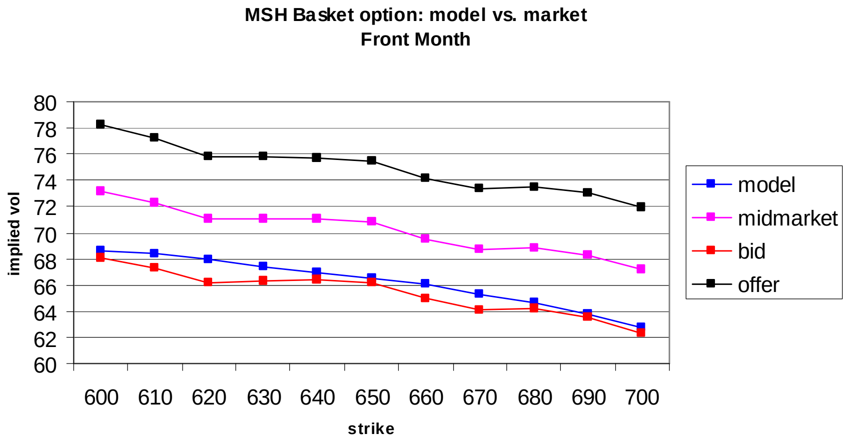
Optimization technique: Quasi-Newton method (explicit gradient)

OptionN	StockTi	ExpDat	Strike	Type	Intrinsic	Bid	Ask	Volume	OpenInt	StockPr	QuoteD
ZQN AC-E	AMZN	1/20/01	15	Call	0	4.125	4.375	13	3058	16.6875	12/20/00
ZQN AT-E	AMZN	1/20/01	16.75	Call	0	3.125	3.375	0	1312	16.6875	12/20/00
ZQN AO-E	AMZN	1/20/01	17.5	Call	0	2.875	3.25	20	10	16.6875	12/20/00
ZQN AU-E	AMZN	1/20/01	18.375	Call	0	2.625	2.875	10	338	16.6875	12/20/00
ZQN AD-E	AMZN	1/20/01	20	Call	0	1.9375	2.125	223	5568	16.6875	12/20/00
ZQN BC-E	AMZN	2/17/01	15	Call	0	5.125	5.625	30	1022	16.6875	12/20/00
ZQN BO-E	AMZN	2/17/01	17.5	Call	0	4	4.375	0	0	16.6875	12/20/00
ZQN BD-E	AMZN	2/17/01	20	Call	0	3.125	3.5	10	150	16.6875	12/20/00
ZQN DC-E	AMZN	4/21/01	15	Call	0	5.875	6.375	0	639	16.6875	12/20/00
ZQN DO-E	AMZN	4/21/01	17.5	Call	0	5	5.375	0	168	16.6875	12/20/00
ZQN DD-E	AMZN	4/21/01	20	Call	0	3.875	4.125	5	1877	16.6875	12/20/00
ZQN DS-E	AMZN	4/21/01	22.5	Call	0	3.125	3.375	20	341	16.6875	12/20/00
ZQN GC-E	AMZN	7/21/01	15	Call	0	6.875	7.375	0	134	16.6875	12/20/00
ZQN GO-E	AMZN	7/21/01	17.5	Call	0	5.625	6.125	0	63	16.6875	12/20/00
ZQN GD-E	AMZN	7/21/01	20	Call	0	4.875	5.25	5	125	16.6875	12/20/00
ZQN GS-E	AMZN	7/21/01	22.5	Call	0	4.125	4.5	0	180	16.6875	12/20/00
ZQN GE-E	AMZN	7/21/01	25	Call	0	3.5	3.875	65	79	16.6875	12/20/00
AOE AZ-E	AOL	1/20/01	32.5	Call	0	6.6	7	20	1972	37.25	12/20/00
AOE AO-E	AOL	1/20/01	33.75	Call	0	5.6	6	0	596	37.25	12/20/00
AOE AG-E	AOL	1/20/01	35	Call	0	4.7	5.1	153	5733	37.25	12/20/00
AOE AU-E	AOL	1/20/01	37.5	Call	0	3.4	3.7	131	3862	37.25	12/20/00
AOE AH-E	AOL	1/20/01	40	Call	0	2.5	2.7	1229	19951	37.25	12/20/00
AOE AR-E	AOL	1/20/01	41.25	Call	0	2	2.3	6	1271	37.25	12/20/00
AOE AV-E	AOL	1/20/01	42.5	Call	0	1.65	1.85	219	4423	37.25	12/20/00
AOE AS-E	AOL	1/20/01	43.75	Call	0	1.3	1.5	44	3692	37.25	12/20/00
AOE AI-E	AOL	1/20/01	45	Call	0	1.2	1.25	817	11232	37.25	12/20/00
AOE BZ-E	AOL	2/17/01	32.5	Call							
AOE BG-E	AOL	2/17/01	35	Call							
AOE BU-E	AOL	2/17/01	37.5	Call							
AOE BH-E	AOL	2/17/01	40	Call							
AOE BV-E	AOL	2/17/01	42.5	Call							
AOE BI-E	AOL	2/17/01	45	Call							
AOE DZ-E	AOL	4/21/01	32.5	Call							
AOE DG-E	AOL	4/21/01	35	Call	0	6.9	7.3	32	179	37.25	12/20/00
AOE DU-E	AOL	4/21/01	37.5	Call	0	5.5	5.9	36	200	37.25	12/20/00
AOE DH-E	AOL	4/21/01	40	Call	0	4.5	4.9	264	2164	37.25	12/20/00
AOE DV-E	AOL	4/21/01	42.5	Call	0	3.6	3.9	209	632	37.25	12/20/00
AOE DI-E	AOL	4/21/01	45	Call	0	2.9	3.1	415	3384	37.25	12/20/00
AOE DW-E	AOL	4/21/01	47.5	Call	0	2.15	2.45	37	1174	37.25	12/20/00
AOO DJ-E	AOL	4/21/01	50	Call	0	1.75	1.95	224	7856	37.25	12/20/00
AOE GZ-E	AOL	7/21/01	32.5	Call	0	9.4	9.8	0	0	37.25	12/20/00

