

**A NEW APPROACH FOR PRICING DERIVATIVE SECURITIES
IN MARKETS WITH UNCERTAIN VOLATILITIES:
A “CASE STUDY” ON THE TRINOMIAL TREE**

by

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ABSTRACT

Given a pair of volatilities $\sigma_{\min}, \sigma_{\max}$, and a parameter μ , we construct a sequence of trinomial trees such that, as the time between trading periods tends to zero, the asset price becomes lognormally distributed with a drift μ and a volatility between σ_{\min} and σ_{\max} . Any volatility in this range can be obtained by specifying different probabilities at each node of the tree. We study the optimal dominating strategies for pricing and hedging derivative securities in this simple model of an incomplete market. We show that, as the time between trading periods tends to zero, the bid or ask prices of a derivative security are given by the solution of a non-linear PDE, which we call the Black-Scholes-Barenblatt equation. In this equation, the input volatility is “dynamically” selected from the two values $\sigma_{\min}, \sigma_{\max}$ according to the sign of the second derivative of the value function with respect to the price of the underlying asset.

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1. Introduction

Arbitrage Pricing Theory is an important tool for derivative security valuation. The theory states that, in the absence of arbitrage opportunities, there exists a probability measure on the ensemble of future price paths such that the present value of any security is given by the expectation of its discounted cash-flows (Duffie [5]). Such a probability is known as a *martingale measure* (Harrison and Kreps [7]), or a *pricing measure*. Determining the appropriate martingale measure associated with a sector of the security space (e.g. the stock of a company and a riskless short-term bond) permits the valuation of contingent claims based on these securities. In practice, such measures are often difficult to calculate and may not be determined uniquely from the observable parameters (prices) defining the economy. Uniqueness of the martingale pricing measure is tantamount to market completeness, a stringent economic assumption¹. It is useful to interpret the non-uniqueness of pricing measures as reflecting different choices for derivative asset prices that the market can assign in an uncertain economy. For instance, the price of an option reflects the market's projection of future volatility, but this projection changes as the market reacts to new information. Hence, in real markets, the implied volatility fluctuates during the option's lifetime. If the future volatility is unknown, a replicating portfolio that gives the "fair" value of a derivative security cannot be determined ahead of time.

This paper addresses the issue of pricing and hedging derivative securities in uncertain volatility environments. In our approach, we shall make the assumption that the future volatilities and correlations of the underlying risky assets remain within given bounds during the lifetime of the derivative security, but are otherwise undetermined. In other words, instead of incorporating into the pricing model a "complete" view of the forward volatility as a single number or as a predetermined function of time and/or price (term-structure of volatilities), or even as a stochastic process with exogenously given statistics, we shall operate under the much less stringent assumption that that volatilities and covariances of future prices are restricted to lie in predetermined "bands". For simplicity of exposition, we restrict our discussion to derivative securities based on a single liquidly traded stock which pays no dividends over the lifetime of the the derivative security and assume also a constant riskless interest rate.

We assume that the paths followed by future stock prices are Itô processes

$$dS_t = S_t \left(\sigma_t dZ_t + \mu_t dt \right), \quad (1)$$

where σ_t and μ_t are non-anticipative functions. The basic parametric assumption of our model is that

$$\sigma_{\min} \leq \sigma_t \leq \sigma_{\max}, \quad (2)$$

¹ A model for a securities market is said to be complete if the associated dividend matrix is invertible or, equivalently, if it is possible to realize any given payoff structure through an appropriate portfolio of traded securities.

where σ_{\min} and σ_{\max} are numbers representing upper and lower bounds on future volatility (see Fig. 1). These numbers must be input by the user of the model following his or her expectations and uncertainty about future price fluctuations. They could be obtained, for instance, from the extreme values of current implied volatilities of liquid options or from high-low peaks in historical stock or option-implied volatilities. These bounds can be viewed as a “confidence interval” for expected volatility values. In addition, the bounds can be used to take into account transaction costs that the user expects to incur in managing an offsetting hedge ². We also point out that, more generally, the upper and lower bounds in (2) can be made to depend on the price level and the time to expiration if necessary (here we shall assume that they remain constant over time and independent of S). Based on these assumptions, we shall derive optimal bounds on the values of derivative securities that are consistent with the volatility bounds (2). We shall also obtain the corresponding hedge-ratios for managing a derivatives position in this environment.

Assume that at a given date t , a derivative security (or a portfolio of derivatives) is characterized by a stream of cash-flows at N future dates, $t_1 \leq t_2 \leq \dots \leq t_N$,

$$F_1(S_{t_1}), F_2(S_{t_2}), \dots, F_N(S_{t_N}), \quad (3)$$

where $F_j(S)$ are known functions of the price of the underlying stock. We wish to find the “value” of this future stream of cash-flows. Under any pricing measure, the stock price should satisfy the modified risk-neutral Itô equation

$$dS_t = S_t (\sigma_t dZ_t + r dt), \quad (4)$$

where r is the riskless interest rate. However, since the volatility is not known exactly, a dynamical strategy that completely offsets price risk and also replicates exactly the future cash-flows will not exist except in trivial cases. Let us denote by P a generic probability measure on the set of paths $\{ S_\tau, 0 \leq \tau \leq T \}$, such that (4) holds for some σ_τ which is non-anticipative and satisfies the bounds (2). If there are no arbitrage opportunities and our assumption on volatility is correct, the value of this derivative should lie somewhere between

$$W^+(S_t, t) = \text{Max}_P \ E_t^P \left[\sum_{j=1}^N e^{-r(t_j - t)} F_j(S_{t_j}) \right] \quad (5)$$

and

$$W^-(S_t, t) = \text{Min}_P \ E_t^P \left[\sum_{j=1}^N e^{-r(t_j - t)} F_j(S_{t_j}) \right], \quad (6)$$

where P ranges over all admissible probability measures and E_t^P is the conditional expectation operator under P conditional on S_t .

² This point will be discussed below in more detail.

A key observation that we make here is that these two functions can be obtained recursively by solving the Hamilton-Jacobi equations, viewing (5) and (6) as stochastic control problems with control variable σ_t [refer to Krylov or maybe Karatzas Shreve]. Accordingly, in the case of a single maturity date ($N = 1$, $t_1 = T$, $F_1 = F$), this valuation problem reduces to solving the final-value problem

$$\frac{\partial W}{\partial t}(S, t) + r(S \frac{\partial W}{\partial S}(S, t) - W(S, t)) + \frac{1}{2} \sigma^2 [\frac{\partial^2 W}{\partial S^2}(S, t)] S^2 \frac{\partial^2 W}{\partial S^2}(S, t) = 0, \quad (7)$$

$$W(S, T) = F(S),$$

where W^+ is obtained setting

$$\sigma[\frac{\partial^2 W}{\partial S^2}] = \begin{cases} \sigma_{\max} & \text{if } \frac{\partial^2 W}{\partial S^2} \geq 0, \\ \sigma_{\min} & \text{if } \frac{\partial^2 W}{\partial S^2} < 0, \end{cases} \quad (8)$$

in (7), and W^- is obtained setting

$$\sigma[\frac{\partial^2 W}{\partial S^2}] = \begin{cases} \sigma_{\max} & \text{if } \frac{\partial^2 W}{\partial S^2} \leq 0, \\ \sigma_{\min} & \text{if } \frac{\partial^2 W}{\partial S^2} > 0. \end{cases} \quad (9)$$

The general case of several payoff dates is similar. Problem (7) is solved for $t_{N-1} < t \leq t_N$ with terminal condition $W(S, t_N) = F_N(S)$ first. At time t_{N-1} the value-function is set to

$$W(S, t_{N-1}) = W(S, t_{N-1} + 0) + F_{N-1}(S), \quad (10)$$

where the first term in the right-hand side represents the limit from the left as $t \rightarrow 0$ (the value t_{N-1} immediately after the cash-flow $F_{N-1}(S_{t_{N-1}})$ is paid out. Equation (7) is then used to discount the price back to time t_{N-2} , and so forth. The non-linear PDE in (7) will be referred to as the *Black-Scholes-Barenblatt* (BSB) equation³. It is a generalization of the classical Black-Scholes PDE, which reduces to it in the special case $\sigma_{\min} = \sigma_{\max}$.

