PRICING AND HEDGING DERIVATIVE SECURITIES IN MARKETS WITH UNCERTAIN VOLATILITIES

by

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ABSTRACT

We present a model for pricing and hedging derivative securities and option portfolios in an environment where the volatility is not known precisely, but is assumed instead to lie between two extreme values $\sigma_{\min}$ and $\sigma_{\max}$. These bounds could be inferred from extreme values of the implied volatilities of liquid options, or from high-low peaks in historical stock- or option-implied volatilities. They can be viewed as defining a confidence interval for future volatility values. We show that the extremal non-arbitrageable prices for the derivative asset which arise as the volatility paths vary in such a band can be described by a non-linear PDE, which we call the Black-Scholes-Barenblatt equation. In this equation, the "pricing" volatility is selected dynamically from the two extreme values $\sigma_{\min}$,$\sigma_{\max}$, according to the convexity of the value-function. A simple algorithm for solving the equation by finite-differencing or a trinomial tree is presented. We show that this model captures the importance of diversification in managing derivatives positions. It can be used systematically to construct efficient hedges using other derivatives in conjunction with the underlying asset.

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1. The uncertain volatility model

According to Arbitrage Pricing Theory, if the market presents no arbitrage opportunities, there exists a probability measure on future scenarios such that the price of any security is the expectation of its discounted cash-flows (Duffie, 1992). Such a probability is known as a *martingale measure* (Harrison and Kreps, 1979), or a *pricing measure*. Determining the appropriate martingale measure associated with a sector of the security space (e.g. the stock of a company and a riskless short-term bond) permits the valuation of any contingent claim based on these securities. However, pricing measures are often difficult to calculate precisely and there may exist more than one measure consistent with a given market.[1] It is useful to view the non-uniqueness of pricing measures as reflecting the many choices for derivative asset prices that can exist in an uncertain economy. For example, option prices reflect the market's expectation about the future value of the underlying asset as well as its projection of future volatility. Since this projection changes as the market reacts to new information, implied volatility fluctuates unpredictably. In these circumstances, fair option values and perfectly replicating hedges cannot be determined with certainty. The existence of so-called "volatility risk" in option trading is a concrete manifestation of market incompleteness.

This paper addresses the issue of derivative asset pricing and hedging in an uncertain future volatility environment. For this purpose, instead of choosing a pricing model that incorporates a complete view of the forward volatility as a single number, or a predetermined function of time and price ("term-structure of volatilities"), or even a stochastic process with given statistics, we propose to operate under the less stringent assumption that that the volatility of future prices is restricted to lie in a bounded set, but is otherwise undetermined. [2]

For simplicity, we restrict our discussion to derivative securities based on a single liquidly traded stock which pays no dividends over the contract's lifetime and assume a constant interest rate. The basic assumption then reduces to postulating that, under all admissible pricing measures, future volatility paths will be restricted to lie within a "band". Accordingly, we assume that the paths followed by future stock prices are Itô processes, *viz.*,

$$dS_t \;=\; S_t \;(\; \sigma_t \; dZ_t \;+\; \mu_t \; dt \;)\;, \tag{1}$$

where σ_t and μ_t are non-anticipative functions such that

$$\sigma_{\min} \;\le\; \sigma_t \;\le\; \sigma_{\max}\;, \tag{2}$$

[1] Uniqueness of the martingale measure is tantamount to market completeness, a stringent economic assumption. A model for a securities market is said to be complete if the associated dividend matrix is invertible or, equivalently, if it is possible to realize any given payoff structure through an appropriate portfolio of traded securities.

[2] The original idea for this paper came from analyzing a simple incomplete market based on a trinomial tree (Levy, Avellaneda and Parás, 1995). For reasons of clarity and generality, we adopt a continuous-time formulation hereafter.

and Z_t is Brownian motion. The constants $\sigma_{\min}$ and $\sigma_{\max}$ represent upper and lower bounds on the volatility that should be input in the model according to the user's expectation and uncertainty about future price fluctuations. These bounds could be obtained, for instance, from the extreme values of implied volatilities of liquid derivative instruments or from peaks in historical stock- or option-implied volatilities. They can be viewed as determining a "confidence interval" for future volatility values. If necessary, $\sigma_{\min}$ and $\sigma_{\max}$ can be modeled as functions of the stock price and time to maturity. (For simplicity, we shall assume here that $\sigma_{\min}$ and $\sigma_{\max}$ are constant over time and independent of S). Based on these assumptions, we shall derive optimal bounds on the values of derivative securities which are consistent with these volatility bounds. We shall also describe strategies for managing a derivatives position in such environment.

Assume that at a given date t, a derivative security (or a portfolio of derivatives) is characterized by a stream of cash-flows at N future dates, $t_1 \le t_2 \le .... \le t_N$,

$$F_1(S_{t_1}) , F_2(S_{t_2}) , ... , F_N(S_{t_N}) , \tag{3}$$

where $F_j(S)$ are known functions of the price of the underlying stock. We would like to find the "present value" of these cash-flows. If there is no arbitrage, the forward stock price dynamics under any pricing measure should satisfy the modified, risk-neutral Itô equation

$$dS_t = S_t (\sigma_t \, dZ_t + r \, dt) , \tag{4}$$

where r is the riskless interest rate (Duffie, 1992). [3] Let us denote by $\mathbf{P}$ the class of all probability measures on the set of paths $\{ S_t , 0 \le t \le T\}$, such that (4) holds for some σ_t which is non-anticipative and satisfies the bounds in (2). If there are no arbitrage opportunities and our assumption on volatility is correct, the value of this derivative should lie somewhere between the bounds

$$W^{+}(S_t , t) = \mathrm{Sup}_P \ \mathrm{E}_t^P \left[\sum_{j=1}^{N} e^{- r (t_j - t)} F_j(S_{t_j}) \right] \tag{5}$$

and

$$W^{-}(S_t , t) = \mathrm{Inf}_P \ \mathrm{E}_t^P \left[\sum_{j=1}^{N} e^{- r (t_j - t)} F_j(S_{t_j}) \right] , \tag{6}$$

where P ranges over all measures in $\mathbf{P}$ and E_t^P is the conditional expectation operator under P conditional on the information up to time t.

