FINANCE CONCEPTS

# Reconstructing Volatility

New techniques for understanding the implied volatility of multi-asset options

Speaker: Marco Avellaneda

Avellaneda, Boyer-Olson, Busca and Friz:
`Reconstructing Volatility', *RISK Oct 2002*; `Large Deviations Methods and the Pricing of Index Options in Finance', *CRAS Paris 2003*

## Outline

- Major US indices and ETFs
- Implied volatility surfaces of single stocks and indices
- Marginalization
- `Reconstructing' the implied volatility of index options
- Steepest-descent Approximation
- The most likely market configurations
- Multivariate stochastic volatility models
- Moment-matching technique: Lee, Wang and Karim
- Cross-currency options

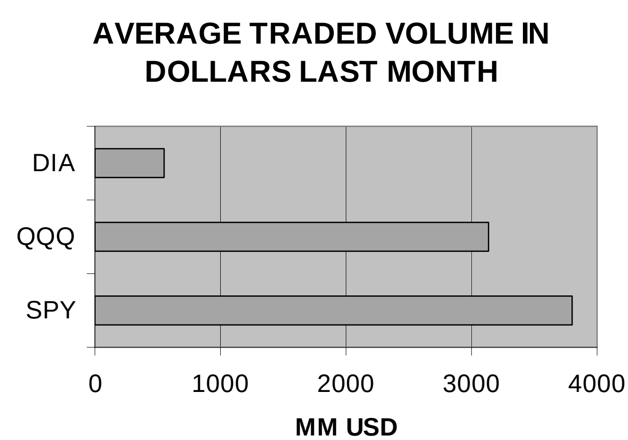
# U.S. Equities: Main Sectors & Their Indices

- Major Indices: SPX, DJX, NDX
  SPY, DIA, QQQ (Exchange-Traded Funds)

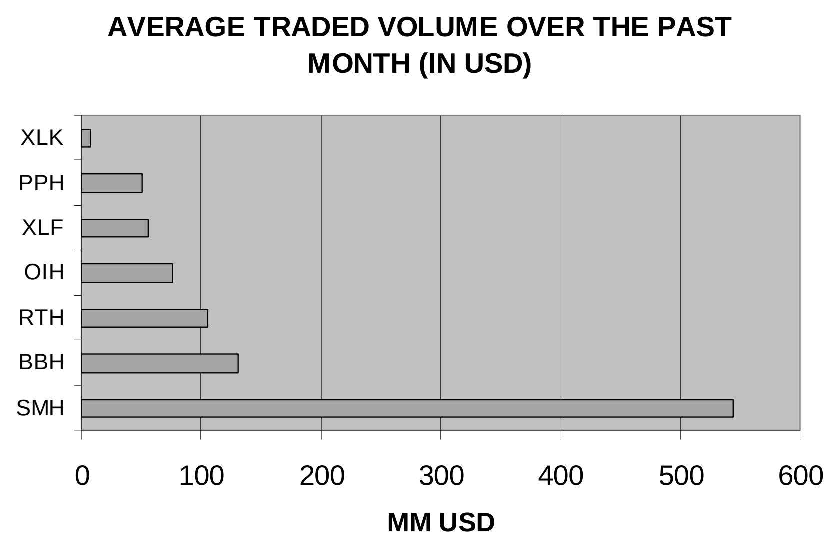
- Sector Indices & Index Trackers:
  Semiconductors: SMH, SOX
  Biotech: BBH, BTK
  Pharmaceuticals: PPH, DRG
  Financials: BKX, XBD, XLF, RKH
  Oil & Gas: XNG, XOI, OSX
  High Tech, WWW, Boxes: MSH, HHH, XBD, XCI
  Retail: RTH

All these indices have options

# Trading statistics (AMEX)



# Sectors




| | | | |
|---|---|---|---|
| COMS | CMGI | LGTO | PSFT |
| ADPT | CNET | LVLT | PMCS |
| ADCT | CMCSK | LLTC | QLGC |
| ADLAC | CPWR | ERICY | QCOM |
| ADBE | CMVT | LCOS | QTRN |
| ALTR | CEFT | MXIM | RNWK |
| AMZN | CNXT | MCLD | RFMD |
| APCC | COST | MEDI | SANM |
| AMGN | DELL | MFNX | SDLI |
| APOL | DLTR | MCHP | SEBL |
| AAPL | EBAY | MSFT | SIAL |
| AMAT | DISH | MOLX | SSCC |
| AMCC | ERTS | NTAP | SPLS |
| ATHM | FISV | NETA | SBUX |
| ATML | GMST | NXTL | SUNW |
| BBBY | GENZ | NXLK | SNPS |
| BGEN | GBLX | NWAC | TLAB |
| BMET | MLHR | NOVL | USAI |
| BMCS | ITWO | NTLI | VRSN |
| BVSN | IMNX | ORCL | VRTS |
| CHIR | INTC | PCAR | VTSS |
| CIEN | INTU | PHSY | VSTR |
| CTAS | JDSU | SPOT | WCOM |
| CSCO | JNPR | PMTC | XLNX |
| CTXS | KLAC | PAYX | YHOO |

## Components of NASDAQ 100 Trust (AMEX:QQQ)

- Capitalization-weighted average of 100 largest stocks in NASDAQ
- QQQ trades as a stock
- QQQ index options are the most heavily traded contract in AMEX

# Morgan Stanley 35 (MSH)

| | |
|---|---|
| ADP | JDSU |
| AMAT | JNPR |
| AMZN | LU |
| AOL | MOT |
| BRCM | MSFT |
| CA | MU |
| CPQ | NT |
| CSCO | ORCL |
| DELL | PALM |
| EDS | PMTC |
| EMC | PSFT |
| ERTS | SLR |
| FDC | STM |
| HWP | SUNW |
| IBM | TLAB |
| INTC | TXN |
| INTU | XLNX |
| | YHOO |

- 35 Underlying Stocks
- Equal-dollar weighted index, adjusted annually
- Each stock has typically O(30) options over a 1yr horizon

# SOX, XNG, XOI

| XNG |
|---|
| |
| APA |
| APC |
| BR |
| BRR |
| EEX |
| ENE |
| EOG |
| EPG |
| KMI |
| NBL |
| NFG |
| OEI |
| PPP |
| STR |
| WMB |

| XOI |
|---|
| |
| AHC |
| BP |
| CHV |
| COC.B |
| XOM |
| KMG |
| OXY |
| P |
| REP |
| RD |
| SUN |
| TX |
| TOT |
| UCL |
| MRO |

| SOX |
|---|
| |
| ALTR |
| AMAT |
| AMD |
| INTC |
| KLAC |
| LLTC |
| LSCC |
| LSI |
| MOT |
| MU |
| NSM |
| NVLS |
| RMBS |
| TER |
| TXN |
| XLNX |

