

# Derivatives Week

INTRODUCING



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## VARIANCE SWAP VOLATILITY AND OPTION STRATEGIES

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Born in the over-the-counter derivatives market, variance swap volatility (VSV) is slowly but surely gaining recognition as a useful tool for managing option positions. Let's begin with this concept by looking at a variance swap contract.

Here's how it works: Counterparty A agrees to pay Counterparty B a fixed notional amount due at the settlement date. In exchange, Counterparty B will pay Counterparty A an amount proportional to the sum of the daily squared returns

$$\frac{1}{n} \sum_{i=0}^{n-1} \left( \frac{S_{i+1} - S_i}{S_i} \right)^2$$

where  $n$  = number of business days until settlement, and  $S_i, i=0,2,\dots,n$  represent daily closing prices of the underlying stock or index. The sum of the daily returns squared can be interpreted as an estimator of the realized variance of stock returns from now to the settlement of the contract.

For instance, a contract can be stipulated as follows: using the **Standard & Poor's 500** as the underlying index, a volatility level of  $\sigma = 23\%$  is fixed for one year. This corresponds to a nominal variance  $\sigma^2 = 5.29\%$ . Counterparty B agrees to pay A USD100,000 for each percentage point of realized variance *above*  $\sigma^2 = 5.29\%$  (where the variance is computed using the above formula) and A agrees to pay B USD100,000 per variance point *below* this value. In this case, the notional value of the contract, or fixed leg payment, is USD529,000.

What is the "fair value" of the parameter  $\sigma$  in such a swap? If no cash flows are exchanged initially,  $\sigma$  should be equal to the *discounted expectation of the realized variance over the time-horizon of the contract*. This is the definition of the variance swap volatility. We will next derive a formula for computing it.

## VARIANCE SWAPS AND LOG-CONTRACTS

A key observation, separately noted by Neuberger and Dupire, is that a variance swap is financially equivalent to--and can be replicated by--a European-style derivatives security with a logarithmic payoff. To see this, we make a Taylor-series expansion of the logarithm of the price up to second-order derivatives, which gives

$$\log S_{i+1} - \log S_i \approx \frac{S_{i+1} - S_i}{S_i} - \frac{1}{2} \left( \frac{S_{i+1} - S_i}{S_i} \right)^2$$

Summing both sides of this equation over the total number of days in the contract, and

rearranging terms, we obtain

$$\sum_{i=0}^{n-1} \left( \frac{S_{i+1} - S_i}{S_i} \right)^2 \approx 2 \log \frac{S_n}{S_0} - 2 \sum_{i=0}^{n-1} \frac{S_{i+1} - S_i}{S_i}$$

It follows that the "floating leg" of the swap (the left-hand side) can be replicated by holding a derivative contract with payoff  $2 \log \left( \frac{S_n}{S_0} \right)$  --a *log-contract*--and a position

in forward contracts. We conclude that the value of the variance swap is equal to that of a log-contract. Secondly, we see that a short position in the variance swap can be hedged by holding a log-contract and maintaining a neutral delta at the close of each trading day, assuming that funding costs are zero.

**A FORMULA FOR THE VSV**

To price the log-contract, we approximate the payoff with a function that is piecewise-linear, that is to say, a function whose graph consists of line segments. We use the fact that a contingent claim with such a payoff can be replicated with a portfolio of plain-vanilla options with the same maturity and different strikes. This leads to a closed-form expression for the VSV

$$\sigma = \sqrt{\frac{2}{T} \left( \int_0^F \frac{P(K,T)dK}{K^2} + \int^{\infty} \frac{C(K,T)dK}{K^2} \right)}$$

where  $T$ =duration of the contract measured in years,  $F$ =forward price of the underlying security, and  $P(K,T)$ ,  $C(K,T)$  represent the prices of European puts and calls with strike  $K$  expiring in  $T$  years. This formula can be interpreted as an arbitrage relationship between the implied volatilities of traded options and the variance swap volatility. Simply put: *the option volatility curve contains the market's expectation on the realized variance of the stock from today to the expiration date of the options.*

**VSV AND VOLATILITY TRADING**

The VSV is not only useful for the over-the-counter market, but serves as a "benchmark" for analyzing options in general and the volatility skew in particular. Often, the implied volatility of at-the-money options is interpreted as the market's expectation of future volatility. We have shown here that this is wrong: the correct value is the VSV. The ATM volatility represents instead the expected volatility conditionally on the underlying price remaining at the same level. Therefore, the VSV is the natural "benchmark" for analyzing option volatility.

