First exit time of a Brownian

path from a strip

(Hopefully right this time!)

Statement of the problem We wish to compute the probability distribution for the first exit time of a Brownian motion from a strip. The idea is to use the result for the first exit time from a **half-space**$(a > 0)$

$$Pr.\{\tau_a \ < \ T\} \ = \ Pr.\left\{ \underset{t \leq \ T}{max} Z(t) \ > \ a \right\} \ = \ 2 \cdot \left(1 \ - \ N\left(\frac{a}{\sqrt{T}} \right) \right) \ . \quad (1)$$

$N(x)$ is the cumulative normal distribution function. Recall that this "first-passage" result is based on the reflection principle for BM.

In the case of a **strip** $(a > 0, \ b > 0)$,

$$S \ = \ \{ z \ : \ -b \ < \ z \ < \ a \ \} \ ,$$

we set

$$\tau_{a,b} \ = \ min \ \{ t \ : \ Z(t) \ > \ a \ \text{ or } Z(t) \ < \ -b \} \ .$$

The problem of interest is to compute $Pr.\{\tau_{a,b} \ < \ T\}$. For this we shall use (1) and the reflection principle in an iterative way.

Part 1. (a) Show that

$$Pr.\{\tau_{a,b} \ < \ T\} \ = \ Pr.\{\tau_a \ < \ T\} \ + \ Pr.\{\tau_{-b} \ < \ T\}$$

$$- \ Pr.\{\tau_a \ < \ T \text{ and } \tau_{-b} \ < \ T\}$$

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(b) Show that

$$Pr.\{\tau_a \ < \ T \text{ and } \tau_{-b} \ < \ T\} \ = \ Pr.\{\tau_a \ < \ \tau_{-b} \ < \ T \ \} + Pr.\{\tau_{-b} \ < \ \tau_a \ < \ T \ \}$$

and that

$$Pr.\{\tau_a \ < \ \tau_{-b} \ < \ T \ \} = \ Pr.\{\tau_{2\,a \ + \ b} \ < \ T \text{ and } \tau_a \ < \ \tau_{-b} \ \}$$

$$= Pr.\{\tau_{2\,a+b} \ < \ T \ \} \ - \ Pr.\{\tau_{-b} \ < \ \tau_{2\,a+b} \ < \ T \ \}$$

This last step uses the reflection principle (draw a picture).

Part 2. Using Part 1, show by induction that

$$Pr.\{\tau_{a,b} \ < \ T\} \ = \ \sum_{k=1}^{infty} \ (-1)^{k+1} \ 2 \cdot \left(2 \ - \ N\left(\frac{a_k}{\sqrt{T}}\right) \ - \ N\left(\frac{b_k}{\sqrt{T}}\right)\right) \ ,$$

where $\{a_k\}$ and $\{b_k\}$ are sequences that converge to infinity. In particular, notice that the partial sums converge exponentially fast.

Part3. Write a code (in C/C++, Matlab or Excel) that computes the above series to an arbitrary order of approximation and run a few examples (e.g., $a = 1, \ b = 2, \ T = 1$).

Part 4. Compare the above results with alternative methods for calculating the probability: a Monte Carlo simulation method of the random walk or a binomial tree approach.