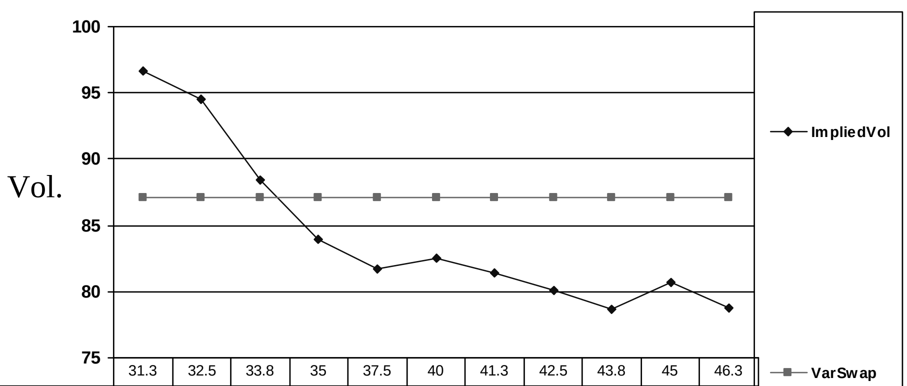
… & many others

# Example: Basket of 20 Biotechnology Stocks ( Components of BBH)

| Ticker | Shares | ATM ImVol | Ticker | Shares | ATM ImVol |
|---|---|---|---|---|---|
| ABI | 18 | 55 | GILD | 8 | 46 |
| AFFX | 4 | 64 | HGSI | 8 | 84 |
| ALKS | 4 | 106 | ICOS | 4 | 64 |
| AMGN | **46** | 40 | IDPH | 12 | 72 |
| BGEN | 13 | 41 | MEDI | 15 | 82 |
| CHIR | 16 | 37 | MLNM | 12 | 92 |
| CRA | 4 | 55 | QLTI | 5 | 64 |
| DNA | 44 | 53.5 | SEPR | 6 | 84 |
| ENZN | 3 | 81 | SHPGY | 6.8271 | 47 |
| GENZ | 14 | 56 | BBH | - | 32 |

# AOL Jan 2001 Options:

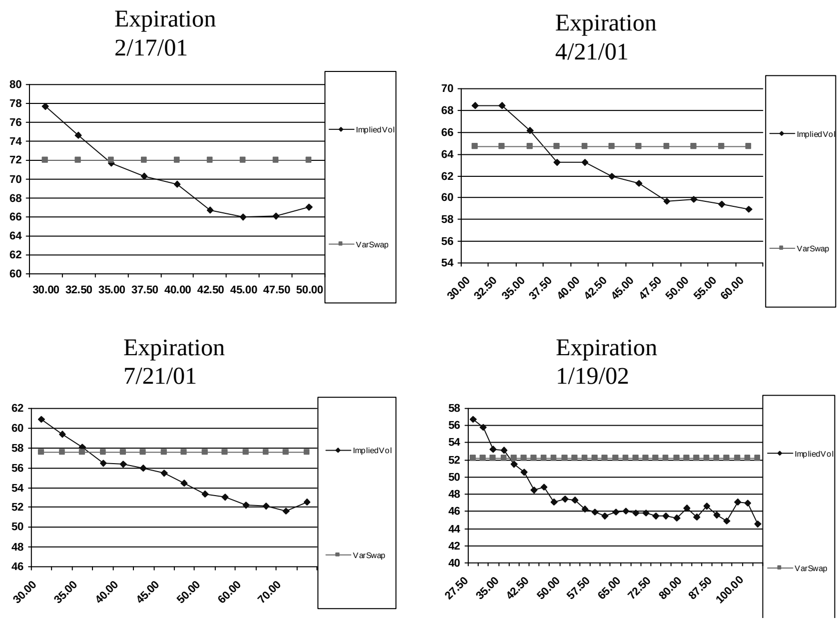
Implied volatility curve on Dec 20,2000
Market close




| | 31.3 | 32.5 | 33.8 | 35 | 37.5 | 40 | 41.3 | 42.5 | 43.8 | 45 | 46.3 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| ImpliedVol | 96.6191 | 94.5071 | 88.4581 | 83.9929 | 81.7033 | 82.5468 | 81.4319 | 80.1212 | 78.6667 | 80.7064 | 78.8035 |
| VarSwap | 87.1215 | 87.1215 | 87.1215 | 87.1215 | 87.1215 | 87.1215 | 87.1215 | 87.1215 | 87.1215 | 87.1215 | 87.1215 |

Strike

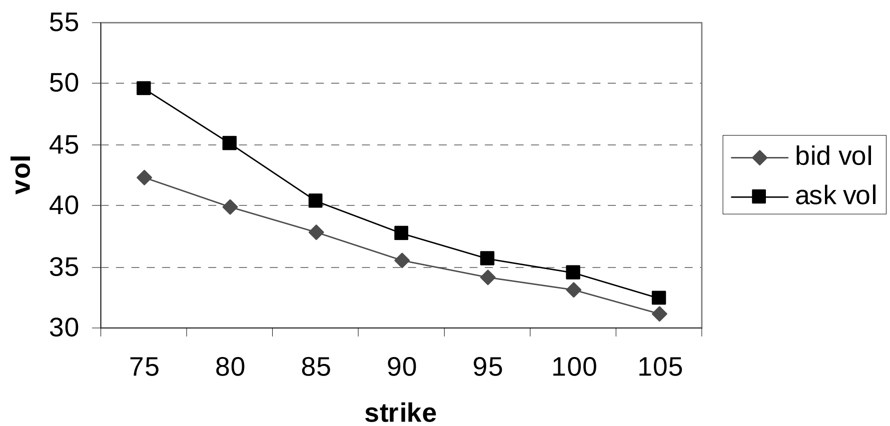
Stock probability is not lognormal



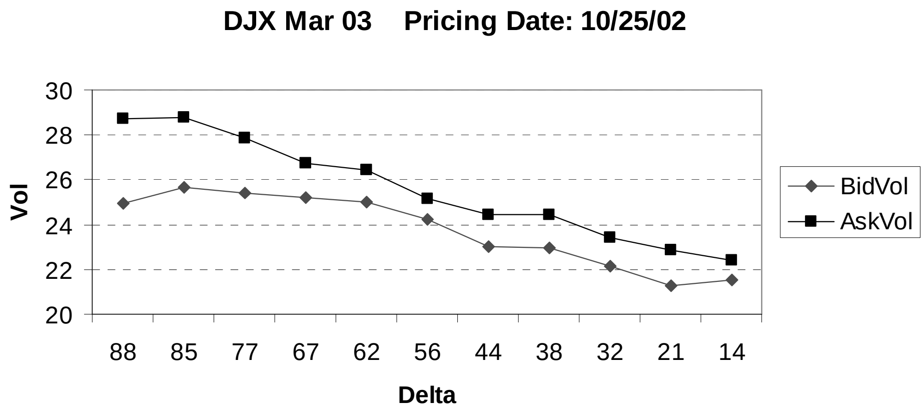

The AOL ``volatility skew'' for several expiration dates

## BBH March 2003 Implied Vols

Pricing Date: Jan 22 03 10:42 AM



## Implied Volatility Curve for Options on Dow Jones Average




## Stylized facts about equity volatility curves

- Implied volatility curves are typically downward sloping
- Counterexamples: precious metal stocks are upward sloping
- There is little curvature (or smile). Skew is important.

## Modeling single-stock volatility curves

$$\frac{dS}{S} = \sigma_t dW$$

Price dynamics not lognormal

$\sigma_t = \sigma(S,t)$ Dupire's Local Volatility

$\frac{d\sigma_t}{\sigma_t} = \kappa dZ_t$ Stochastic Volatility

$\sigma_{\text{implied}}(K,T) = \sigma_{\text{implied}}(S,T)\cdot(1 + a\ln(K/S))$ Empirical Fit

Jumps …

## What is the relation between index options and options on the components?

Standard (log-normal) Volatility Formula for Index Options

$$\sigma_I^2 = \sum_{j=1}^{N} p_j^2 \sigma_j^2 + \sum_{i \neq j} p_i\, p_j \sigma_i\, \sigma_j \rho_{ij} \qquad (*)$$

Does not apply when volatilities are strike-dependent

How can we incorporate volatility skew information into (*)?