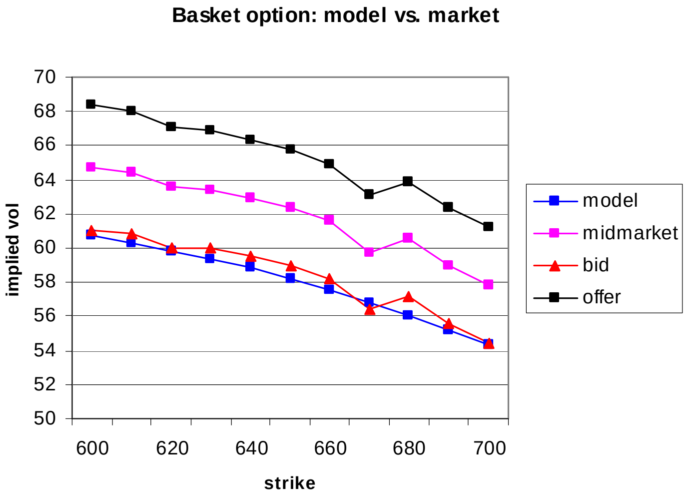
Fragment of data for calibration with 876 constraints

Near-month options

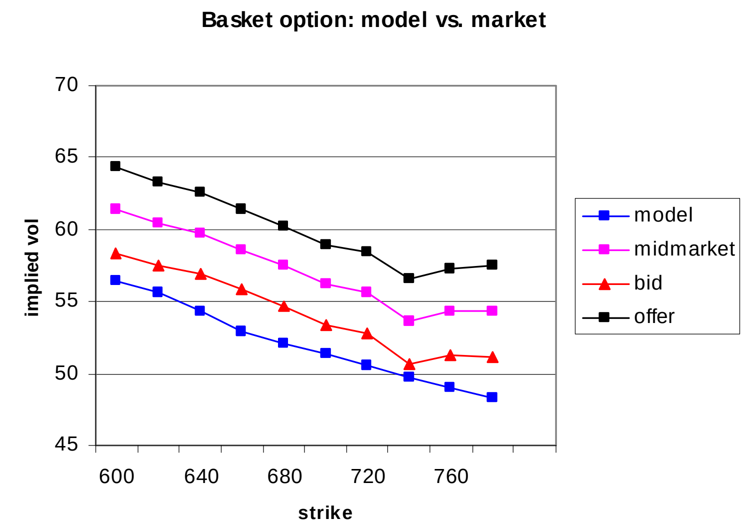
(Pricing Date: Dec 2000)