[3] Equation (4) is obtained from eq. (1) by substituting the return μ_t by the riskless rate r. We assume that the latter is constant.

We now make the key observation that these two functions can be obtained by solving dynamical programming partial differential equations, viewing (5) and (6) as stochastic control problems with control variable σ_t (Krylov, 1980). Accordingly, in the case of a single maturity date ($N = 1$, $t_1 = T$, $F_1 = F$), the two extreme functions are obtained by solving the final-value problem

$$\frac{\partial W(S,t)}{\partial t} + r\left(S \frac{\partial W(S,t)}{\partial S} - W(S,t)\right) + \frac{1}{2}\sigma^2\left[\frac{\partial^2 W(S,t)}{\partial S^2}\right]S^2 \frac{\partial^2 W(S,t)}{\partial S^2} = 0 , \tag{7}$$

$$W(S, T) = F(S) ,$$

where W^+ is obtained setting

$$\sigma\left[\frac{\partial^2 W}{\partial S^2}\right] = \begin{cases} \sigma_{\max} & \text{if } \frac{\partial^2 W}{\partial S^2} \geq 0, \\ \sigma_{\min} & \text{if } \frac{\partial^2 W}{\partial S^2} < 0 , \end{cases} \tag{8}$$

in (7), and W^- with

$$\sigma\left[\frac{\partial^2 W}{\partial S^2}\right] = \begin{cases} \sigma_{\max} & \text{if } \frac{\partial^2 W}{\partial S^2} \leq 0, \\ \sigma_{\min} & \text{if } \frac{\partial^2 W}{\partial S^2} > 0 . \end{cases} \tag{9}$$

The case of multiple payoffs is similar. Problem (7) is first solved for $t_{N-1} < t \leq t_N$ with terminal condition $W(S, t_N) = F_N(S)$. At time t_{N-1} the value-function is set to

$$W(S, t_{N-1}) = W(S, t_{N-1} + 0) + F_{N-1}(S) , \tag{10}$$

where the first term in the right-hand side represents the limit from the right as $t \to 0$ (the value at the date t_{N-1} immediately after the cash-flow $F_{N-1}(S_{t_{N-1}})$ is paid out). Equation (7) is then used to discount the price back to time t_{N-2}, and so forth[4].

We shall refer to the non-linear PDE in (7) as the *Black-Scholes-Barenblatt* (BSB) equation [5]. It is a generalization of the classical Black-Scholes PDE [3], and reduces to it in the case of constant volatility ($\sigma_{\min} = \sigma_{\max}$). We note that the "volatility" $\sigma\,[\cdot]$ implied by equation (7) is a function of $\frac{\partial^2 W}{\partial S^2}(S, t)$. This has following consequence: if the future volatility path was actually given by

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[4] A justification of the dynamical programming equations and a more compact formulation of the multiple-payoff problem are given in the Appendix.

[5] The physicist G. I. Barenblatt [2] introduced a diffusion equation with a similar nonlinearity to model flow in porous media; hence our terminology.

$$\sigma_t \;=\; \sigma\left[\; \frac{\partial^2\; W^+(S_t\;,\;t)}{\partial S^2} \;\right]\;, \tag{11}$$

with $\sigma\,[\cdot]$ satisfying (8), then the standard Black-Scholes argument shows that a portfolio composed of Δ_t shares and B_t bonds, where

$$\Delta_t \;=\; \frac{\partial W^+(S_t\;,\;t)}{\partial S} \tag{12}$$

and

$$B_t \;=\; W^+(S_t\;,\;t) \;-\; S_t \cdot \frac{\partial W^+(S_t\;,\;t)}{\partial S} \tag{13}$$

would be self-financing and would replicate exactly all future cash-flows. On the other hand, if σ_t was an arbitrary function satisfying (2), then a self-financing portfolio of stocks and bonds worth initially $V_t \;=\; W^+(S_t, t)$ and subsequently constrained to satisfy (12) (but not (13)) will have a non-negative final value after paying out all the derivative's cash-flows, almost surely [6]. Therefore, a self-financed trading strategy which uses the hedge-ratio (12) with σ given in (8), will hedge risklessly a short position in the derivative security. This strategy is optimal in the sense that it has the least possible initial cost within the class of all other dominating strategies that use only stocks and bonds. In fact, the initial cost cannot be reduced further since we have perfect matching of the cash-flows (3) when the volatility path is given by (11). Equation (11) represents the "worst-case scenario" volatility path in the context of this model. Similarly, $W^-(S_t\;, t)$ and $\frac{\partial W^-(S_t\;,\;t)}{\partial S}$ can be interpreted, respectively, as the maximal initial bid value and the hedge-ratio for managing a long position in this derivative security.

2. Risk-diversification

The uncertain volatility model gives us a means of quantifying the diversification of volatility risk in portfolios of derivative securities. In fact, let

$$\Phi \;=\; \sum_{j=1}^{N} e^{-\;r\;(t_j\;-\;t)}\;\; F_j(\;S_{t_j}\;) \tag{14}$$

and

[6] See the the Appendix.