# "Marginalization"

Consider an *n+1* dimensional diffusion:

$$(X,Y)\in R^{n+1}$$

$$\begin{cases} dX^i = \sum_{j=1}^{m} \sigma_j^i(X,Y,t)dW^j + \mu^i(X,Y,t)dt \quad i=1,\dots,n, \\ dY = \sum_{j=1}^{m} \kappa_j(X,Y,t)dW^j + \nu(X,Y,t)dt \end{cases}$$

Problem: given a starting point, $(X_0,Y_0,t_0)$, find a <u>one-dimensional diffusion</u>

$$d\overline{Y} = \overline{\kappa}(\overline{Y},t)dW + \overline{\kappa}(\overline{Y},t)dt$$

such that $Y(t)$ and $\overline{Y}(t)$ have the <u>same probability distributions</u> for all $t > t_0$

# Fokker-Planck Equation

$$\pi(x,y,t) \equiv \pi(X_0,Y_0,t_0;x,y,t)$$

transition probability function

$$\frac{\partial \pi(x,y,t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\kappa^2 \pi(x,y,t)\right) + \sum_i \frac{\partial^2}{\partial y \partial x^i}(\dots) + \sum_{ij} \frac{\partial^2}{\partial x^i \partial x^j}(\dots) - \frac{\partial}{\partial y}\left(\nu \pi(x,y,t)\right) - \sum_i \frac{\partial}{\partial x^i}(\dots)$$

# Computing the marginal distributions…

$$\bar{\pi}(y,t) \equiv \int \pi(x,y,t)\, d^n x$$

integration with respect to x eliminates terms with x-derivatives

$$\frac{\partial \bar{\pi}(y,t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\overline{\kappa^2 \pi(\cdot,y,t)}\right) - \frac{\partial}{\partial y}\left(\overline{\nu \pi(\cdot,y,t)}\right)$$

$$\therefore$$

$$\frac{\partial \bar{\pi}(y,t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\frac{\overline{\kappa^2 \pi(\cdot,y,t)}}{\bar{\pi}(y,t)}\bar{\pi}(y,t)\right) - \frac{\partial}{\partial y}\left(\frac{\overline{\nu \pi(\cdot,y,t)}}{\bar{\pi}(y,t)}\bar{\pi}(y,t)\right)$$

new 1D diffusion diffusion equation

# Equivalent 1-D diffusion

$$\bar{\kappa}^2(y,t) \equiv \frac{\overline{\kappa^2(\cdot,y,t)\pi(\cdot,y,t)}}{\bar{\pi}(y,t)} = \frac{\int \kappa^2(x,y,t)\pi(x,y,t)\, d^n x}{\int \pi(x,y,t)\, d^n x}$$

$$\bar{\nu}\ (y,t) \equiv \frac{\overline{\nu(\cdot,y,t)\pi(\cdot,y,t)}}{\bar{\pi}(y,t)} = \frac{\int \nu(x,y,t)\pi(x,y,t)\, d^n x}{\int \pi(x,y,t)\, d^n x}$$

$$\bar{\kappa}^2(y,t) = E\left\{\kappa^2(X(t),Y(t),t)\middle| Y(t)=y\right\}$$

$$\bar{\nu}(y,t) = E\left\{\nu(X(t),Y(t),t)\middle| Y(t)=y\right\}$$

Equivalent parameters = conditional expectations of local parameters

# Application to Index Options

$$I = \sum_{i=1}^{n} w_i S_i$$

Index = weighted sum of stock prices (constant weights)

Diffusion eq.
for each stock
reflects vol skew

$$\begin{cases} \dfrac{dS_i}{S_i} = \sigma_i(S_i, t)dW_i + \mu_i dt, & \mu_i = r - d_i, \\ E(dW_i dW_j) = \rho_{ij} dt \end{cases}$$

$$\frac{dI}{I} = \sigma_{\text{loc}}(S,t)dZ + \mu_{\text{loc}}(S,t)dt$$

$$\sigma^2{}_{\text{loc}}(S,t) = \frac{\sum_{ij} \sigma_i(S_i,t)\sigma_j(S_j,t) w_i S_i w_j S_j \rho_{ij}}{I^2} \qquad \mu_{\text{loc}}(S,t) = \frac{\sum_i \mu_i w_i S_i}{I}$$

# Characterization of the equivalent volatility for the index

$$\overline{\sigma}^2(I,t) = E\left\{ \frac{\sum_{ij} \sigma_i(S_i(t),t)\sigma_j(S_j(t),t) S_i(t) S_j(t) w_i w_j \rho_{ij}}{I^2} \middle| \sum_i w_i S_i(t) = I \right\}$$

$$= E\left\{ \sum_{ij} p_i(S(t)) p_j(S(t)) \sigma_i(S_i(t),t) \sigma_j(S_j(t),t) \rho_{ij} \middle| \sum_i w_i S_i(t) = I \right\}$$

$$p_i(S) = \frac{w_i S_i}{\sum_j w_j S_j}, \qquad i = 1, \ldots, n.$$

- sigma_loc can be seen as
 a ‘stochastic vol’ driving the index

- sigma_bar is then the ``averaged vol”

# Varadhan's Formula

$$\begin{cases} dX_i = \sum_{j=1}^{n} \sigma_i^j (X,t) dW_j \\ X_i(0) = x_i \end{cases} \qquad E\{dW_j dW_k\} = \rho_{jk} dt$$

Dupire local volatility model for each stock

$$\log \text{Prob.}\{X(t) = y | X(0) = x\} \approx -\frac{d^2(x,y)}{2t}, \qquad (\bar{\sigma})^2 t << 1$$

$$d^2(x,y) = \inf_{\gamma(0)=x, \gamma(1)=y} \int_0^1 \sum_{ij=1}^{n} g_{ij}(\gamma(s)) \dot{\gamma}^i(s) \dot{\gamma}^j(s) ds$$

Riemannian metric

$$g(x) = a^{-1}(x) \qquad a_{ij}(x) = \sigma_i(x,0)\sigma_j(x,0)\rho_{ij}$$

In practice: dimensionless time ~ 0.02

# Steepest-descent approximation

Change to log-scale: $x_i \equiv \log\left(\frac{S_i}{S_i(0)e^{\mu_i t}}\right) = \log\left(\frac{S_i}{F_i(t)}\right) \quad i = 1,2,\ldots,n.$

Formally, $$\bar{\sigma}^2(I,t) = \frac{E\{\sigma_{\text{loc}}^2 \delta(I(t) - I)\}}{E\{\delta(I(t) - I)\}}$$

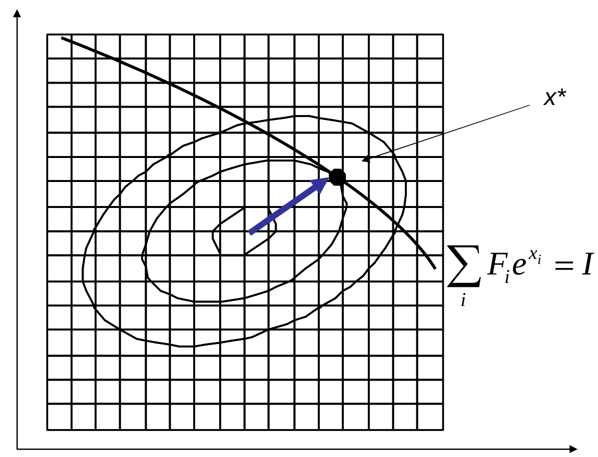
Applying Varadhan's Formula,

$$\bar{\sigma}^2(I,t) \cong \sigma^2_{\text{loc}}(S^*,t) \qquad S_i^* = S_i(0)e^{\mu_i t} e^{x_i^*}$$

where

$$x^* = \arg\min\left\{ d^2(0,x) \middle| \sum_i w_i S_i(0) e^{\mu_i t} e^{x_i} = I \right\}$$