In general, the “volatility” $\sigma[\cdot]$ appearing in the equation is a function of $\frac{\partial^2 W}{\partial S^2}(S_t, t)$. This has following important implication: if the future volatility path was actually given by

$$\sigma_t = \sigma \left[\frac{\partial^2 W^+(S_t, t)}{\partial S^2} \right], \quad (11)$$

³ The physicist G. I. Barenblatt [B] introduced a diffusion equation with a similar nonlinearity to model flow in porous media; hence our terminology.

with $\sigma [\cdot]$ satisfying (8), then the standard Black-Scholes argument shows that a portfolio consisting of Δ_t shares and B_t bonds, where

$$\Delta_t = \frac{\partial W^+(S_t, t)}{\partial S} \quad (12)$$

and

$$B_t = W^+(S_t, t) - S_t \cdot \frac{\partial W^+(S_t, t)}{\partial S}, \quad (13)$$

would be self-financing and would replicate exactly all future cash-flows. On the other hand, if σ_t were arbitrary, then a self-financing portfolio of stocks and bonds worth initially $P_t = W^+(S_t, t)$ and subsequently constrained to satisfy (12) (but not (13)) can be shown to have almost surely a non-negative final value, after paying out all the derivative's cash-flows⁴. Therefore, a self-financed trading strategy which uses the hedge-ratio (12), where $W = W^+$ solves equation (7) with σ given in (8), will risklessly hedge a short position in the derivative security. Moreover, this strategy is optimal in the sense that it has the least possible initial cost within the class of all other dominating strategies that use only stocks and bonds. In fact, since it matches exactly the cash-flows (3) when the volatility path is given by (11), the initial cost cannot be reduced any further without risk⁵. Similarly, $W^-(S_t, t)$ and $\frac{\partial W^-(S_t, t)}{\partial S}$ can be interpreted as the optimal initial hedging cost and hedge-ratio for managing a long position in this derivative security. Thus, the extreme values W^+ and W^- in (5) and (6) can be computed simply using a PDE and can be used to construct optimal riskless hedges, assuming that the volatility remains between the expected bounds.

Despite the simplicity of the model, we believe that it captures an important feature derivative valuation and market-making, namely, the some attention. Notice that standard In fact, since

$$\text{Max}_P \ E_t^P \left[\sum_{j=1}^N e^{-r(t_j - t)} F_j(S_{t_j}) \right] \quad (14)$$

We offer the following financial interpretation to this result. Given that only the “extreme” values σ_{\min} and σ_{\max} are known, the estimate of volatility that should be input in the pricing equation for contingent claims should be σ_{\min} or σ_{\max} , according to whether the hedger is “short” or “long” $\Gamma = \frac{\partial^2 V}{\partial S^2}$. For standard options, this is intuitively obvious: the bounds are the Black-Scholes option prices using the two extreme values for the volatility. For more complex derivatives combining short and long positions in options, the *BSB* equation dynamically selects the input volatility so as to generate optimal bid/ask prices. The *BSB* equation values a diversified

⁴ This is shown in the next section of the paper.

⁵ We argue later that the initial cost can be reduced if one uses other derivative securities to limit risk-exposure.

option portfolio on a single underlying asset in a consolidated fashion. Due to the non-linearity of the solution with respect to the payoff $F(S)$, pricing the “whole” is more efficient than pricing the “parts” individually. In Figure 1, we demonstrate this by comparing the bid/ask prices for a call spread using the *BSB* equation with the corresponding prices obtained by undoing the spread and pricing the options individually. The *BSB* equation gives an unequivocal method to estimate bid-ask spreads for trading complex derivative securities in markets with uncertain volatilities.

Perhaps the simplest mathematical example of an incomplete market is the *trinomial tree model*: it consists of a single risky asset and a riskless bond, in which the asset price S can change with uncertainty to one of three values DS , MS or US after each trading period. Here D , M and U are given positive numbers satisfying $D < M < U$. The *single-node probability measures* which are compatible with no-arbitrage in the trinomial situation must satisfy

$$P_D + P_M + P_U = 1$$

and

$$P_D D + P_M M + P_U U = R, \tag{15}$$

where R is one plus the riskless lending rate per trading period ². In contrast with the binomial model, for which the single-node risk-neutral probability is uniquely determined, in this case there is a one-parameter family of admissible probabilities at each node. Due to this non-uniqueness, (P_D, P_M, P_U) can vary from node to node in the trinomial tree. Consequently, the collection of admissible martingale measures on the set of paths of a trinomial tree is vast. If we assume that all market participants know the values D , M , and U , the problem is to guess which martingale measure should be used for asset pricing. Clearly, this question cannot be answered without introducing additional assumptions. Alternatively, we can look for bounds on the derivative asset price based on the information available to all.

In order to find an upper bound for the price of a given derivative security, we construct a portfolio of shares of the underlying asset and riskless bonds which has a cash-flow that dominates (is larger than) the payoff for all paths. Any such portfolio constitutes a riskless hedging strategy for being short the derivative security. Similarly, a “dominated” portfolio (i.e., with cash-flow less than or equal to the derivative payoff) gives a riskless strategy for being long the derivative. We can optimize these riskless portfolios by selecting the dominating portfolio with the smallest initial cost and, similarly, the dominated portfolio with the largest initial cost. The initial costs for such optimal “dominating” and “dominated” portfolios are, respectively, the best upper and lower bounds on the derivative asset price which follow from no-arbitrage considerations, given that the parameters U , M and D are known. They represent the bid-ask spread for the derivative price which is acceptable by a totally risk-averse agent.

The main question of interest to us is to characterize the bounds in the asymptotic limit of a very large trinomial tree, to simulate a “continuous-time” economy. Following Cox, Ross

² The first equation corresponds to the assumption that (P_D, P_M, P_U) is a probability. The latter one states that the present value of a risky asset is equal to its discounted expected value. Of course, the probabilities are nonnegative.

and Rubinstein, we scale the node parameters U, D , and M in such a way that the annualized volatility and the return of the underlying asset remains fixed as the tree is refined. To wit, by specifying an (identical) subjective probability measure at each node and assuming independent increments, a scaling can be implemented in such a way that the *subjective* price of the underlying asset follows a lognormal distribution with constant volatility σ . An interesting property of the trinomial world (as opposed to the binomial) is that, due to the non-uniqueness of the single-node measures compatible with no-arbitrage, this normalization does not fix the volatility of the underlying asset under the martingale measure. In fact, in the asymptotic limit of a large tree, the probability distribution of the price of the underlying asset under an admissible martingale measure can be a geometric Brownian motion with a volatility different than σ or even with a *variable* volatility depending on the local asset price or even on its past history. The situation is richer than in the CRR and Black-Scholes “worlds” since in those models the volatilities associated with the subjective and the objective probabilities are identical.