$$\Psi \ = \ \sum_{k=1}^{N'} e^{- \ r \ (t'_k \ - \ t)} \ G_k(\ S_{t'_k} \ ) \tag{15}$$

denote the sums of the discounted cash-flows of two "generic" derivative products. Clearly, we have

$$\mathrm{Sup}_P \ \ \mathrm{E}_t^P \ \left[\ \Phi \ + \ \Psi \ \right] \ \le \ \mathrm{Sup}_P \ \ \mathrm{E}_t^P \ \left[\ \Phi \ \right] \ + \ \mathrm{Sup}_P \ \ \mathrm{E}_t^P \ \left[\ \Psi \ \right] \tag{16}$$

and

$$\mathrm{Inf}_P \ \ \mathrm{E}_t^P \ \left[\ \Phi \ + \ \Psi \ \right] \ \ge \ \mathrm{Inf}_P \ \ \mathrm{E}_t^P \ \left[\ \Phi \ \right] \ + \ \mathrm{Inf}_P \ \ \mathrm{E}_t^P \ \left[\ \Psi \ \right] \ . \tag{17}$$

Typically, the inequalities in (16) and (17) will be strict, except when the pricing measures realizing the supremums (or infimums) separately for each derivative security are identical. Therefore, in general, the optimal risk-averse "offer" price of the portfolio "$\Phi \ + \ \Psi$" will be lower than the sum of the individual optimal offer prices for "Φ" and "Ψ". Similarly, the "bid" price for "$\Phi \ + \ \Psi$" portfolio will be higher than the sum of the separate bid prices.

This can be understood more concretely in terms of option portfolios. Since the "volatility" in the BSB equation is equal to σ_{min} or σ_{max} according to the convexity of the value function, and that option values are convex in S, the upper and lower bounds in (5) and (6) are simply Black-Scholes prices obtained by using the extreme volatilities. This result is intuitively clear: a risk-averse agent that plans to delta- hedge his position in this uncertain volatility environment would have to sell options at the highest volatility and buy them at the lowest one. On the other hand, more complex derivatives and option portfolios that combine short and long option positions (and thus have mixed convexity), are priced differently. The BSB equation selects the volatility path that generates the most efficient non-arbitrageable bid/offer values. Pricing the "whole" is more efficient than adding the prices of the individual "parts". This is illustrated in Figures 1 and 2 and Tables 1 and 2, where we study this "pooling" effect on vertical and horizontal call-spreads.

Since the extreme prices for options are obtained by using the two extreme volatilities, one might believe that the extremal prices for option portfolios would be given by Black-Scholes prices with *some* constant volatility σ in the range $\sigma_{min} \ \le \ \sigma \ \le \ \sigma_{max}$. However, the theory presented here shows that such valuation is always subject to volatility risk if the portfolio has mixed convexity. Theoretical premiums calculated with the Black-Scholes formula with a constant volatility will be too low to enter into a delta-hedging strategy that protects against future volatility movements within the band. To illustrate this, we present in Figure 3 comparison between the extreme prices that would be obtained by pricing with a constant volatility and those obtained from the BSB equation for the call spread of Figure 1/Table 1. It is noteworthy that the volatility risk is significant despite the fact that the Vega of the position (using the BS formula) can be quite small[7].

[7] Vega measures the sensitivity with respect to infinitesimal, rather than large, volatility movements.

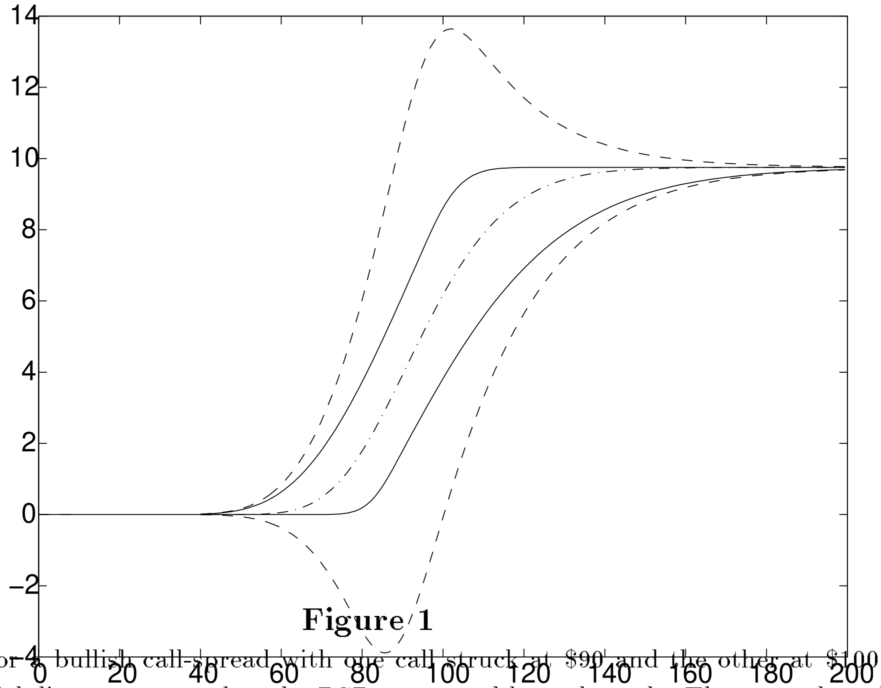

Figure 1

Different pricing curves for a bullish call-spread with one call struck at \$90 and the other at \$100, both with 6 months to expiry. The thick lines correspond to the BSB upper and lower bounds. The outer dotted lines are the bid/ask values obtained by pricing the options separately using the Black-Scholes formula. The middle line is the Black-Scholes price with the mid volatility. The parameters are r=0.05, $\sigma_{\rm min}$=0.1 and $\sigma_{\rm max}$=0.4.

Table 1

Numerical values corresponding to Figure 1 for different values of the stock S.