# Steepest Descent=Most Likely Stock Price Configuration




Replace conditional distribution by “Dirac function” at most likely configuration

# Characterization of MLC

Euler-Lagrange equations: find $(x^*, \lambda)$ such that

$$\begin{cases} \displaystyle\int_0^{x_i^*} \frac{du}{\sigma_i(u)} = \lambda \sum_{j=1}^{n} p_j(x^*)\sigma_j(x_j^*)\rho_{ij} \quad i = 1,\ldots,n \\ \displaystyle\sum_{i=1}^{n} w_i S_i(0) e^{x_i^* + \mu_i t} = I \end{cases}$$

$$\bar{\sigma}^2(I,t) = \sum_{ij=1}^{n} p_i(x^*) p_j(x^*) \sigma_i(x_i^*) \sigma_j(x_j^*) \rho_{ij}$$

## Solution of linearized system in terms of the stock betas

$$\sigma_I^2(0) \equiv \sum_{ij=1}^{n} p_i(0) p_j(0) \sigma_i(0) \sigma_j(0) \rho_{ij}$$

$$\bar{x} \equiv \ln\left(\frac{I}{I(0)e^{\bar{\mu}t}}\right)$$

$$x_i^* \cong \frac{\bar{x}}{\sigma_I^2(0)} \sum_{j=1}^{n} \rho_{ij} p_j(0) \sigma_i(0) \sigma_j(0) = \frac{\bar{x}}{\sigma_I^2(0)} Cov(x_i, \bar{x})$$

$$x_i^* = \hat{\beta}_i \bar{x}$$

$$\hat{\beta} = Cov\left(\frac{\Delta S}{S}, \frac{\Delta I}{I}\right) / \left[Var\left(\frac{\Delta I}{I}\right)\right]$$

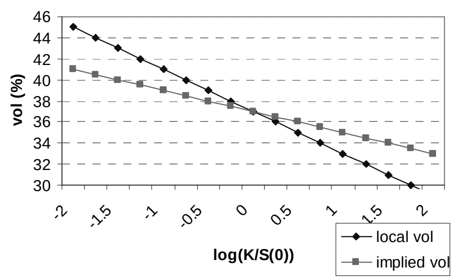
Most likely config. : described by the risk-neutral regression coefficients of stock returns with the index return (``micro" CAPM)

## Expression in terms of Implied Volatilities and Black- Scholes Deltas

- Seek direct relation between implied volatilities of single-stock options and implied volatility of index options
- Tool: Berestycki-Busca-Florent large-deviations result for single-stock ("1/2 slope rule")

$$\sigma^{\text{impl.}}(x) \approx \left(\frac{1}{x}\int_0^x \frac{du}{\sigma(u)}\right)^{-1}$$

$$\sigma^{\text{impl.}}(x) \approx \frac{1}{2}\left(\sigma^{\text{impl.}}(0) + \sigma(x)\right)$$



# The Formula In Terms of Black Scholes Deltas

From Euler-Lagrange equations:

$$\frac{\ln\left(S_i^*/F_i\right)}{\sigma_i^{\text{impl.}}\left(S_i^*\right)} \approx \frac{\ln\left(I/I_f\right)}{\sigma_{\text{I}}^{\text{impl}}(I)} \times \sum_{j=1}^{n} \rho_{ij} p_j \left(\frac{\sigma_j^{\text{impl.}}\left(F_j\right)}{\sigma_{\text{I}}^{\text{impl}}\left(I_f\right)}\right) = \frac{\ln\left(I/I_f\right)}{\sigma_{\text{I}}^{\text{impl}}(I)} \cdot Corr\left(\frac{\Delta S_i}{S_i}, \frac{\Delta I}{I}\right)$$

Translate into Black-Scholes deltas

$$\Delta \equiv N\left(\frac{1}{\sigma\sqrt{t}}\ln\left(\frac{F}{S^*}\right) + \frac{1}{2}\sigma\sqrt{t}\right), \qquad N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{z^2}{2}}dz$$

$$\Delta_i = N\left(N^{-1}(\Delta_I) \times \sum_{j=1}^{n} \rho_{ij} p_j \left(\frac{\sigma_j^{\text{impl.}}\left(F_j\right)}{\sigma_{\text{I}}^{\text{impl}}\left(I_f\right)}\right)\right)$$

Delta of each stock is a function of delta of index option

# Reconstruction Formula for the Index Volatility

$$\sigma_I^{\text{impl.}}(\Delta_I) \cong \frac{\sigma_I^{\text{impl.}}(0.5) + \overline{\sigma}(\Delta_I)}{2}$$

$$\overline{\sigma}(\Delta_I) = \sqrt{\sum_{ij=1}^{n} \rho_{ij} p_i p_j \left(2\sigma_i^{\text{impl.}}(\Delta_i) - \sigma_i^{\text{impl.}}(0.5)\right)\left(2\sigma_j^{\text{impl.}}(\Delta_j) - \sigma_j^{\text{impl.}}(0.5)\right)}$$

$p_i =$ capitalization of stock in index (%)

# Correlation & Optimal Configurations

- If the stocks are perfectly correlated,

$$\sum_{j=1}^{n} p_j \sigma_j^{\text{impl}}\left(F_j\right) = \sigma_I^{\text{impl}}\left(I_f\right) \quad \therefore \quad \boxed{\Delta_i \equiv \Delta_I}$$

$$\frac{\ln\left(S_i^* / F_i\right)}{\sigma_i^{\text{impl.}}\left(S_i^*\right)} \approx \frac{\ln\left(I / I_f\right)}{\sigma_{\text{I}}^{\text{impl}}\left(I\right)} \quad \forall i$$

" Equal-delta approximation "

- If the stocks are uncorrelated,

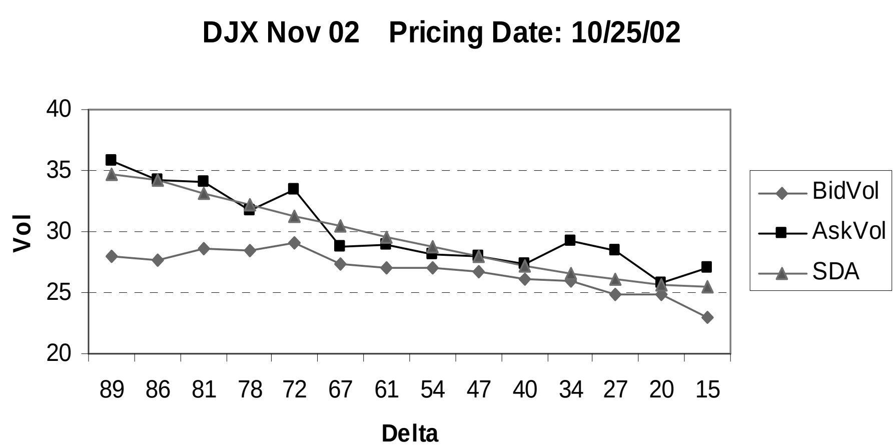
$$\frac{\ln\left(S_i^* / F_i\right)}{\sigma_i^{\text{impl.}}\left(S_i^*\right)} \approx \frac{\ln\left(I / I_f\right)}{\sigma_{\text{I}}^{\text{impl}}\left(I\right)} \times p_i \times \frac{\sigma_i^{\text{impl.}}\left(F_j\right)}{\sigma_I^{\text{impl}}\left(I_f\right)}$$

$$\Delta_i = N\left( N^{-1}\left(\Delta_I\right) \times p_i \times \frac{\sigma_i^{\text{impl.}}\left(F_i\right)}{\sigma_I^{\text{impl}}\left(I_f\right)} \right)$$