We show that the class of all possible martingale measures for normalized trinomial tree in the limit $N \gg 1$ (limit martingale measures) includes all lognormal processes with volatilities $\tilde{\sigma}$ within a band $\sigma_{\min} \leq \tilde{\sigma} \leq \sigma_{\max}$ and with a drift equal to the to the riskless lending rate, r . The extreme values for the volatility, σ_{\min} and σ_{\max} are determined by the parameters of the trinomial tree. Conversely, given any pair $(\sigma_{\min}, \sigma_{\max})$, there exists a normalized trinomial tree for which the class of admissible limit martingale measures contains all lognormal processes with volatilities $\tilde{\sigma}$ satisfy $\sigma_{\min} \leq \tilde{\sigma} \leq \sigma_{\max}$ and with average rate or return r . Furthermore, if a limit martingale measure corresponds to a price process with variable volatility, e.g. with a volatility term-structure, this volatility must necessarily lie between the bounds σ_{\min} and σ_{\max} .

In the limit of a large trinomial tree, the bounds on the prices of a European contingent claim depend only on $\sigma_{\min}, \sigma_{\max}$, the riskless lending rate, the payoff function $F(S)$, the spot price of the underlying asset and the time to expiration. For standard calls and puts, the upper and lower bounds are given, respectively, by the Black-Scholes formula with volatilities σ_{\max} and σ_{\min} . For more complicated payoffs corresponding to option spreads, binaries, etc., which are neither convex or concave, the upper bound is obtained by solving an initial-value problem for the *Black-Scholes-Barenblatt (BSB) equation*³, viz.,

$$\frac{\partial W}{\partial t}(S, t) = r \left[S \frac{\partial W}{\partial S}(S, t) - W(S, t) \right] + \frac{1}{2} \sigma^2 \left[\frac{\partial^2 W}{\partial S^2}(S, t) \right] S^2 \frac{\partial^2 W}{\partial S^2}(S, t), \quad (16)$$

Here S is the price of the underlying security, t is the time-to-expiration and r is the riskless rate of interest. The “volatility” $\sigma[\cdot]$ is defined by

$$\sigma[\Gamma] = \begin{cases} \sigma_{\max} & \text{if } \Gamma \geq 0, \\ \sigma_{\min} & \text{if } \Gamma < 0. \end{cases} \quad (17)$$

The lower bound for the price of the contingent claim is obtained by solving the same equation but interchanging the values of σ_{\max} and σ_{\min} in the definition of the Γ -dependent volatility (17).

³ The physicist G. I. Barenblatt [B] introduced a nonlinear diffusion equation with a similar nonlinearity to model flow in porous media; hence our terminology.

The proof of this theorem is done by showing that the optimal strategies/ prices on the trees correspond to difference equations which formally approximate the *BSB* equation. The prices given by these difference schemes are then shown to converge to the the *BSB* price as the number of trading periods increases.

We offer the following financial interpretation to this result. Given that only the “extreme” values σ_{\min} and σ_{\max} are known, the estimate of volatility that should be input in the pricing equation for contingent claims should be σ_{\min} or σ_{\max} , according to whether the hedger is “short” or “long” $\Gamma = \frac{\partial^2 V}{\partial S^2}$. For standard options, this is intuitively obvious: the bounds are the Black-Scholes option prices using the two extreme values for the volatility. For more complex derivatives combining short and long positions in options, the *BSB* equation dynamically selects the input volatility so as to generate optimal bid/ask prices. The *BSB* equation values a diversified option portfolio on a single underlying asset in a consolidated fashion. Due to the non-linearity of the solution with respect to the payoff $F(S)$, pricing the “whole” is more efficient than pricing the “parts” individually. In Figure 1, we demonstrate this by comparing the bid/ask prices for a call spread using the *BSB* equation with the corresponding prices obtained by undoing the spread and pricing the options individually. The *BSB* equation gives an unequivocal method to estimate bid-ask spreads for trading complex derivative securities in markets with uncertain volatilities.

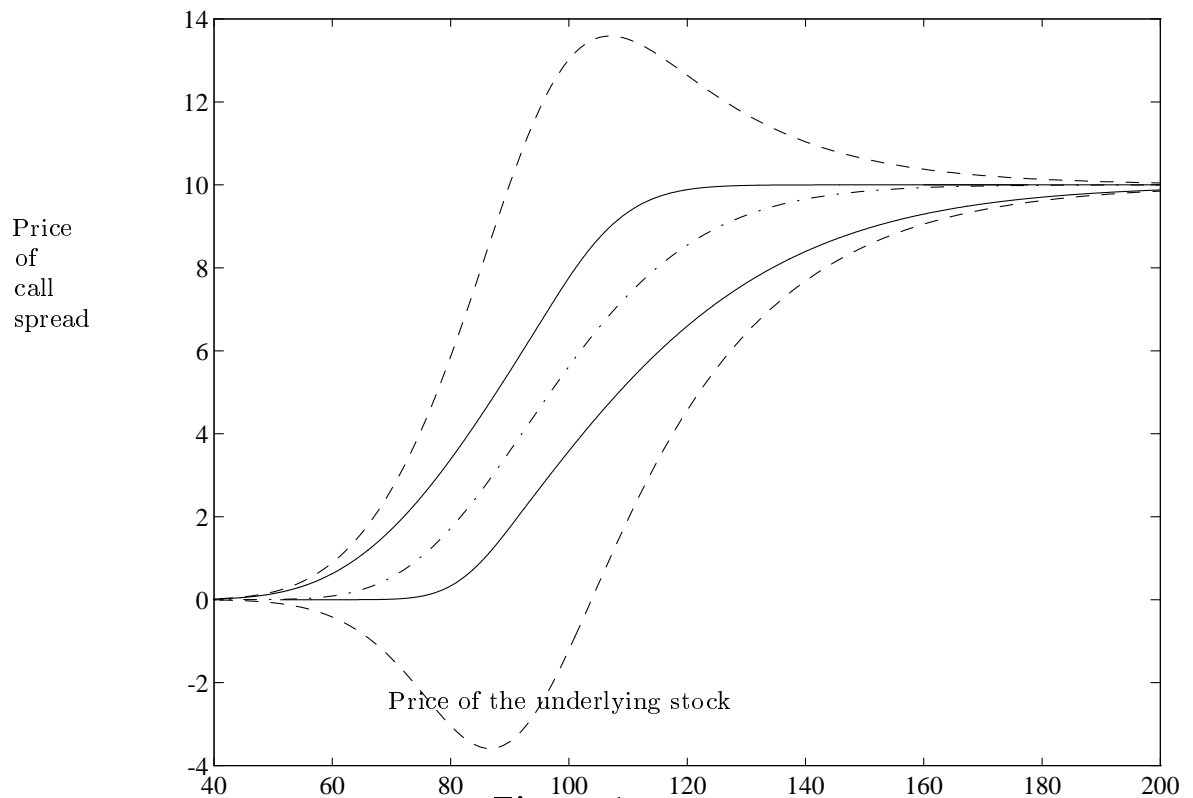


Figure 1

Bid and ask prices for a call spread between \$90 and \$100 with one year to maturity and volatility $\sigma = 0.2 \pm 0.1$. The full lines correspond to the *BSB* upper and lower bounds. The dashed lines represent the bounds that would be obtained by adding the individual ask (bid) prices for the short \$90 call and the long \$100 call. The curve $-\cdot-$ corresponds to the Black-Scholes price with $\sigma = 0.2$.