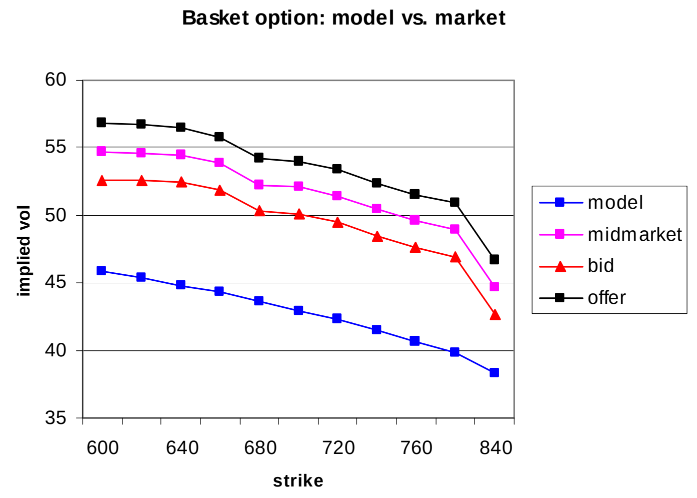
Second-month options



Third-month options



Six-month options



Entropy as a Measure of Information Content

Claude Shannon, 1941

$$0 \le H(p) \equiv -\sum_{k=1}^{N} p_k \log p_k \le \log N$$

$H(p) = \log \nu \Rightarrow \;\; p_k = 1/N$ — No discrepancy between current market and prior distribution

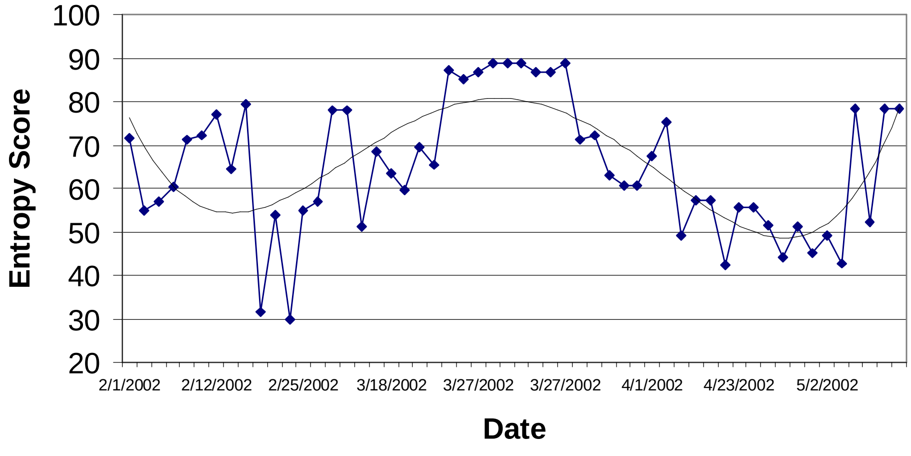
$H(p) \approx 0 \quad \Rightarrow$ — Extreme departure from prior distribution: signals internal mispricing ☺ (or data entry error ☹)