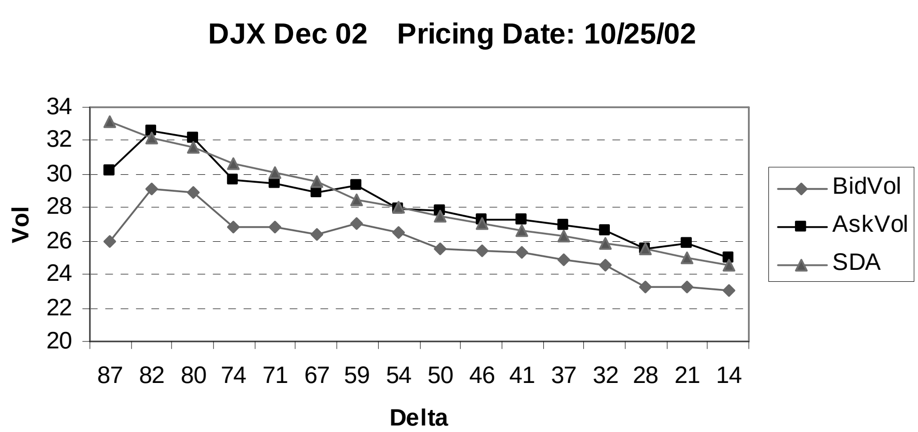
Less volatile & small stocks have smaller deltas closer to 50% (their skew does not affect the index as much)