There are many works in the Finance literature which deal with pricing derivative securities with random or otherwise unknown volatilities; see for example Hull and White [8], Eisenberg and Jarrow [6] and references therein. Hull and White propose a system of stochastic differential equations to model the joint evolution of the stock price and its volatility. Eisenberg and Jarrow propose a method based on hedging the volatility risk by introducing an additional correlated security, such as an index future. Rubinstein [11] and Derman and Kani [4] consider option pricing models in which the volatility of the underlying asset depends on the price of the underlying asset and the time to expiration, *i. e.*, $\sigma = \sigma(S, t)$. In contrast to those approaches, the present theory gives upper and lower bounds on the prices of derivatives which are non-parametric, since we make no assumptions on the probability distribution of the volatility aside from postulating the existence of *a-priori* bounds σ_{\min} and σ_{\max} . A comparison between the present method and others based on a random distribution of volatility will be presented in a separate paper.

2. The trinomial tree

We make the following assumptions:

1. Our model for the evolution of security prices consists of a large trinomial tree with discrete coordinates of time and security price. There are $N = \frac{T}{\Delta t}$ trading periods, Δt represents the time-lag between trades and T represents the longest possible duration for a contingent claim. We assume that there is one security with price S_t (t = time to expiration of a contingent claim), which pays no dividends. The “root” of the tree corresponds to time T and the “leaves” to time 0. The price of the security at the root is S_T . From the root emerge three branches which reach the next time level, $T - \Delta t$. These branches end at the nodes with coordinates $(US_T, T - \Delta t)$, $(MS_T, T - \Delta t)$, and $(DS_T, T - \Delta t)$. $U > 1$ corresponds to a maximum increase in the price of the security during the time step, $D < 1$ corresponds to maximum decrease in the price of the security during a time step and $D < M < U$ corresponds to an intermediate change in the price. From each of the three nodes emerge three different branches, terminating at the next time level, $T - 2\Delta t$. For example, $(US_T, T - \Delta t)$ branches to $(U^2S_T, T - 2\Delta t)$, $(MUS_T, T - 2\Delta t)$ and $(DUS_T, T - 2\Delta t)$. This construction is iterated until the tree reaches the time level $t = 0$ (see Figure 2.)

2. U, M and D , are given by the following expressions:

$$\begin{aligned}U &= e^{\mu\Delta t + u\sqrt{\Delta t}}, \\M &= e^{\mu\Delta t + m\sqrt{\Delta t}}, \\D &= e^{\mu\Delta t + d\sqrt{\Delta t}}.\end{aligned}\tag{18}$$

$\mu, u > 0, d < 0$ and m are constants, satisfying $d < m < u$.

3. The payoff of the derivative security, $F(S)$, is a function of the price of the underlying security at the expiration date, $t = 0$.

4. The return on lending \$1 for a single trading period is

$$R = e^{r\Delta t}.\tag{19}$$

We will assume throughout that Δt is sufficiently small, say, $\Delta t < \delta$, so that $D < R < U$, and that $M - R$ has a constant sign for $\Delta t < \delta$.

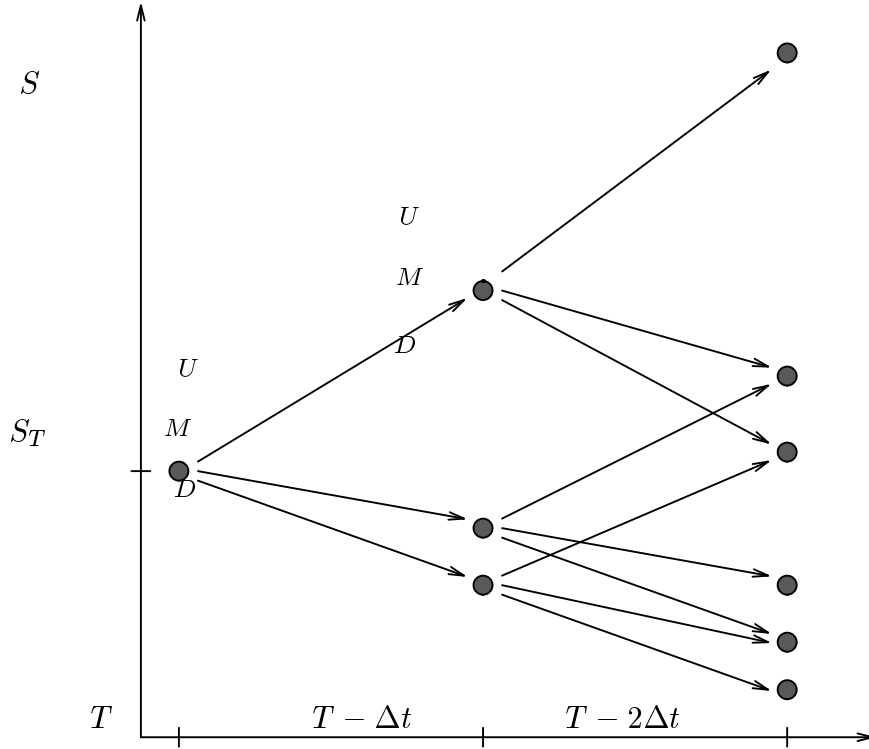


Figure 2.
The trinomial tree.

3. Asymptotic lognormality and limit martingale measures

Consider a sequence of trinomial trees satisfying assumptions 1 and 2, with $\Delta t = \Delta t_N = \frac{T}{N}$, where N runs over the positive integers. We compute all single-node probabilities (P_U, P_M, P_D) which are independent of N and for which the price of the underlying security is asymptotically lognormal as $N \nearrow \infty$. More precisely, assuming that successive price movements are statistically independent and that

$$\frac{S_{t-\Delta t}}{S_t} = \begin{cases} U & \text{with probability } P_U, \\ M & \text{with probability } P_M, \\ D & \text{with probability } P_D \end{cases} \quad (20)$$

we seek conditions on (P_U, P_M, P_D) that ensure that $\ln \frac{S_0}{S_T}$ converges in distribution to a Gaussian as $N \nearrow \infty$.

For each t , the expected value of $\ln \frac{S_{t-\Delta t}}{S_t}$ is

$$\begin{aligned} E\left[\ln \frac{S_{t-\Delta t}}{S_t}\right] &= P_D \ln D + P_M \ln M + P_U \ln U \\ &= (P_D + P_M + P_U)\mu\Delta t + (P_D d + P_M m + P_U u)\sqrt{\Delta t} \\ &= \mu\Delta t + (P_D d + P_M m + P_U u)\sqrt{\Delta t} \end{aligned} \quad (21)$$

Therefore, $\ln \frac{S_0}{S_T} = \sum \ln \frac{S_{t-\Delta t}}{S_t}$ has finite mean as $\Delta t \searrow 0$ if and only if

$$P_D d + P_M m + P_U u = 0. \quad (22)$$

Using the fact that $P_D, P_M, P_U \geq 0$ and $P_D + P_M + P_U = 1$, it is not hard to see that P_U, P_M and P_D satisfy (22) if and only if

$$P_U = \frac{P_D(m-d) - m}{u-m}, \quad (23)$$

$$P_M = \frac{-P_D(u-d) + u}{u-m}, \quad (24)$$

and

$$\max\left\{\frac{m}{m-d}, 0\right\} \leq P_D \leq \frac{u}{u-d}. \quad (25)$$

The variance of $\ln \frac{S_0}{S_T}$ is

$$\sigma^2 = P_D d^2 + P_M m^2 + P_U u^2, \quad (26)$$

from (22). Thus by the Central Limit Theorem, if the single-node probability (P_U, P_M, P_D) satisfies (23), (24) and (25), $\ln \frac{S_0}{S_T}$ converges in distribution to a normal random variable with mean μ and variance σ^2 .