Writer and money-manager **Nassim Taleb**, from the Greenwich, Conn.-based fund **Empirica Capital**, has been a strong proponent of this interpretation of the variance swap volatility, recognizing the implications that it has for option trading. In his Ph. D. thesis, he describes a hypothetical conversation between two traders that runs as follows:

"What do you believe the volatility will be for the next month?"

"20%."

"And what market are you making on one month at-the-money calls?"

"18% bid, 18.50% offered."

This exchange seems paradoxical from the point of view of option pricing theory but it is not. In fact, "Taleb's paradox" is resolved by observing that the expectations of volatility correspond to the VSV--not the implied volatility of any given (fixed-strike)

option.

Traders often view option-implied volatility as a "break-even level" for daily movements of the underlying stock price (e.g. if the implied volatility is 16%, the daily break-even is an absolute move of  $16/255=1\%$ ). However, they cannot be certain of the levels of implied volatility that they will face when they revise their portfolios as time passes. For example, if a trader wants to run an options book with a constant time-decay exposure for the next three months, he or she could compare his or her forecast of future implied volatility with the current three-month VSV to determine the profitability of such trading strategy.

Another application is to dynamic hedging. Option pricing theory tells us that profits/losses proportional to the stock returns *squared* can be achieved by holding a delta-neutral option position. Because a variance swap delivers a similar payoff, the VSV can be used for relative-value analysis of implied volatilities and for designing option strategies. For example, liquid contracts trading at levels deeply below VSV should be attractive for buyers of volatility, while contracts trading above the VSV should be viewed as expensive from this perspective.

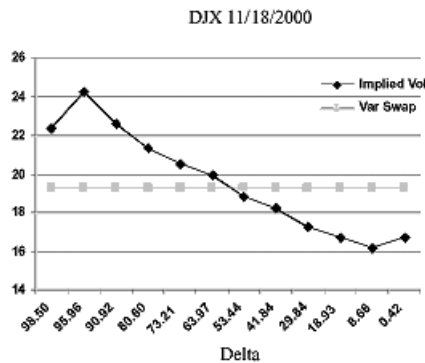
As a general rule, shifts in the overall levels of volatility (i.e. the entire volatility curve) play a crucial role in option trading. These shifts cannot be predicted in advance. Notwithstanding this uncertainty, the VSV provides an interesting tool to analyze the volatility skew and to detect buying/selling opportunities using available market information. Indices like the S&P 500 and the Nasdaq 100 have large skews and hence are interesting from the point of view of VSV. Options on many individual stocks trade with a large skew and can also be analyzed in this way.

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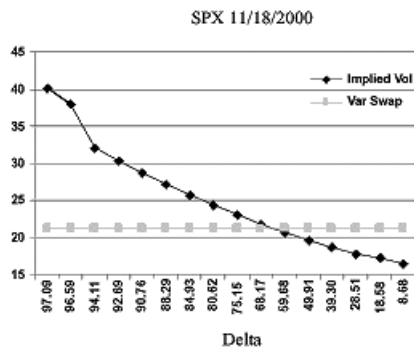
References: Dupire, B, 1993, Model Art, Risk, Sept.1993, 118-120 Neuberger, A., 1990, Volatility Trading, London Business School working paper Taleb, N., 1998, Ph.D. Dissertation, University of Paris

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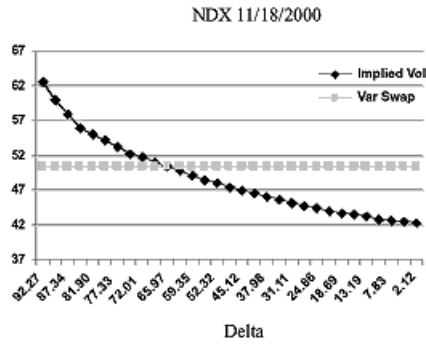
This week's Learning Curve was written by Marco Avellaneda, a professor of mathematics and director of the division of financial mathematics at New York University in New York.



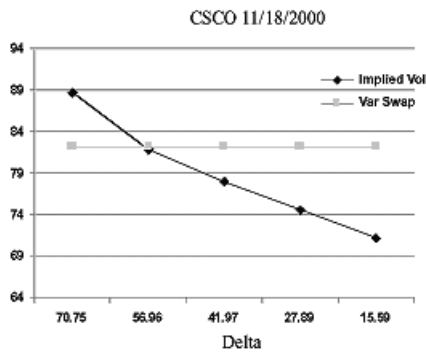
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