# DJX: Dow Jones Industrial Average

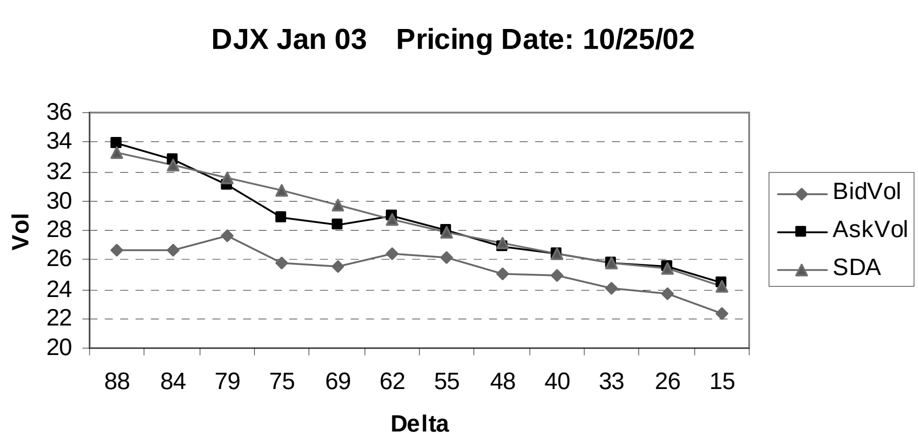
T=1 month



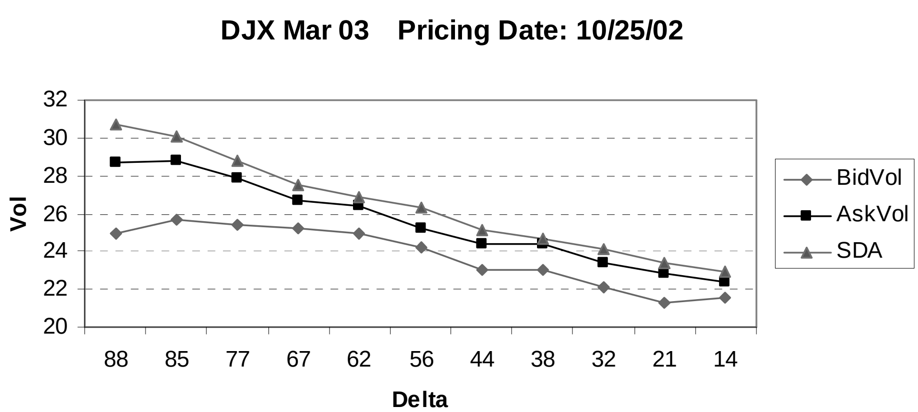
T= 2 months



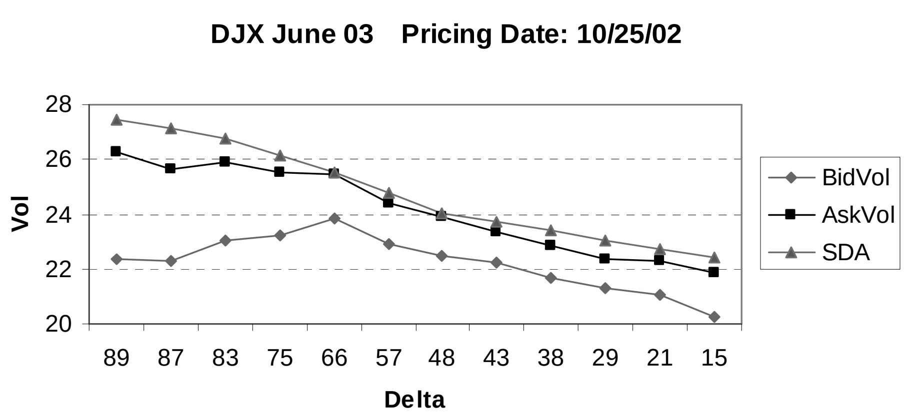

T=3 months



T= 5 months

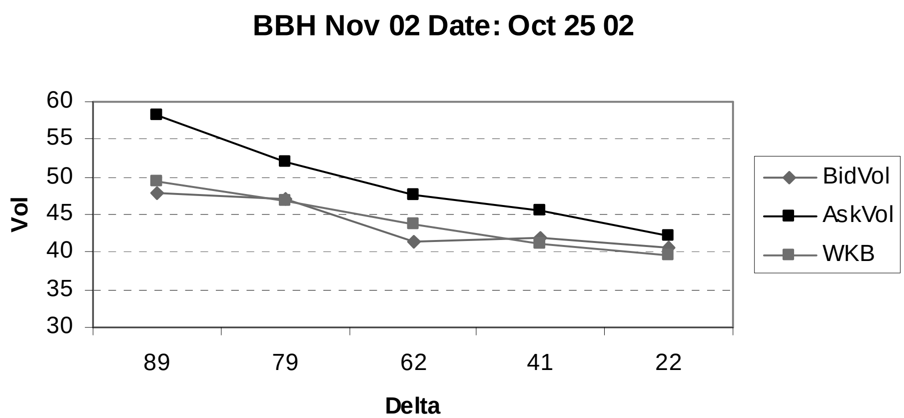



T=7 months

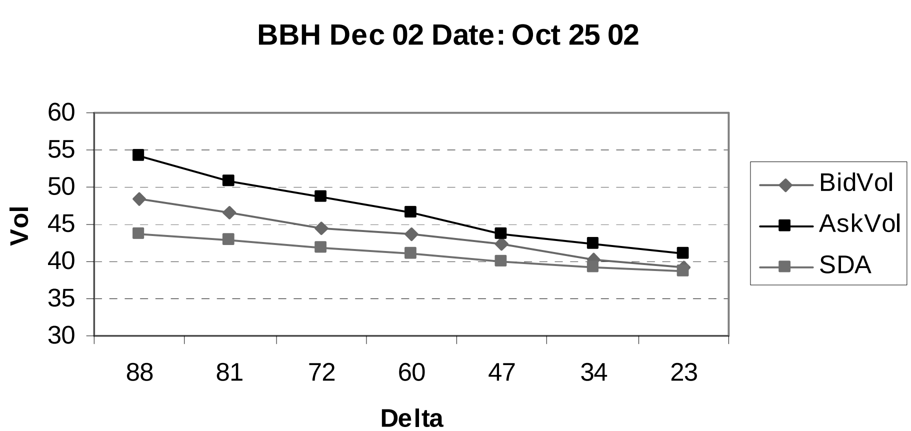


# BBH: Biotechnology HLDR

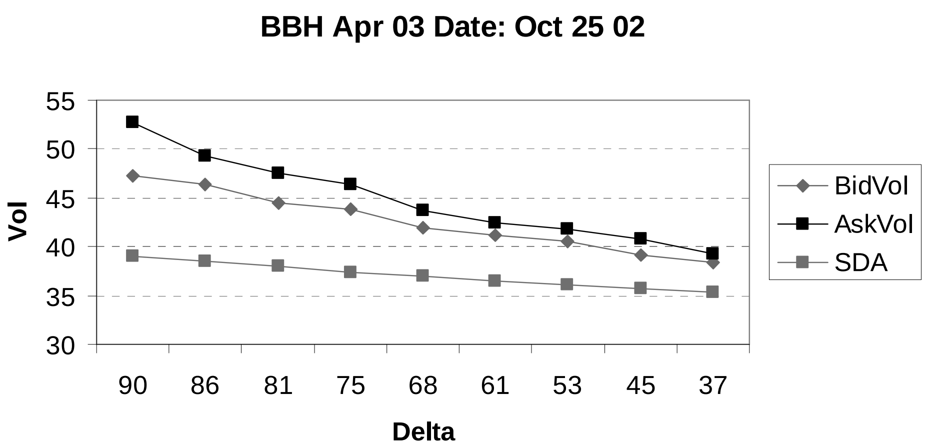
T = 1 month




T = 2 months



T = 6 months

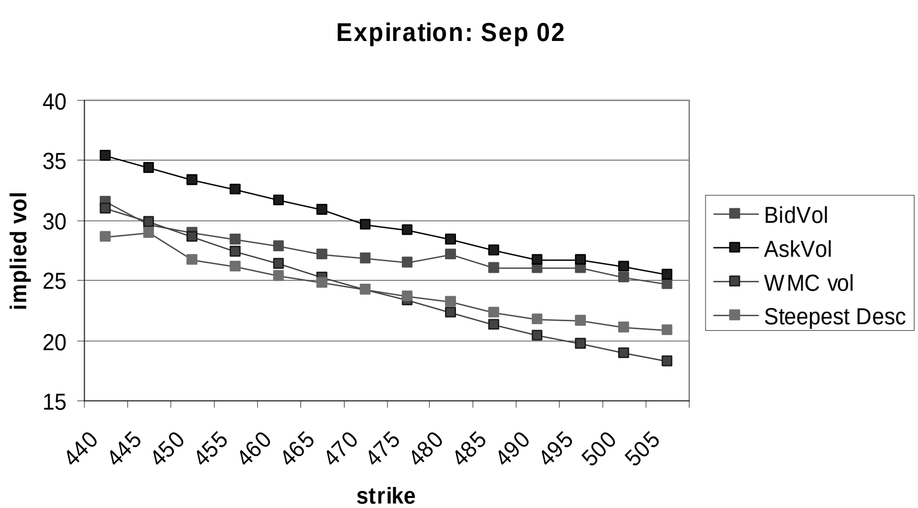



Is dimensionless time is too long? (Error bars: Juyoung Lim)

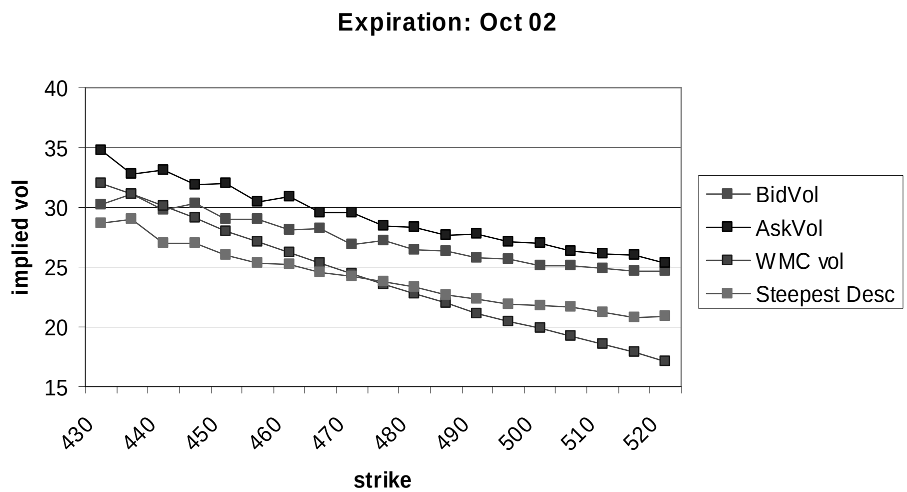
Is correlation causing the discrepancy?

# S&P 100 Index Options

(Quote date: Aug 20, 2002)



# S&P 100 Index Options
(Quote date: Aug 20, 2002)

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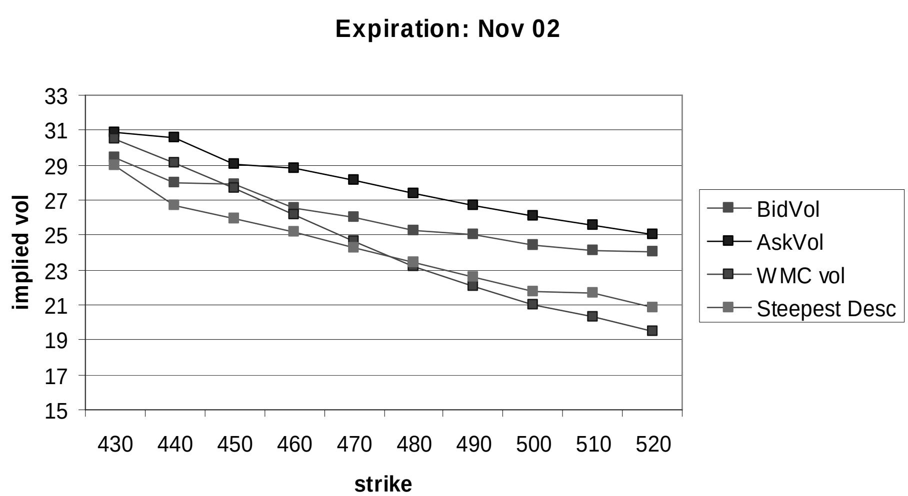