S	W^+ / W^-	$C_{90}^+ - C_{100}^-$ / $C_{90}^- - C_{100}^+$	$C_{90} - C_{100}$ (midvol.)
75	2.69	4.13	1.01
	0.02	– 2.26	
80	3.73	6.04	1.79
	0.19	– 3.28	
85	4.90	8.33	2.79
	0.79	– 3.88	
90	6.15	10.72	3.93
	1.79	– 3.43	
95	7.44	12.75	5.09
	2.83	– 1.96	

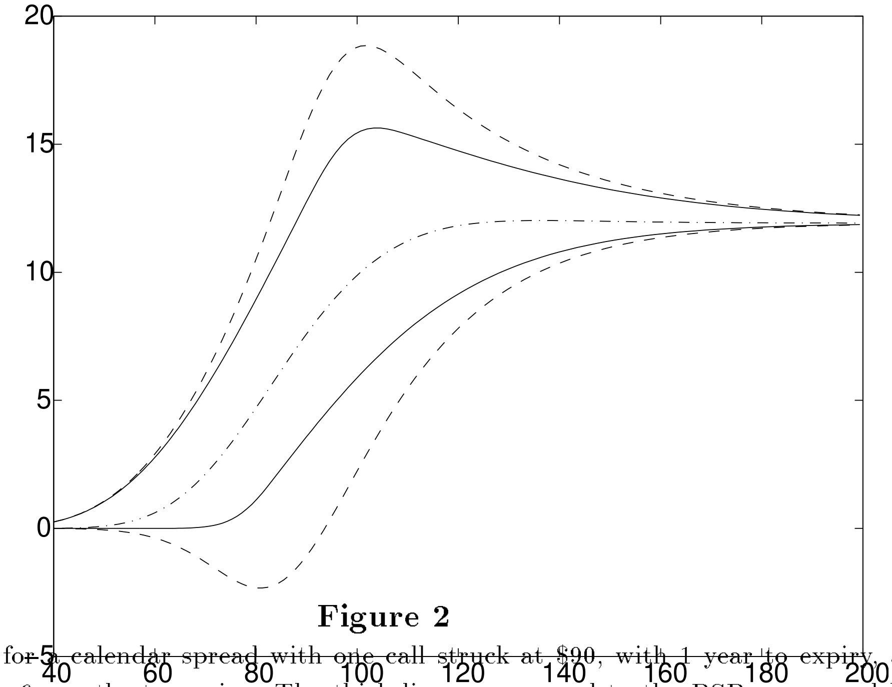

Figure 2

Different pricing curves for a calendar spread with one call struck at \$90, with 1 year to expiry, and the other call struck at \$100, with 6 months to expiry. The thick lines correspond to the BSB upper and lower bounds. The outer dotted lines are the bid/ask values obtained by pricing the options separately using the Black-Scholes formula. The middle line is the Black-Scholes price with the mid volatility. The parameters are r=0.05, $\sigma_{\min}$=0.1 and $\sigma_{\max}$=0.4.

Table 2

Numerical values corresponding to Figure 2 for different values of the stock S.

S	W^+ / W^-	$C_{90}^+ - C_{100}^-$ / $C_{90}^- - C_{100}^+$	$C_{90} - C_{100}$ (midvol.)
75	7.14	8.11	3.31
	0.34	− 1.94	
80	8.94	10.50	4.71
	1.11	− 2.32	
85	10.83	13.26	6.18
	2.33	− 2.07	
90	12.75	15.80	7.60
	3.58	− 1.07	
95	14.47	17.85	8.85
	4.78	0.48	

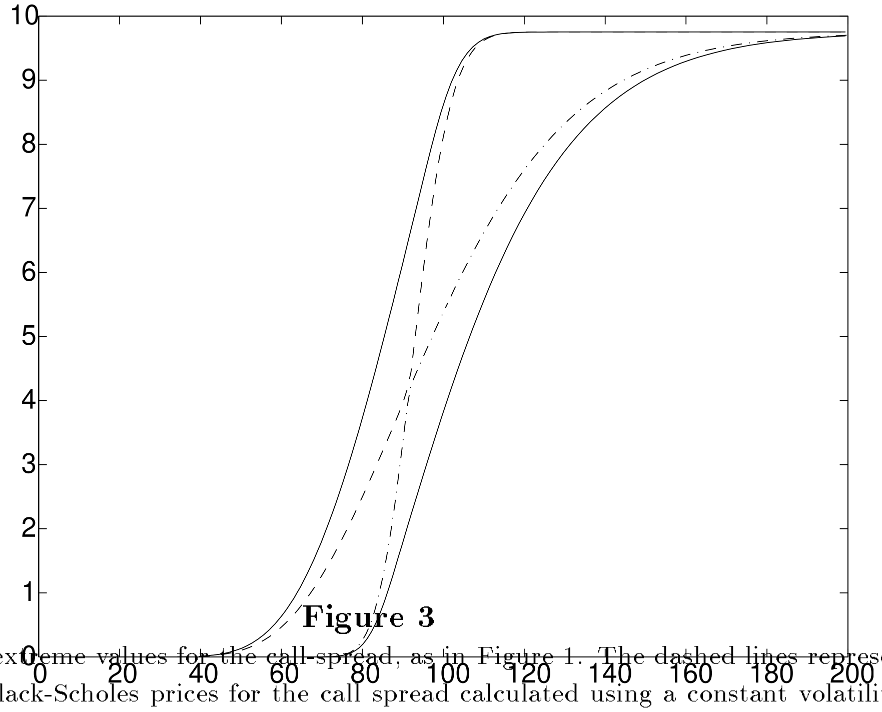

Figure 3

Solid lines represent the extreme values for the call-spread, as in Figure 1. The dashed lines represent the upper and lower envelopes of Black-Scholes prices for the call spread calculated using a constant volatility σ between $\sigma_{\min}=0.1$ and $\sigma_{\max}=0.4$.

3. Hedging with stochastic volatility

Many interesting papers in Finance study stochastic volatility and incomplete markets, cf. Johnson & Shanno (1987), Hull and White (1987), Scott (1987), Wiggins (1987), Dupire (1992) and Eisenberg and Jarrow (1994). Hull and White, for instance, considered a system of stochastic differential equations to model the joint statistics of the stock price and its volatility. Their method consists in calculating the present value of a security by averaging over the joint distribution of stock price and volatility. In this section, we consider a similar stochastic volatility model and compare the BSB price/hedging strategy – which corresponds to extreme volatility scenarios – with the prices/strategies obtained by averaging over volatility. The main interest of this comparison lies in the issue of hedging. Since volatility is not a traded asset, delta-hedging against volatility movements using the underlying stock and risk-neutral valuation is always risky (Dupire, 1992).