Notice that the asymptotic variance of $\ln \frac{S_0}{S_T}$ depends on the choice of (P_U, P_M, P_D) . From equations (23), (24) and (25), we obtain

$$\begin{aligned} \sigma^2 &= P_D d^2 + (1 - P_U - P_D) m^2 + P_U u^2 = P_D(d^2 - m^2) + m^2 + P_U(u^2 - m^2) \\ &= P_D(d^2 - m^2) + m^2 + [P_D(m-d) - m](u+m) \\ &= P_D(m-d)(u-d) - mu \end{aligned} \quad (27)$$

Since $(m-d)(u-d) > 0$, the extreme values of the asymptotic variance are obtained when P_D takes the extreme values in (25). These values are

$$\sigma_{\max}^2 = -d u$$

and

$$\sigma_{\min}^2 = \begin{cases} -m u & m \leq 0, \\ -m d & m \geq 0. \end{cases} \quad (28)$$

These relations show that, given any pair of “extremal” volatilities with $0 \leq \sigma_{\min} \leq \sigma_{\max}$, by choosing u, m and d so that $u = -d = \sigma_{\max}$ and $m = -\sigma_{\min}^2 / \sigma_{\max}$, we can construct a sequence of trinomial trees that will have these values for the extremal volatilities in the limit $N \gg 1$. To fix ideas, we assume that a particular value of σ represents our subjective estimate of annualized volatility, satisfying $\sigma_{\min} \leq \sigma \leq \sigma_{\max}$.

Next, we characterize the martingale measures associated with the trinomial tree. The no-arbitrage condition requires that

$$S_t = \frac{1}{R} [P_U U S_t + P_M M S_t + P_D D S_t],$$

which is equivalent to (15). Using (18) and (19), we find that this implies that

$$P_D = P_D^{(0)} + \pi_D \sqrt{\Delta t} + o(\sqrt{\Delta t}),$$

$$P_M = P_M^{(0)} + \pi_M \sqrt{\Delta t} + o(\sqrt{\Delta t}),$$

and

$$P_U = P_U^{(0)} + \pi_U \sqrt{\Delta t} + o(\sqrt{\Delta t}).$$

Here, $P_D^{(0)}$, $P_M^{(0)}$ and $P_U^{(0)}$ satisfy conditions (23), (24), (25) and π_D, π_M, π_U are real numbers satisfying

$$\pi_D + \pi_M + \pi_U = 0$$

and

$$\mu + \frac{1}{2}(P_D^{(0)} d^2 + P_M^{(0)} m^2 + P_U^{(0)} u^2) + \pi_D d + \pi_M m + \pi_U u = r.$$

Thus, each single-node probability associated with a martingale measure is a perturbation of order $\sqrt{\Delta t}$ of a triad satisfying the conditions for asymptotic lognormality. In particular, the range of possible volatilities for the martingale measures is also $\sigma_{\min} \leq \sigma \leq \sigma_{\max}$. The class of all martingale measures includes Geometric Brownian Motions (with constant drift and volatility) as well as more general diffusions such that $\sigma_t = \sigma(S_t, t)$ with $\sigma_{\min} \leq \sigma(S_t, t) \leq \sigma_{\max}$. This is due to the fact that single-node probabilities can vary from node to node.

Remark . If the tree is to be constructed in a computer, it is desirable that $UD = M^2$, since in this case the number of nodes on the tree is proportional to N^2 instead of N^3 . The latter estimate is the number of nodes for a general trinomial tree with fixed U, M and D (note that the tree recombines partially since, for example, $UD = DU$). This additional constraint implies that $\frac{u+d}{2} = m$. It restricts the extremal volatilities that can be realized on the tree to satisfy $0 \leq 2\sigma_{\min}^2 < \sigma_{\max}^2$.

4. Finding the optimal dominating strategy

Given a European-style derivative security with payoff $F(S)$, we would like to find the hedging strategy that will dominate the payoff at expiration with probability one having the lowest possible initial cost. The initial cost of this strategy, $V(S_t, t)$, corresponds to the ask price for a contract with time-to-expiration t , for $0 \leq t \leq T$. Moreover, the “discrete” S -derivative of $V(S_t, t)$ gives the hedge-ratio or number of shares that should be held at each time in the optimal dominating portfolio ⁴.

The value function is constructed recursively, from the “leaves” to the “root”. At the expiration date (i.e. $n = 0$) we have

$$V(S, 0) = F(S). \quad (29)$$

Suppose now that $V(S, n\Delta t)$ has been determined for all values of S . Let $(S, (n+1)\Delta t)$ be a node of the tree at time $(n+1)\Delta t$. This node branches into the three nodes $(US, n\Delta t)$, $(MS, n\Delta t)$ and $(DS, n\Delta t)$ (see Figure 3(a)). In order to hedge in a riskless way these three outcomes, a hedger should hold a portfolio of bonds and shares of the underlying stock, such that its value at the next time-step will exceed $V(US, n\Delta t)$, $V(MS, n\Delta t)$ and $V(DS, n\Delta t)$ according to the outcome. We say that the triad $(V(DS, n\Delta t), V(MS, n\Delta t), V(US, n\Delta t))$ is *convex (concave)* if the piecewise linear function obtained by interpolating the points $(DS, V(DS, n\Delta t))$, $(MS, V(MS, n\Delta t))$ and $(US, V(US, n\Delta t))$ is convex (concave) (see Figures 3(b) and 3(c)). To find the least costly dominating portfolio at the node $(S, (n+1)\Delta t)$, we have to distinguish between three cases:

Convex case. In this case the optimal portfolio will be such that at time n its value as a function of S will correspond to the straight line passing through the points $(DS, V(DS, n\Delta t))$ and $(US, V(US, n\Delta t))$ (Figure 3(b)). This means that the amount of stock $\Delta(S, (n+1)\Delta t)$ that has to be held is given by

$$\Delta(S, (n+1)\Delta t) = \frac{V(US, n\Delta t) - V(DS, n\Delta t)}{S(U - D)}. \quad (30)$$

The corresponding investment in riskless bonds $B(S, (n+1)\Delta t)$ which is needed satisfies

$$RB(S, (n+1)\Delta t) = V(DS, n\Delta t) - \Delta(S, (n+1)\Delta t)DS. \quad (31)$$

After a bit of algebra we obtain that the value of this portfolio at $(S, (n+1)\Delta t)$, is

$$\begin{aligned} V(S, (n+1)\Delta t) &= B(S, (n+1)\Delta t) + \Delta(S, (n+1)\Delta t)S \\ &= \frac{1}{R} \left[V(DS, n\Delta t) \frac{U - R}{U - D} + V(US, n\Delta t) \frac{R - D}{U - D} \right] \\ &\equiv L_{\text{conv}}[V](S, n\Delta t). \end{aligned} \quad (32)$$

⁴ Similarly, the “dominated” portfolio (with value $\leq F(S_0)$) which has the largest initial price corresponds to the bid price.