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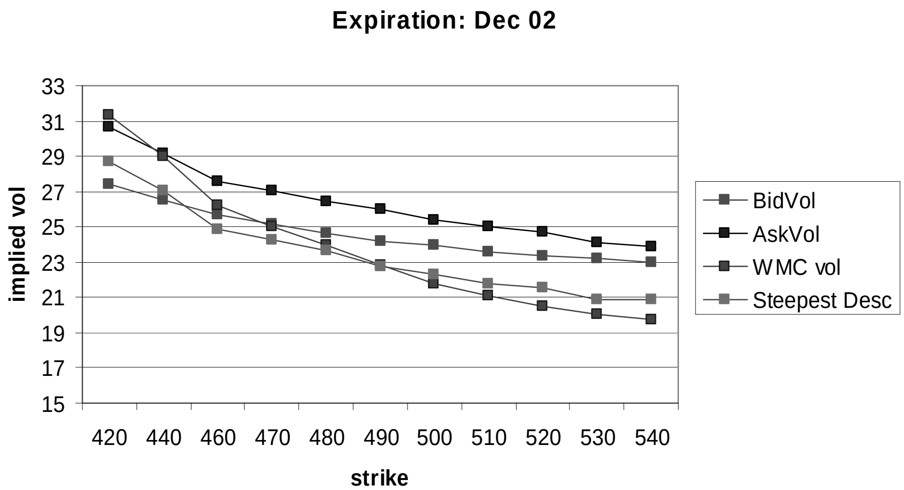



# General Stochastic Volatility Systems

$$\frac{dS_i}{S_i} = \sigma_i dW_i \qquad \frac{d\sigma_i}{\sigma_i} = \kappa_i dZ_i$$

$$E\left(dW_i dW_j\right) = \rho_{ij} dt \qquad E\left(dW_i dZ_j\right) = r_{ij} dt$$

$$\bar{x} = \frac{dI}{I}, \qquad x_i = \frac{dS_i}{S_i} \qquad y_i = \frac{d\sigma_i}{\sigma_i}$$

Look for most likely configuration of <u>stocks and vols</u> $(x_1,...,x_n,y_1,...,y_n)$ corresponding to a given index displacement $\bar{x}$

# Most likely configuration for Stochastic Volatility Systems

$$x_i^* = \beta_i \bar{x} \qquad \beta_i = \frac{\sigma_i \rho_{iI}}{\sigma_I}$$

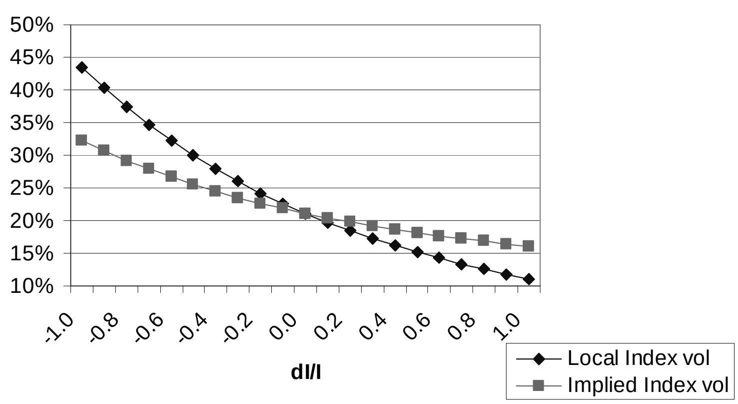
$$y_i^* = \gamma_i \bar{x} \qquad \gamma_i = \frac{\kappa_i r_{iI}}{\sigma_I}$$

Most likely configuration for stocks moves and volatility moves, given the index move

$$\sigma_I^2(\bar{x},t) \cong \sum_{ij=1}^{n} p_i p_j \sigma_i(0,t) \sigma_j(0,t) e^{\gamma_i \bar{x}} e^{\gamma_j \bar{x}} \rho_{ij}$$

SDA

# Numerical Example

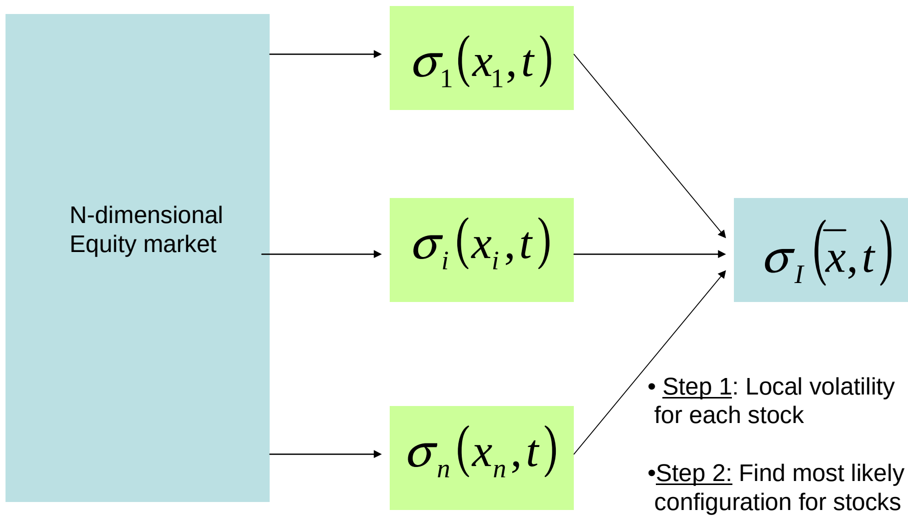



$n = 2 \qquad p_1 = p_2 = 0.5$

$\sigma_1 = 20\%, \quad \gamma_1 = -1 \qquad \sigma_2 = 30\%, \quad \gamma_2 = -0.5$

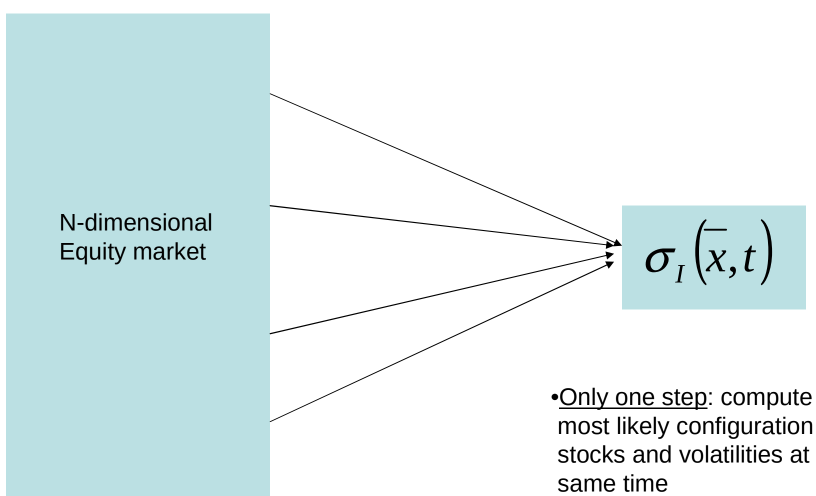
$\rho = 40\%$

## Method I: Dupire & Most Likely Configuration for Stock Moves




- Step 1: Local volatility for each stock
- Step 2: Find most likely configuration for stocks

## Method II: Stochastic Volatility System and joint MLC for Stocks and Volatilities




- Only one step: compute the most likely configuration of stocks and volatilities at the same time

## Are methods I and II ``equivalent''?

Answer is NO, in general.