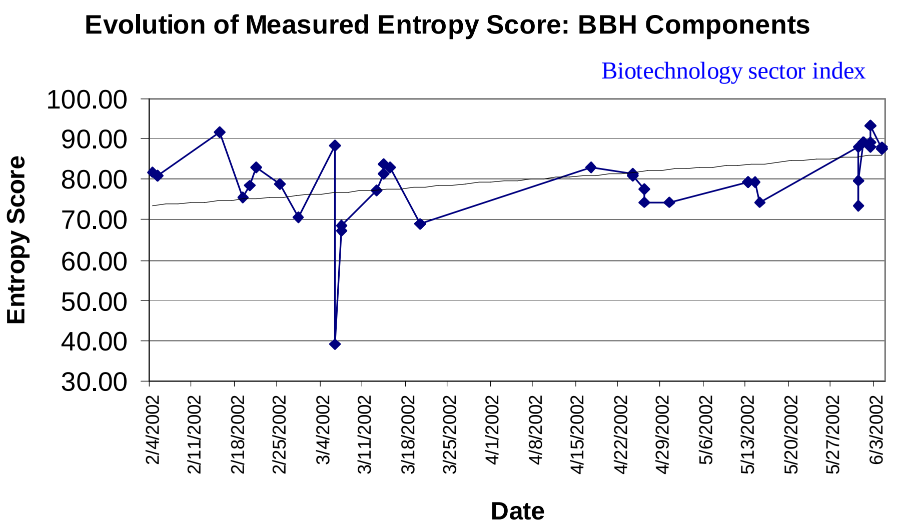
$$\text{Entropy score} = 100 \cdot \left(\frac{H(p)}{\log N} \right)$$

Measures ``information content'' of the data

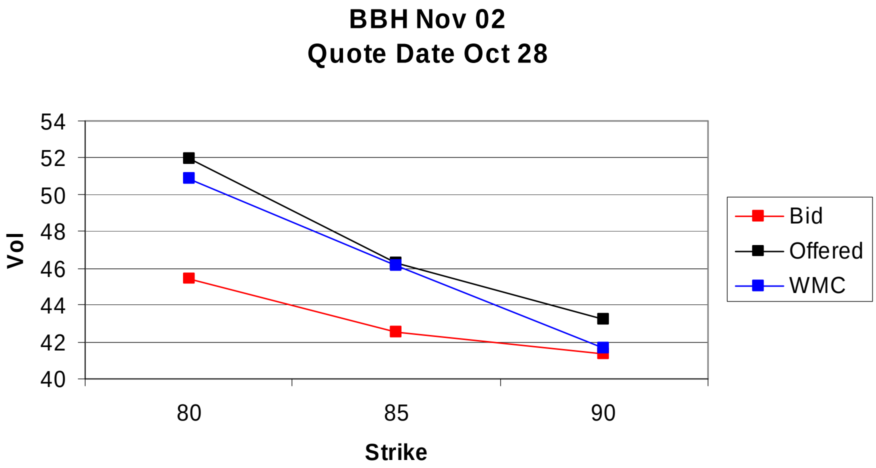
Evolution of Measured Entropy Score : DJX components