To analyze the potential losses arising from hedging a derivative product in a stochastic volatility environment, we implemented the following volatility dynamics:

$$\sigma_t \;=\; \exp\left(X_t \right) \quad , \tag{18}$$

where X_t is the mean-reverting Gaussian diffusion satisfying

$$d X_t \;=\; \alpha \left(\gamma \;-\; X_t \right) dt \;+\; \rho \, dW_t \; . \tag{19}$$

Here W_t is a Brownian motion independent of Z_t. The parameter values were taken to be $\alpha = 0.693 = \ln 2$, $\gamma = -1.609 = \ln 0.2$, $\rho = 2 \times (\ln 2)^{1/2}/Z_{95}$, where $Z_{95} = 1.64$ represents the 95% percentile of the standard Gaussian. With these parameter values, the volatility band $0.1 - 0.4$ is then a centered 90% confidence interval for the volatility under the equilibrium measure. Monte-Carlo simulations corresponding to price evolutions over 6-month periods show that volatility paths initialized at the mean value $< \sigma_t >= 0.2$ spent on average 1.9% of the time above 0.4 and 1.9% of the time below 0.1. These slightly better statistics are due to the finite 6 month time-horizon.

We considered the problem of delta-hedging a call spread consisting of a short call with strike price \$90 and a long call with strike price \$100, both with six months to maturity. To construct a hedging strategy consistent with the average price, we observed that, in the long run, the price corresponding to averaging with respect to the mean-reverting volatility paths coincides with the standard Black-Scholes price with constant volatility $\sigma_{eff} \approx < \sigma_t^2 >^{1/2}$. This is due to the ergodic theorem, which equates the temporal and ensemble averages of σ_t. We verified that this approximation gives essentially the same price six months from expiry as as the one obtained by averaging over the joint statistics of stock price and volatility. The exact value of σ_{eff} is 0.2214. Consistently with this observation, we assumed that one agent would delta-hedge his position using the Black-Scholes formula with $\sigma = \sigma_{eff}$ over the six-month period and another one would use the BSB hedge. Adjustments in Delta take place at regular intervals with about 100 rehedgings over the six-month period. In Figures 4, 5 and 6 we present various histograms made by running $100,000$ Monte-Carlo simulations of stock prices following the (4) with the stochastic volatility in (18), (19). These results show that the BSB hedge provides strong protection against catastrophic losses in a stochastic volatility environment, comparable to that of the Black-Scholes strategy in the ideal case of constant volatility.

4. Using derivatives to hedge volatility risk

The uncertain volatility model can be used to construct hedging portfolios that use other liquid derivatives in addition to the underlying stock. To see this, consider the hypothetical case of an agent that wishes to offer the derivative security "Φ" to a client and hedge his or her risk using another derivative asset "Ψ", contingent on S, which we assume can be bought in the market at price G.[8] If the agent buys λ units of this derivative to hedge the short position and delta-hedges the remaining exposure, the cost of establishing the hedge, including the additional coverage for the risk due to the mismatch between a "Φ" and "Ψ" will be at most

$$\lambda\, G \;+\; \mathrm{Sup}_P \;\; \mathrm{E}_t^P \;[\; \Phi \;-\; \lambda\, \Psi \;]\;. \tag{20}$$

The second term can be calculated using the BSB equation. To find the optimal the number of

[8] We use the same conventions as in Section 2. The functions Φ and Ψ are defined in equations (14) and (15).

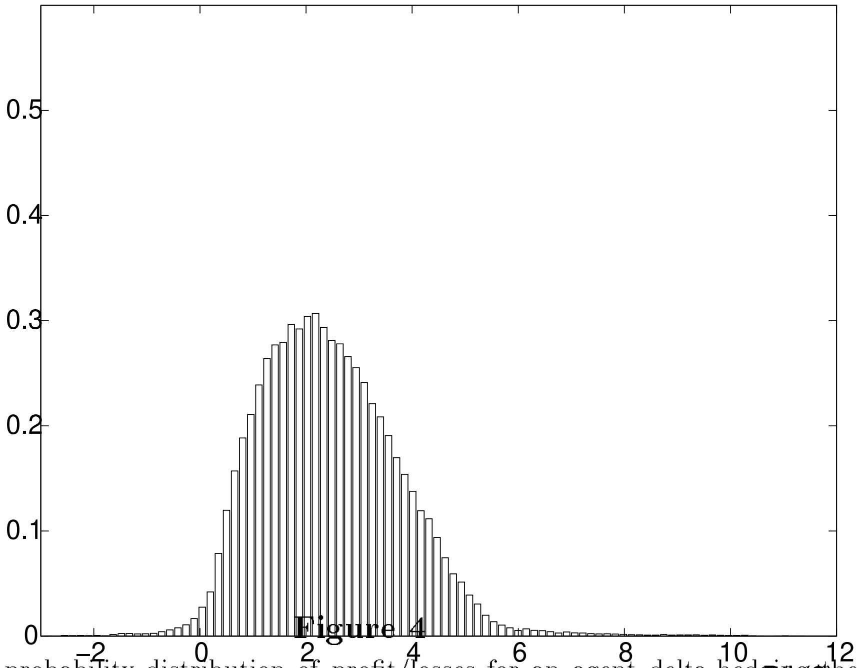

Figure 4

Histogram describing the probability distribution of profit/losses for an agent delta-hedging the call-spread of Figure 1 /Table 1 using BSB in a lognormal volatility environment. The interval $\sigma_{min}=0.1 < \sigma_t < \sigma_{max}=0.4$ is a 90%-96% centered confidence interval for σ. The tail of the distribution to the left of 0 represents the probability of losses.