Concave case. This case is itself divided into two subcases according to $M \leq R$ or $R < M$ (Figure 3(c)).

In the former case, we have $M \leq R < U$, and thus a similar argument as the one for the convex case will show that the least costly portfolio that dominates $V(MS, n\Delta t)$ and $V(US, n\Delta t)$ has

$$\Delta(S, (n+1)\Delta t) = \frac{V(US, n\Delta t) - V(MS, n\Delta t)}{S(U - M)}. \quad (33)$$

securities, and

$$B(S, (n+1)\Delta t) = \frac{1}{R} [V(MS, n\Delta t) - \Delta(S, (n+1)\Delta t)MS] \quad (34)$$

bonds. Since we are in the concave case this portfolio will dominate also $V(DS, n\Delta t)$, as desired. From (33) and (34) the value at $(S, (n+1)\Delta t)$ of this portfolio is given by

$$V(S, (n+1)\Delta t) = \frac{1}{R} \left[V(MS, n\Delta t) \frac{U - R}{U - M} + V(US, n\Delta t) \frac{R - M}{U - M} \right] \equiv L_{\text{conc}}^{M \leq R}[V](S, n\Delta t). \quad (35)$$

In the case $R < M$, the optimal portfolio that will dominate $V(DS, n\Delta t)$ and $V(MS, n\Delta t)$ has

$$\Delta(S, (n+1)\Delta t) = \frac{V(MS, n\Delta t) - V(DS, n\Delta t)}{S(M - D)}. \quad (36)$$

shares and

$$B(S, (n+1)\Delta t) = \frac{1}{R} [V(MS, n\Delta t) - \Delta(S, (n+1)\Delta t)MS]. \quad (37)$$

bonds. Due to the concavity of the function that interpolates linearly between the three values at time $(n+1)\Delta t$, the portfolio also dominates $V(US, n\Delta t)$. From (36) and (37) the value at $(S, (n+1)\Delta t)$ of this portfolio is given by

$$V(S, (n+1)\Delta t) = \frac{1}{R} \left[V(DS, n\Delta t) \frac{M - R}{M - D} + V(MS, n\Delta t) \frac{R - D}{M - D} \right] \equiv L_{\text{conc}}^{M > R}[V](S, n\Delta t). \quad (38)$$

Combining (32), (35) and (38) one obtains the recurrence relation

$$\begin{aligned} & V(S, (n+1)\Delta t) \\ &= \max \left\{ L_{\text{conv}}[V](S, n\Delta t), \min \left\{ L_{\text{conc}}^{M \leq R}[V](S, n\Delta t), L_{\text{conc}}^{M > R}[V](S, n\Delta t) \right\} \right\}. \end{aligned} \quad (39)$$

The last equation determines recursively the function $V(S, n\Delta t)$ on the entire tree. This function represents the minimum value of a portfolio which dominates the payoff under all possible stochastic paths described by the price over the duration of the contract.

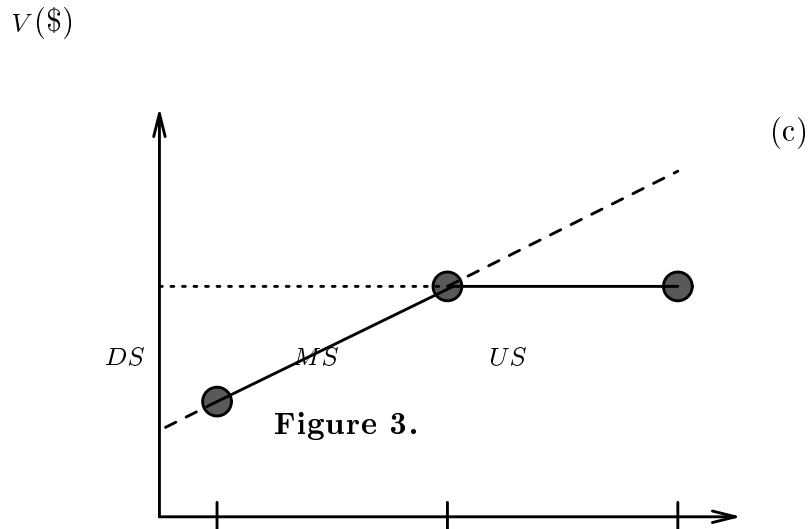
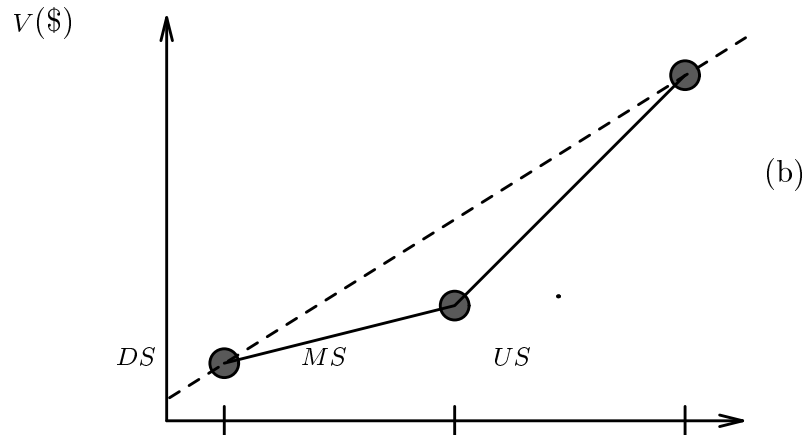
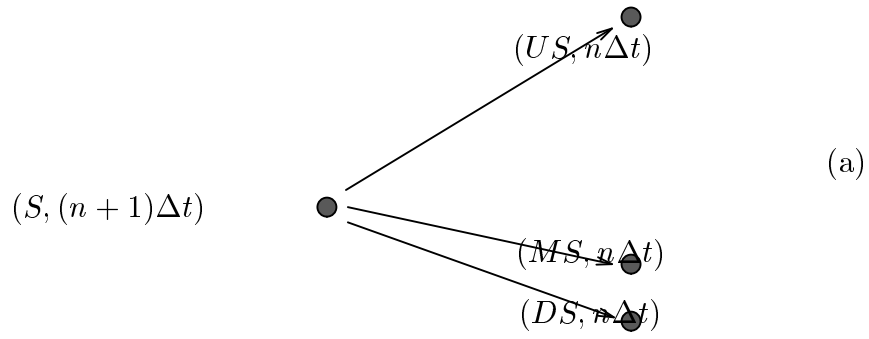


Figure 3.

- (a) A node on the trinomial tree and its branches.
- (b) Convex case: The full line represents the linear interpolation between the values at the three outcomes. The dashed line represents the value of the optimal dominating portfolio at time $n\Delta t$.
- (c) Concave case(s): The full line is the linear interpolation. The dashed line represents the value of the optimal dominating portfolio at time $n\Delta t$ for the case $R < M$, and the dotted line the value of the optimal dominating portfolio at time $n\Delta t$ in the case $R > M$. To see this, consider the present values of the three outcomes, i.e. we discount them by R . The optimal portfolio depends on the relation between $\frac{M}{R}S$ and S .

5. Convergence to the solution of the Black-Scholes-Barenblatt equation

We assume henceforth that $0 < \sigma_{\min} \leq \sigma_{\max}$.