Industry-diversified blue-chip index

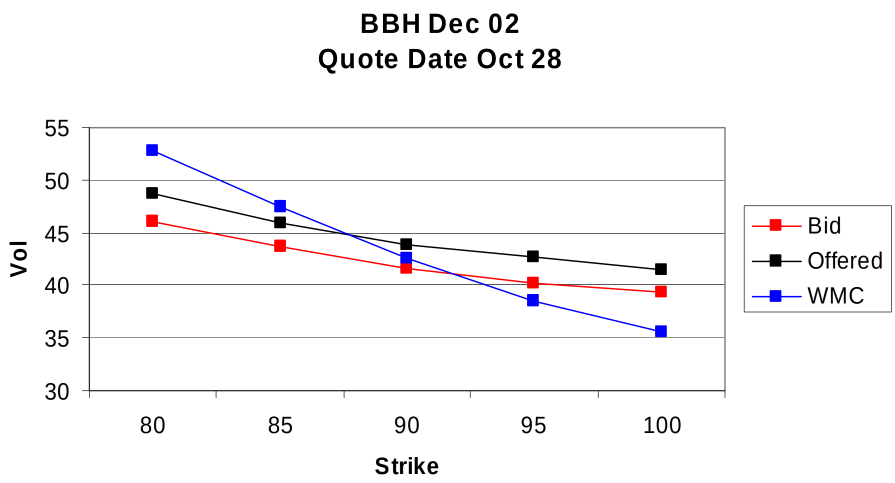




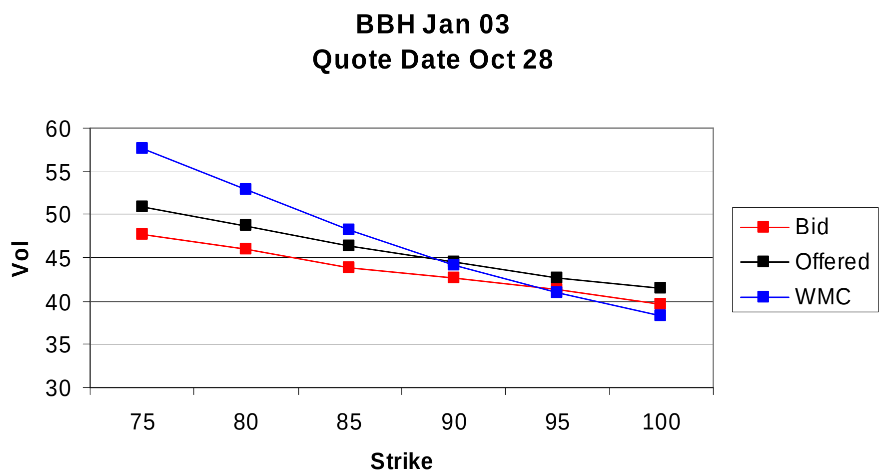
BBH Index Options



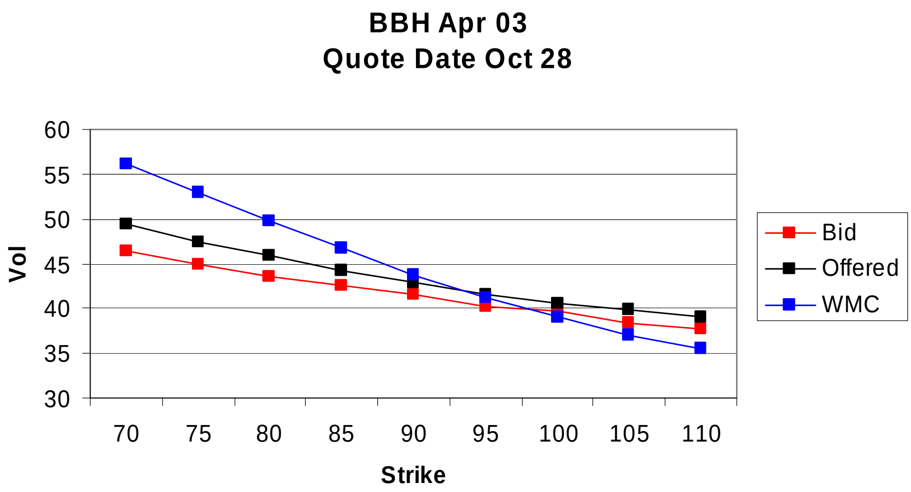
BBH Index Options



BBH Index Options



BBH Index Options



Comparison With Steepest-Descent Approximation

Avellaneda, Boyer-Olson, Busca, Friz: RISK 2002, CRAS Paris 2003

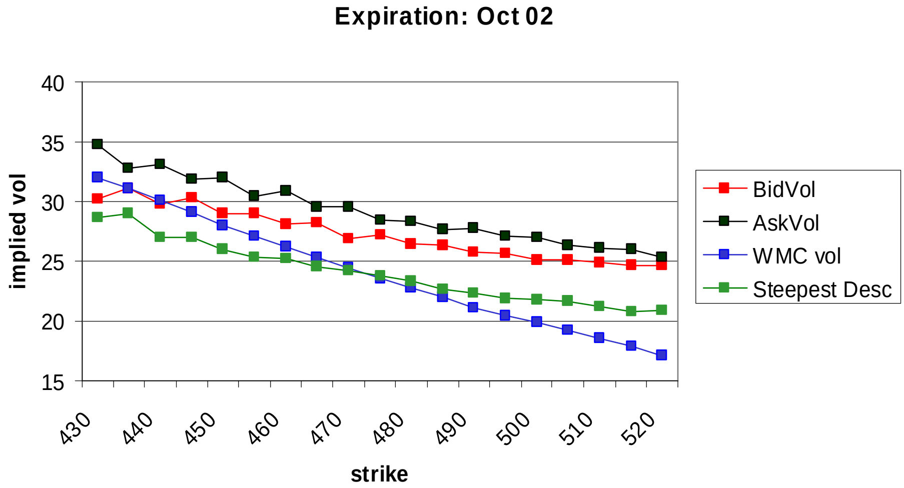
- Based on Steepest Descent Approximation, or short-time asymptotics for diffusion kernels
- Use implied volatilities (co-terminal) of options on underlying stocks
- Use historical or estimated correlation matrix