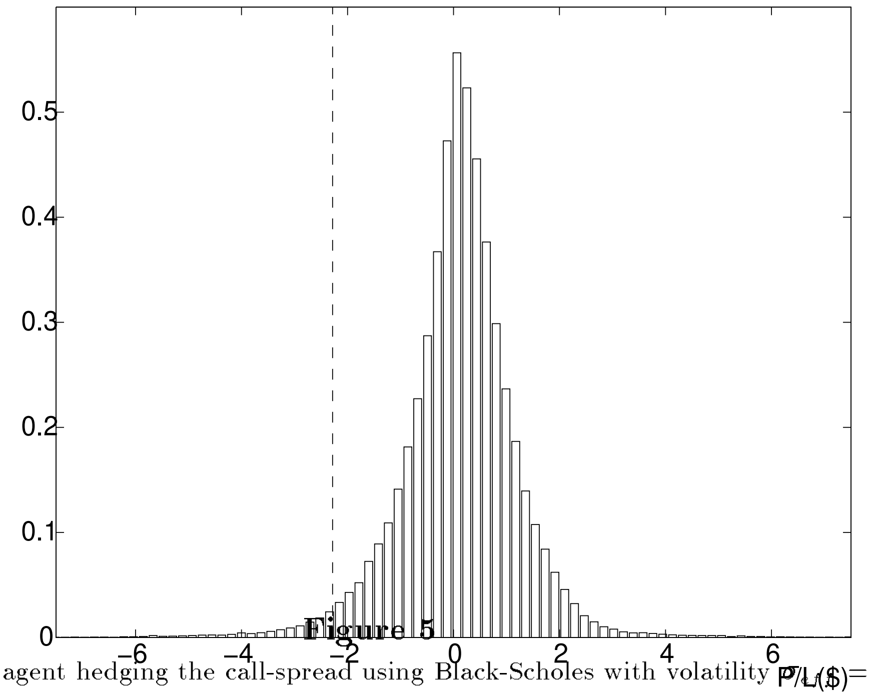

Figure 5

Same as Figure 4, for an agent hedging the call-spread using Black-Scholes with volatility $\sigma=0.2214$. Notice that the probability of losses, determined by the portion of the density to the left of zero, is significant. The vertical dashed line indicates the break-even point for an agent with initial endowment equal to the BSB upper price.

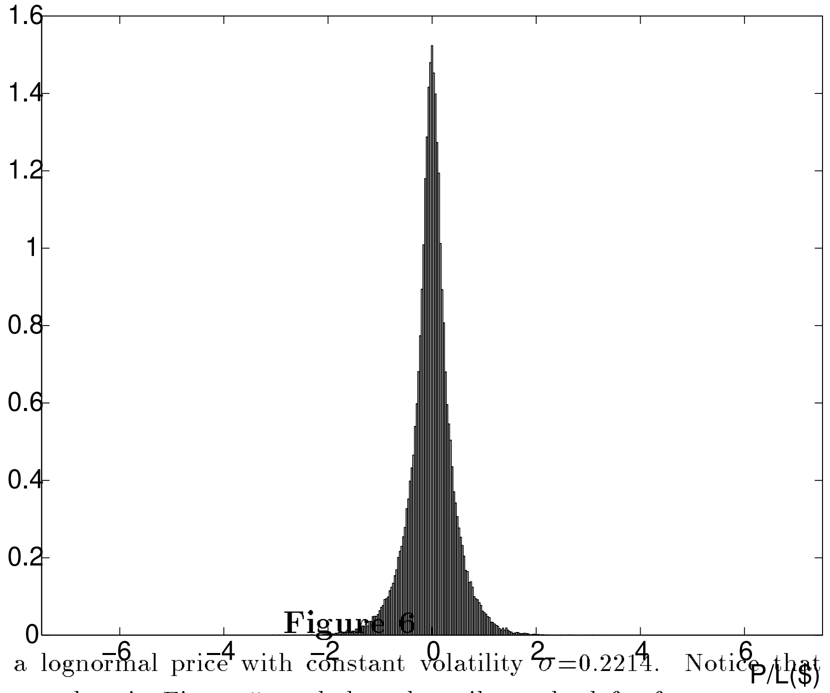

Figure 6

Same as Figure 5, but for a lognormal price with constant volatility σ=0.2214. Notice that the tails of the distribution are much narrower than in Figure 5, and that the tails to the left of zero are comparable to those of Figure 4. This suggests that the BSB hedge gives a similar degree of protection in an uncertain volatility environment than does Black-Scholes in a constant volatility environment.

contracts for such hedge, we must therefore solve the minimization problem

$$\mathrm{Inf}_{\lambda} \left\{ \lambda\, G \;+\; \mathrm{Sup}_{P} \;\; \mathrm{E}\left[\, \Phi \;-\; \lambda\, \Psi \, \right] \right\} . \tag{21}$$

Similarly, an agent that wants to take improve the value of his portfolio and holds the derivative security with payoff Φ can solve the maximization problem

$$\mathrm{Sup}_{\lambda} \left\{ \lambda\, G \;+\; \mathrm{Inf}_{P} \;\; \mathrm{E}\left[\, \Phi \;-\; \lambda\, \Psi \, \right] \right\} , \tag{22}$$

to hedge away some of the volatility risk of his position. In practice, the calculation of the optimal hedge requires solving the BSB equation repeatedly with different λs to perform the optimization. The same procedure can be be implemented to determine an optimal hedge using several derivatives (in which case Ψ , G and λ should be regarded as vectors).

We observe that dynamically complete models or models that incorporate a random volatility with known statistics cannot distinguish between hedges that use derivatives and those which use only the underlying stock. Therefore, unlike the present model, they cannot be used to construct hedges that involve derivative securities. Risk-diversification through derivatives is a natural and important tool in derivative market-making, precisely because of the uncertain nature of volatility[9].