Dupire local vol. for single names → $\sigma_i(x_i,t) \approx \sigma_i(0,t) e^{\varpi_i x_i}$ $\qquad \varpi_i = \dfrac{\kappa_i r_{ii}}{\sigma_i}$

Index vol., Method I → $$\sigma_I^2(\bar{x},t) = \sum_{ij} p_i p_j \sigma_i(0,t)\sigma_j(0,t)\rho_{ij} e^{\varpi_i \beta_i \bar{x}} e^{\varpi_j \beta_j \bar{x}}$$

Index vol., Method II → $$\sigma_I^2(\bar{x},t) = \sum_{ij} p_i p_j \sigma_i(0,t)\sigma_j(0,t)\rho_{ij} e^{\gamma_i \bar{x}} e^{\gamma_j \bar{x}}$$

## Equivalence holds only under additional assumptions on stock-volatility correlations

$$\varpi_i \beta_i = \frac{\kappa_i r_{ii}}{\sigma_i}\frac{\sigma_i \rho_{iI}}{\sigma_I} = \frac{\kappa_i r_{ii} \rho_{iI}}{\sigma_I}$$ Method I

$$\gamma_i = \frac{\kappa_i r_{iI}}{\sigma_I}$$ Method II

$$r_{iI} = r_{ii}\rho_{iI}$$

$$r_{ij} = r_{ii}\rho_{ij}$$

Conditions under which both methods give equivalent valuations

## The corresponding stock-volatility correlation structure is `diagonal'

$$dz_i = \sum_k m_{ik} dW_k + \alpha_i d\varsigma_i \qquad E(dW_k, d\varsigma_l) = 0$$

$$\therefore$$

$$r_{ij} = \sum_k m_{ik} \rho_{kj}$$

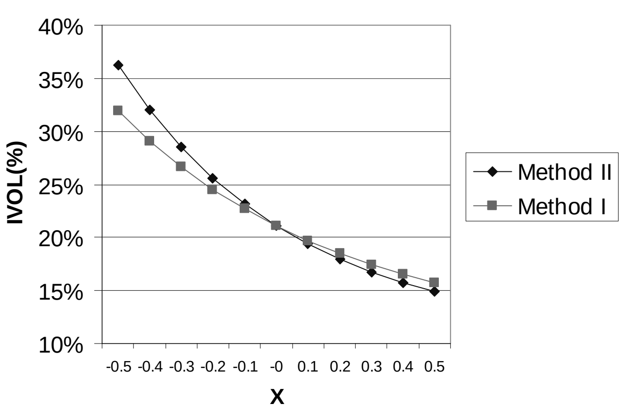
$$r_{ij} = r_{ii}\rho_{ij} \qquad \Rightarrow \qquad dz_i = r_{ii} dW_i + \alpha_i d\varsigma_i$$

Examples: Dupire, CEV, uncoupled SV mode

## Numerical Example

$$\sigma_1 = 20\%, \sigma_2 = 30\%, \rho = 40\%$$

$$r = \begin{bmatrix} -0.7 & -0.5 \\ -0.6 & -0.7 \end{bmatrix}, \quad \kappa_1 = \kappa_2 = 50\%$$



# Lee, Wang and Karim

Gram-Charlier expansion *RISK*, Dec 2003

$$x = \log\left(\frac{K}{F}\right),\ \sigma^2 = Var(x),\quad s = \frac{E\left(x^3\right)}{\sigma^3},\ k = \frac{E\left(x^4\right)}{\sigma^4} - 3$$

$$\sigma_{\text{atm}} = \sigma\left[1 + \frac{s}{3!}\sigma\sqrt{T} + \frac{k}{4!}\left(\frac{7}{4}\sigma^2 T - 1\right)\right]$$

$$Skew = \frac{1}{\sqrt{T}}\left(\frac{s}{3!} + \frac{2k\sigma\sqrt{T}}{4!}\right),\qquad Smile = \frac{1}{T}\frac{k}{4!\sigma}$$

$$\sigma(x) = \sigma_{\text{atm}} + x \cdot Skew + x^2 \cdot Smile$$

# Calculate Index Moments

$$E\left(\overline{x}^2\right) = \sum_{ij} p_i p_j \sigma_i \sigma_j \rho_{ij}$$

$$E\left(\overline{x}^3\right) = \sum_{ijk} p_i p_j p_k \sigma_i \sigma_j \sigma_k E\left(z_i z_j z_k\right)$$

$$E\left(\overline{x}^4\right) = \sum_{ijk} p_i p_j p_k p_l \sigma_i \sigma_j \sigma_k \sigma_l E\left(z_i z_j z_k z_l\right)$$

``Gaussian'' closure for third and 4th moments

# 3rd Moments

$$E(z_i z_j z_k) = Sym[Corr(z_i, z_k)Corr(z_i, z_k)s_i] = Sym[\rho_{ij}\rho_{ik}s_i] \quad \text{if } i \neq j \neq k$$

$$E(z_i z_i z_k) = Corr(z_i, z_k)E(z_i^3) = \rho_{ik}s_i$$

$$E(z_i z_i z_i) = E(z_i^3) = s_i$$

$$s_I = \frac{1}{\sigma^3}\sum_{ijk} p_i p_j p_k \sigma_i \sigma_j \sigma_k \rho_{ij}\rho_{ik}s_i$$

Consistent with the SDA approximation formula!

# Stochastic Correlation Model

- Lee, Wang and Karim: stochastic correlation
- Linear fit

$$\overline{\rho} = \alpha + \beta \ln I + \varepsilon$$

Rho_bar is the `average' correlation

$$OEX: \quad \beta = -0.66$$
$$BKX: \quad \beta = -0.34$$

This model can be used to improve the accuracy of the SDA.

# Cross-Currency Options

Major currency triad: USD/JPY, USD/EUR, JPY/EUR

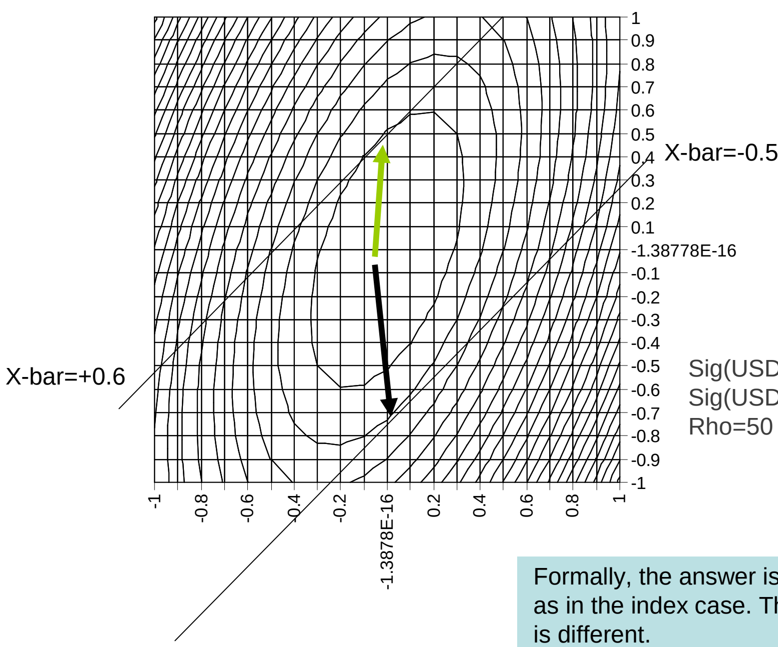
$$S_1 = JPY/USD \qquad x_1 = \ln \frac{S_1}{S_1(0)e^{\mu_1 t}}$$

$$S_2 = EUR/USD \qquad x_2 = \ln \frac{S_2}{S_2(0)e^{\mu_2 t}}$$

$$I = EUR/JPY \qquad \bar{x} = \ln \frac{I}{I(0)e^{(\mu_2-\mu_1)t}}$$

$$\bar{x} = x_2 - x_1$$

$$\sigma_{I,\text{atm}}^2 = \sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2$$




Sig(USD/JPY)=20 pct
Sig(USD=11.2 pct
Rho=50 pct

Formally, the answer is the same as in the index case. The ``geometry'' is different.