Theorem 1 *Assume that F is continuous and that $F(S) = AS + B$ for some constants A and B and S large enough. Let $V_N(S, n\Delta t)$ be defined as in section 4, (39) with initial condition $V_N(S, 0) = F(S)$. Here, the subscript N is used to indicate the number of periods of the tree. Let $W(S, t)$ be the solution to the BSB equation (16) with the same initial conditions. Then*

$$\lim_{N \rightarrow \infty} \max_{(S,t)} |V_N(S, t) - W(S, t)| = 0.$$

The proof requires the use of regularity properties of the solutions to the BSB equation (see for, for instance Krylov [10] and Kamin *et al* [9]). If $F(S)$ is continuous and has linear growth as $S \nearrow \infty$ then the BSB equation admits a unique solution. Furthermore, if we assume in addition that F' and F'' are uniformly continuous and F is linear at infinity, it can be shown that $\frac{\partial W}{\partial t}$, $\frac{\partial W}{\partial S}$, $\frac{\partial^2 W}{\partial S^2}$ are uniformly continuous, bounded functions and that $S^2 \frac{\partial^2 W}{\partial S^2}$ converges to zero as $S \nearrow +\infty$ uniformly in t for $0 \leq t \leq T$. In the course of the proof, we shall assume that F satisfies these extra regularity assumptions. This entails no loss of generality, since an arbitrary continuous payoff $F(S)$ which is linear at infinity can be approximated uniformly by a sequence of smooth payoffs $F_n(S)$ satisfying the above conditions.

To simplify notation, we shall omit the superscript N when writing the value function. We shall also assume, without loss of generality, that $M < R$.⁵ In this case the operator $L_{\text{conc}}^{M>R}$ does not appear in the recursion relation (39). Therefore, we can use the simpler notation $L_{\text{conc}} \equiv L_{\text{conc}}^{M \leq R}$. Equation (39) reduces to

$$V(S, (n+1)\Delta t) = \max \{ L_{\text{conv}}[V](S, n\Delta t), L_{\text{conc}}[V](S, n\Delta t) \}. \quad (40)$$

The following lemma gives an estimate of the truncation error of the difference scheme (40), seen as an discrete approximation to the the BSB equation.

Lemma 2.

$$W(S, (n+1)\Delta t) - \max \{ L_{\text{conv}}[W](S, n\Delta t), L_{\text{conc}}[W](S, n\Delta t) \} = o(\Delta t), \quad (41)$$

where $o(\Delta t)$ is a quantity that depends on F and on u, m, d and tends to zero faster than Δt .

The proof of this lemma is given in the Appendix. As a corollary of this lemma, we obtain the following error estimates:

⁵ This amounts to assuming that $m < 0$. The case $M > R$ is analogous and corresponds to $m > 0$. We shall not treat the case $m = 0$, when the lower volatility σ_{\min} vanishes and the BSB equation becomes degenerate.

Corollary 3 . *The functions $W(S, n\Delta t)$ and $V(S, n\Delta t)$ satisfy the inequalities*

$$W(S, (n+1)\Delta t) - V(S, (n+1)\Delta t) \leq \max \left\{ \max_{(S, n\Delta t)} \left\{ W(S, n\Delta t) - V(S, n\Delta t) \right\}, 0 \right\} + o(\Delta t)$$

and

$$W(S, (n+1)\Delta t) - V(S, (n+1)\Delta t) \geq \min \left\{ \min_{(S, n\Delta t)} \left\{ W(S, n\Delta t) - V(S, n\Delta t) \right\}, 0 \right\} + o(\Delta t).$$

Proof. From the Lemma, we know that at an arbitrary node $(S, (n+1)\Delta t)$ we have

$$\begin{aligned} W(S, (n+1)\Delta t) - V(S, (n+1)\Delta t) &= \max \left\{ L_{\text{conv}}[W](S, n\Delta t), L_{\text{conc}}[W](S, n\Delta t) \right\} \\ &\quad - \max \left\{ L_{\text{conv}}[V](S, n\Delta t), L_{\text{conc}}[V](S, n\Delta t) \right\} + o(\Delta t). \end{aligned}$$

Recalling that $L_{\text{conv}}, L_{\text{conc}}$ are linear operators, it follows that ⁶

$$\begin{aligned} W(S, (n+1)\Delta t) - V(S, (n+1)\Delta t) &\leq \max \left\{ L_{\text{conv}}[W - V](S, n\Delta t), L_{\text{conc}}[W - V](S, n\Delta t) \right\} + o(\Delta t), \\ W(S, (n+1)\Delta t) - V(S, (n+1)\Delta t) &\geq \min \left\{ L_{\text{conv}}[W - V](S, n\Delta t), L_{\text{conc}}[W - V](S, n\Delta t) \right\} + o(\Delta t). \end{aligned}$$

Finally, since the operators $L_{\text{conv}}, L_{\text{conc}}$ are convex combinations of values of the function in their argument discounted by $\frac{1}{R} \leq 1$, we have

$$\frac{1}{R} \min_{(S, n\Delta t)} \left\{ W(S, n\Delta t) - V(S, n\Delta t) \right\} \leq L_{\text{conv}}[W - V](S, n\Delta t)$$

and

$$L_{\text{conv}}[W - V](S, n\Delta t) \leq \frac{1}{R} \max_{(S, n\Delta t)} \left\{ W(S, n\Delta t) - V(S, n\Delta t) \right\},$$

with similar inequalities for L_{conc} . The Corollary is established by substituting these estimates into the previous inequalities.

We are in a position to conclude the proof of Theorem 1.

⁶ Given arbitrary constants a, b, c, d , the inequalities $\max\{a, b\} - \max\{c, d\} \leq \max\{a - c, b - d\}$ and $\max\{a, b\} - \max\{c, d\} \geq \min\{a - c, b - d\}$ always hold.

Consider an arbitrary node $(S, (n + 1)\Delta t)$ in the tree. From Corollary 3, by induction on n , we have

$$\begin{aligned} \min \left\{ \min_{(\hat{S}, 0) \in T_{\Delta t_N}} \left\{ W(\hat{S}, 0) - V(\hat{S}, 0) \right\}, 0 \right\} + o(\Delta t)n \\ \leq W(S, (n + 1)\Delta t) - V(S, (n + 1)\Delta t) \\ \leq \max \left\{ \max_{(\hat{S}, 0) \in T_{\Delta t_N}} \left\{ W(\hat{S}, 0) - V(\hat{S}, 0) \right\}, 0 \right\} + o(\Delta t)n . \end{aligned}$$

Since W and V agree at the time level zero, and since $0 \leq n \leq \frac{T}{\Delta t}$, $|o(\Delta t)n| \leq |\frac{o(\Delta t)}{\Delta t}T| \rightarrow 0$ as Δt tends to zero, we conclude that $V(S, t) = V_N(S, t)$ converges uniformly to $W(S, t)$ as $N \rightarrow +\infty$.

This theorem characterizes the optimal value function for *dominating* strategies. The corresponding result for the *dominated* strategies, which gives the optimal bid price, can be obtained as a corollary. In fact, if $\tilde{V}(S, t)$ represents the value of the optimal dominated strategy for the payoff $F(S)$, it can easily be shown that $-\tilde{V}(S, t)$ is the value of the optimal dominating strategy for the payoff $-F(S)$. It follows from this that $\tilde{V}(S, t)$ is a solution of the *BSB* equation with payoff $F(S)$ and nonlinear volatility

$$\sigma[\Gamma] = \begin{cases} \sigma_{\max} & \text{if } \Gamma \leq 0, \\ \sigma_{\min} & \text{if } \Gamma > 0. \end{cases}$$

Thus, the equation for the bid price is obtained by interchanging σ_{\min} and σ_{\max} .