Recently, Rubinstein (1994), Shimko (1993), Derman and Kani (1994) and Dupire (1994) considered the problem of calculating "implied trees", or implied martingale measures consistent with prices of liquidly traded derivatives. The model presented here is closely related to this point of view. In fact, the starting point of the uncertain volatility model consists of an environment with a multiplicity of pricing measures. As shown above, the existence of derivative products in the market offers the possibility of further risk-diversification, resulting in narrower bounds for the prices of OTC derivative products than those corresponding to delta-hedging with the underlying security alone. A systematic application of the the procedure in (21), using the BSB equation, would yield the optimal price range consistent with the market, and thus the corresponding "implied trees" and hedging portfolios. Notice that the range between the upper and lower bounds will be progressively narrowed if the number of derivative products available for hedging increases and the market becomes more complete. Thus, the optimization problem in (21) can also be viewed as a constructive algorithm for finding the "implied tree" if it exists, i.e. if the market is complete.

5. Numerical implementation

In practice, we implement the two-volatility model using a discretization of the BSB equation. This discretization can be viewed as a "trinomial tree" that approximates the stochastic differential equation (4) with adjusted drift. Thus, we consider a model with N trading periods. After each trading period, the asset price S can change with uncertainty to one of three values: $D\,S$ $M\,S$ or $U\,S$. Here D, M and U are positive numbers satisfying $D < M < U$ and $U\,D = M^2$. This last condition is imposed so that the set of all possible trajectories forms a recombining tree. To calibrate the model, we assume a time horizon $0 \le t \le T$, (time is measured in years) and divide it into N trading periods of $\Delta t = T / N$ with $\Delta t \ll 1$. We set

$$U = e^{\sigma_{max} \sqrt{\Delta t} + r\,\Delta t},$$

$$M = e^{r\,\Delta t}$$

and

[9] It is also noteworthy that the BSB equation is formally similar to an equation proposed recently to hedge option portfolios in the presence of transaction costs, Leland (1985), Whalley and Wilmott (1993), Hoggard, Whalley and Wilmott (1994), Flesacker and Houghton (1994) and Avellaneda and Parás (1994). The benefits of risk-diversification captured by using the nonlinear equation with Gamma-dependent volatility were pointed out by Hoggard *et al.*. Thus, *a posteriori*, the uncertain volatility model could also incorporate the agent's expected delta-hedging costs due to bid-offer spreads.

$$D \; = \; e^{- \; \sigma_{max} \; \sqrt{\Delta t} \; + \; r \; \Delta t} \; . \tag{23}$$

We define a one-parameter family of probabilities at each node of the tree, given by

$$P_U \; (\; p \;) \; = \; p \cdot \; (\; 1 \; - \; \frac{\sigma_{\max} \; \sqrt{\Delta t}}{2} \;) \; ,$$

$$P_M \; (\; p \;) \; = \; 1 \; - \; 2 \, p$$

and

$$P_D \; (\; p \;) \; = \; p \cdot \; (\; 1 \; + \; \frac{\sigma_{\max} \; \sqrt{\Delta t}}{2} \;) \; , \tag{24}$$

where p varies in the range

$$\frac{\sigma^2_{\min}}{2 \, \sigma^2_{\max}} \; \leq p \; \leq \; 1/2 \; . \tag{25}$$

For a given value of p, the variance of the logarithm of the price shock is given by

$$2 \; p \cdot \; \sigma^2_{\max} \; \Delta t \; . \tag{26}$$

Therefore, as the parameter p varies at different nodes of the tree, the set of probabilities defined in (26) span a family of stochastic processes which approximate the solutions of the stochastic differential equation (4) and satisfy the local volatility bounds in (2).[1]0

The finite-difference scheme approximating the BSB equation can also be viewed as the solution of the extremal problems (5) and (6) in this discrete setting. To describe it in detail, we label the nodes of the trinomial tree by pairs of integers $(n \; , \; j)$, where n indicates the time coordinate and j the price level at the node, with j increasing with price. We make the convention that each node $(n \; , \; j)$ has three "offspring" $(n+1 \; , \; j)$(up), $(n+1 \; , \; j)$ (middle) and $(n+1 \; , \; j-1)$ (down). There are $2n + 1$ nodes at stage n. According to (23), the price at node $(n \; , \; j)$ is

$$S_n^j \; = \; S_0^0 \; e^{j \cdot \; \sigma_{max} \; \sqrt{\Delta t} \; + \; n \cdot \; r \; \Delta t} \; . \tag{27}$$

Assuming that the cash-flows of the derivative security of interest are of the form (3) with $t_n \; = \; n \cdot \; \Delta t$, we set

[1]0 It is easy to verify that the probabilities (26) make the model risk-neutral in the limit $\Delta t \; \ll \; 1$.

$$F_n^j \;=\; F_n\,(\,S_n^j) \qquad \text{and} \;\; W_n^{\pm,j} \;=\; \text{Sup or Inf} \quad \mathrm{E}\,\Big[\,\sum_{k=j+1}^{N} \; e^{-r\,(t_k\,-\,t_n)}\;F_k\,(\,S_k\,)\,\Big]\,. \tag{28}$$