$$\sigma_I(\Delta) \cong \frac{1}{2}\sigma_{I,ATM} + \frac{1}{2}\sqrt{\sum_{ij=1}^{M} p_i p_j \rho_{ij} \left(2\sigma_i(\Delta_i) - \sigma_{i,ATM}\right)\left(2\sigma_j(\Delta_i) - \sigma_{j,ATM}\right)}$$

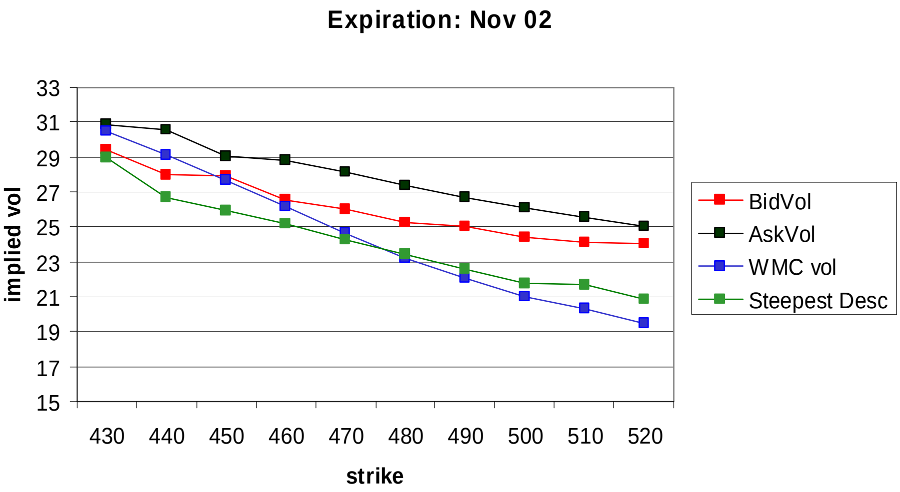
$$\Delta_i = \mathrm{N}\left[\mathrm{N}^{-1}(\Delta) \times \sum_{j=1}^{M} p_j \rho_{ij} \frac{\sigma_{i,ATM}}{\sigma_{I,ATM}}\right] \qquad \mathrm{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$$

S&P 100 Index Options
(Quote date: Aug 20, 2002)

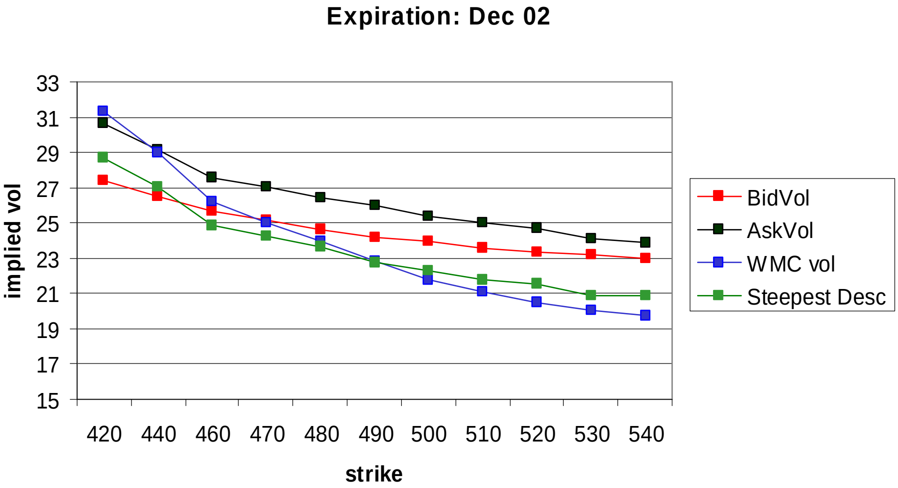
S&P 100 Index Options
(Quote date: Aug 20, 2002)