The equations for $W_n^{\pm,j}$ are then

$$W_n^{\pm,j} \;=\; F_n^j \;+\; e^{-r\Delta t}\;\times$$

$$\text{Sup}_p \text{ or } \text{Inf}_p \;\Big[\,P_U\,(\,p\,)\,W_{n+1}^{\pm,j+1} \;+\; P_M\,(\,p\,)\,W_{n+1}^{\pm,j} \;+\; P_D\,(\,p\,)\,W_{n+1}^{\pm,j+1}\,\Big]\,, \tag{29}$$

where the parameter p in the interval (25). The equations can be recast in the more explicit form

$$W_n^{+,j} \;=\; F_n^j \;+\; e^{-r\;\Delta t}\begin{cases} W_{n+1}^{+,j} \;+\; \frac{1}{2}\,L_{n+1}^{+,j} & \text{if } L_{n+1}^{+,j} \;\geq 0 \\ \\ W_{n+1}^{+,j} \;+\; \frac{\sigma_{min}^2}{2\sigma_{max}^2}\,L_{n+1}^{+,j} & \text{if } L_{n+1}^{+,j} \;<\; 0 \end{cases} \tag{30}$$

and

$$W_n^{-,j} \;=\; F_n^j \;+\; e^{-r\;\Delta t}\begin{cases} W_{n+1}^{-,j} \;+\; \frac{1}{2}\,L_{n+1}^{-,j} & \text{if } L_{n+1}^{-,j} \;<\; 0 \\ \\ W_{n+1}^{-,j} \;+\; \frac{\sigma_{min}^2}{2\sigma_{max}^2}\,L_{n+1}^{-,j} & \text{if } L_{n+1}^{-,j} \;\geq 0\;, \end{cases} \tag{31}$$

where

$$L_{n+1}^{\pm,j} \;=\; \Big(\,1\;-\;\frac{\sigma\text{max}\,\sqrt{\Delta t}}{2}\,\Big)\,W_{n+1}^{\pm,j+1} \;+\; \Big(\,1\;+\;\frac{\sigma\text{max}\,\sqrt{\Delta t}}{2}\,\Big)\,W_{n+1}^{\pm,j-1} \;-\; W_{n+1}^{\pm,j}\,. \tag{32}$$

This last expression can be viewed as a discretization of the second-derivative operator. In particular, the right-hand sides of these equations depend on the local convexity/concavity of the solution. Finally, we note that this discrete scheme can be shown to converge to the solution of the BSB equation as the mesh-size Δt tends to zero.

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Appendix: Dominating Strategies and the BSB Equation

We establish here that the solutions to the BSB equation give rise to optimal dominating strategies for derivative securities with cash-flows given by (3) (using only shares and cash). This result implies, in particular, that the BSB equation is the dynamical programming equation for the control problems (5) and (6). For simplicity, we shall assume that the riskless rate is zero or, alternatively that we measure prices in dollars-at-expiration. The BSB equation corresponding to the function W^+ in (5) can then be written in the form

$$\frac{\partial W^+(S,t)}{\partial t} + \frac{1}{2} S^2 \sigma^2 \left[\frac{\partial^2 W^+(S,t)}{\partial S^2} \right] \cdot \frac{\partial^2 W^+(S,t)}{\partial S^2} = \sum_{t_k > t}^{N-1} F_k(S) \cdot \delta(t - t_k) \tag{33}$$

for $t < t_N$, where δ is the Dirac function, with the final condition

$$W^+(S,t_N) = F_N(S) . \tag{34}$$

Notice that we represented the stream of intermediate cash-flows as singular term in the right-hand side of (33).

Consider an agent that has the following position at time t:

i) short one derivative security with payoff (3),

ii) long $\frac{\partial W^+(S_t,t)}{\partial S}$ shares of stock,

iii) $W^+(S_t,t) - S_t \frac{\partial W^+(S_t,t)}{\partial S}$ in cash deposited in a money-market account.

Suppose also that this agent adopts a trading strategy which consists in maintaining a hedge-ratio of

$$\Delta_\tau = \frac{\partial W^+(S_\tau,\tau)}{\partial S} \tag{35}$$

shares at each future time, financing these trades using the money market account and paying out the obligations implied by the short position in the derivative security. Let V_τ represent the combined value of the stock portfolio and money market account at time τ. Recalling that $r = 0$, it is clear that after the last payoff date t_N, the combined value of the position would be

$$V_{t_N+0} = W^+(S_t\ ,t) + \int_t^{t_N} \Delta_\tau \, dS_\tau - \sum_{t_k > t}^{N} F_k(S_{t_k}) . \tag{36}$$

On the other hand, by Itô's Lemma, we have

$$F_N(S_{t_N}) \;=\; W^+(S_{t_N}, t_N) \;=\; W^+(S_t, t) \;+$$

$$\int_t^{t_N} \Delta_\tau \, dS_\tau \;+\; \int_t^{t_N} \left[\frac{\partial W^+(S_\tau,\, \tau)}{\partial \tau} \;+\; \frac{1}{2}\, \sigma_\tau^2\, S_\tau^2\, \frac{\partial^2 W^+(S_\tau,\, \tau)}{\partial S^2} \right] \, d\tau \;. \tag{37}$$

Now, since

$$\sigma_t^2 \;\frac{\partial^2 W^+(S,\tau)}{\partial S^2} \;\leq\; \sigma^2 \left[\frac{\partial^2 W^+(S,\tau)}{\partial S^2} \right] \cdot \frac{\partial^2 W^+(S,\tau)}{\partial S^2} \;, \tag{38}$$

for $\sigma_{\min} \;\leq\; \sigma_t \;\leq\; \sigma_{\max}$, we have, using (33) and (37),

$$F_N(S_{t_N}) \;\leq\; W^+(S_t, t) \;+\; \int_t^{t_N} \Delta_\tau \, dS_\tau \;-\; \sum_{t_k \,>\, t}^{N-1} F_k(S_{t_k}) \;\;, \tag{39}$$

and thus

$$0 \;\leq\; W^+(S_t, t) \;+\; \int_t^{t_N} \Delta_\tau \, dS_\tau \;-\; \sum_{t_k \,>\, t}^{N} F_k(S_{t_k}) \;=\; V_{t_N+0} \;. \tag{40}$$

This proves that the strategy is riskless if the volatility remains between the bounds. On the other hand, the final position would be worth exactly zero if the volatility was such that we had equality in (39), i.e. if equation (11) held. Therefore, the strategy is the best possible for delta-hedging a short position. A similar argument applies to long positions and to $W^-(S\;, t\;)$.