S&P 100 Index Options
(Quote date: Aug 20, 2002)



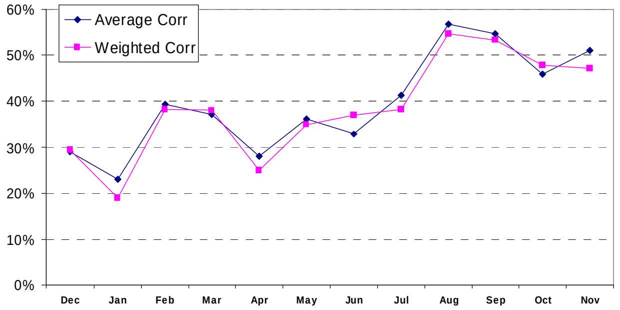
S&P 100 Index Options
(Quote date: Aug 20, 2002)



Dow Industrial Average (DJX)

Volatility

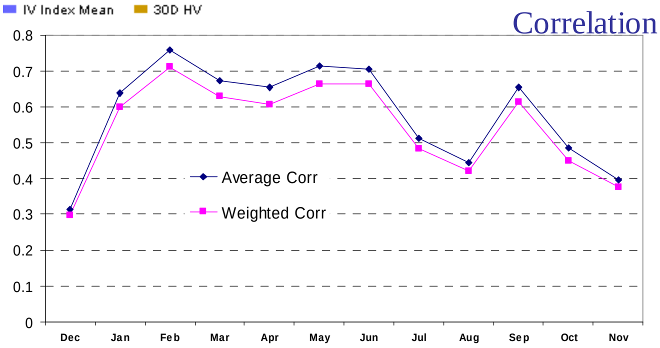
Correlation



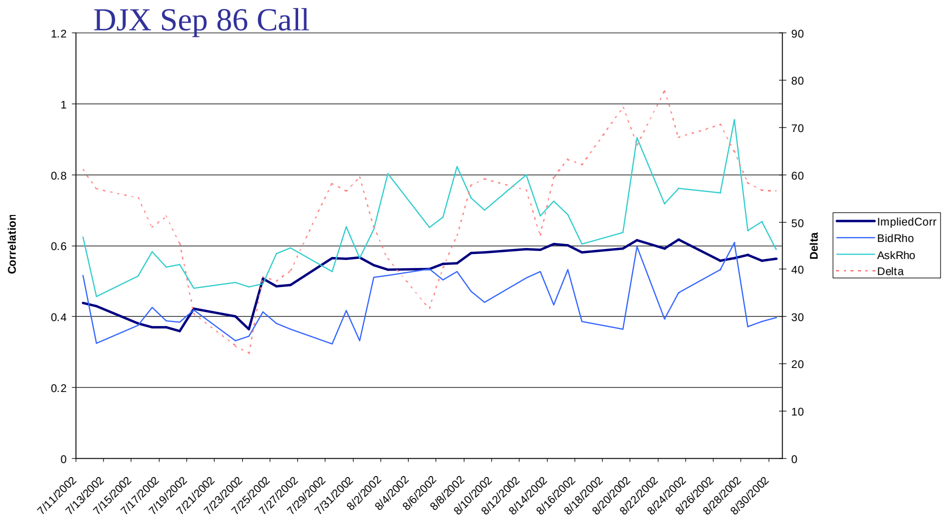
Amex Biotechnology Index (BTK)

Volatility

Correlation



DJX Correlation Blowout, July 2002



Conclusions

- Weighted Monte Carlo: convenient framework for pricing and hedging derivatives with many underlying assets
- Non-parametric: avoids the use of latent variables (`market model') and is able to fit econometric and price data in detail
- Results are consistent with Steepest Descent reconstruction algorithm for pricing index options
- Entropy ~ information content of the market. Allows for monitoring the information content in a large group of quotes and to search for arbitrage opportunities
- Obvious important extensions: credit derivatives and capital structure arbitrage (joint model for equity, corporate debt and credit derivatives)