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Appendix: Proof of Lemma 2.

The proof of Lemma 2 consists in analyzing the Taylor expansion for the solution of the *BSB* equation, $W(S, t)$, in order to evaluate the operators L_{conv} and L_{conc} applied to its “discretization” on the trinomial tree.

Expanding W in a Taylor series we have

$$W(\alpha S, t) = W(MS, t) + \frac{\partial W}{\partial S}(MS, t)(\alpha - M)S + \frac{1}{2} \frac{\partial^2 W}{\partial S^2}((M + \theta)S, t)(\alpha - M)^2 S^2$$

Where θ is between 0 and $\alpha - M$. Noting that under our assumptions⁷, $\frac{\partial^2 W}{\partial S^2}$ is uniformly continuous, and $S^2 \frac{\partial^2 W}{\partial S^2} \rightarrow 0$ as $S \nearrow +\infty$, it follows that

$$W(\alpha S, t) = W(MS, t) + \frac{\partial W}{\partial S}(MS, t)(\alpha - M)S + \frac{1}{2} \frac{\partial^2 W}{\partial S^2}((MS, t)(\alpha - M)^2 S^2 + o((\alpha - M)^2)).$$

Thus

$$\begin{aligned} L_{conv}[W](S, n\Delta t) &= \frac{1}{R} \left[W(DS, n\Delta t) \frac{U - R}{U - D} + W(US, n\Delta t) \frac{R - D}{U - D} \right] \\ &= \frac{1}{R} \left\{ \left[W(MS, t) + \frac{\partial W}{\partial S}(MS, t)(D - M)S + \frac{1}{2} \frac{\partial^2 W}{\partial S^2}(MS, t)(D - M)^2 S^2 \right] \frac{U - R}{U - D} \right. \\ &\quad \left. + \left[W(MS, t) + \frac{\partial W}{\partial S}(MS, t)(U - M)S + \frac{1}{2} \frac{\partial^2 W}{\partial S^2}(MS, t)(U - M)^2 S^2 \right] \frac{R - D}{U - D} \right\} \\ &\quad + o((M - D)^2 + (U - M)^2) \\ &= \frac{1}{R} \left\{ W(MS, t) + S \frac{\partial W}{\partial S}(MS, t) \frac{(D - M)(U - R) + (U - M)(R - D)}{U - D} \right. \\ &\quad \left. + \frac{1}{2} S^2 \frac{\partial^2 W}{\partial S^2}(MS, t) \frac{(D - M)^2(U - R) + (U - M)^2(R - D)}{U - D} \right\} \\ &\quad + o((M - D)^2 + (U - M)^2) \\ &= \frac{1}{R} \left\{ W(MS, t) + S \frac{\partial W}{\partial S}(MS, t)(R - M) \right. \\ &\quad \left. + \frac{1}{2} S^2 \frac{\partial^2 W}{\partial S^2}(MS, t)[M^2 + (U + D - 2M)R - UD] \right\} \\ &\quad + o((M - D)^2 + (U - M)^2). \end{aligned} \tag{A.1}$$

⁷ See discussion after the statement of Theorem 1.

Now, from the expansion of (18),(19) in Taylor series we have

$$\begin{aligned}
\frac{1}{R} &= 1 - r\Delta t + o(\Delta t), \\
1 - \frac{M}{R} &= -m\sqrt{\Delta t} - \mu\Delta t + r\Delta t, \\
(D + U - 2M)^2 &= (u + d - 2m)^2 \Delta t + o(\Delta t), \\
\frac{M^2}{R} + (U + D - 2M) - \frac{UD}{R} &= 1 - r\Delta t + 2\mu\Delta t + 2m\sqrt{\Delta t} + m^2 \Delta t + (u + d - 2m)\sqrt{\Delta t} \\
&\quad - (1 + u\sqrt{\Delta t} + \mu\Delta t)(1 + d\sqrt{\Delta t} + \mu\Delta t)(1 - r\Delta t) + o(\Delta t) \\
&= (-du)\Delta t + m^2 \Delta t + o(\Delta t).
\end{aligned}$$

Substituting these relations (28) and (18) into (A.1) we obtain

$$\begin{aligned}
&L_{\text{conv}}[W](S, n\Delta t) \\
&= W(MS, t) + r \left[S \frac{\partial W}{\partial S}(MS, t) - W(MS, t) \right] \Delta t - (m\sqrt{\Delta t} + \mu\Delta t) S \frac{\partial W}{\partial S}(MS, t) \\
&\quad + \frac{1}{2} \sigma_{\text{max}}^2 S^2 \frac{\partial^2 W}{\partial S^2}(MS, t) \Delta t + \frac{1}{2} m^2 S^2 \frac{\partial^2 W}{\partial S^2}(MS, t) \Delta t + o(\Delta t) \\
&= W(MS, t) + r \left[MS \frac{\partial W}{\partial S}(MS, t) - W(MS, t) \right] \Delta t + \frac{1}{2} \sigma_{\text{max}}^2 (MS)^2 \frac{\partial^2 W}{\partial S^2}(MS, t) \Delta t \\
&\quad - (m\sqrt{\Delta t} + \mu\Delta t) S \frac{\partial W}{\partial S}(MS, t) - r(M-1) S \frac{\partial W}{\partial S}(MS, t) \Delta t + \frac{1}{2} m^2 S^2 \frac{\partial^2 W}{\partial S^2}(MS, t) \Delta t \\
&\quad - \frac{1}{2} \sigma_{\text{max}}^2 (M^2 - 1) S^2 \frac{\partial^2 W}{\partial S^2}(MS, t) \Delta t + o(\Delta t) \tag{A.2} \\
&= W(MS, t) + r \left[MS \frac{\partial W}{\partial S}(MS, t) - W(MS, t) \right] \Delta t + \frac{1}{2} \sigma_{\text{max}}^2 (MS)^2 \frac{\partial^2 W}{\partial S^2}(MS, t) \Delta t \\
&\quad - (m\sqrt{\Delta t} + \mu\Delta t) S \frac{\partial W}{\partial S}(MS, t) + \frac{1}{2} m^2 S^2 \frac{\partial^2 W}{\partial S^2}(MS, t) \Delta t + o(\Delta t).
\end{aligned}$$

By Taylor expansion, we have

$$\begin{aligned}
W(S, t + \Delta t) &= W(MS, t) + \frac{\partial W}{\partial t}(MS, t) \Delta t + \frac{\partial W}{\partial S}(MS, t) (1 - M) S \\
&\quad + \frac{1}{2} \frac{\partial^2 W}{\partial S^2}(MS, t) (1 - M)^2 S^2 + o(\Delta t + (1 - M)^2) \\
&= W(MS, t) + \frac{\partial W}{\partial t}(MS, t) \Delta t - S \frac{\partial W}{\partial S}(MS, t) (m\sqrt{\Delta t} + \mu\Delta t) \\
&\quad + \frac{1}{2} m^2 S^2 \frac{\partial^2 W}{\partial S^2}(MS, t) \Delta t + o(\Delta t). \tag{A.3}
\end{aligned}$$

It thus follows from (16),(A.2) and (A.3) that

$$\begin{aligned} W(S, t + \Delta t) - L_{\text{conv}}[W](S, n\Delta t) \\ = \frac{1}{2} \left[\sigma^2 \left(\frac{\partial^2 W}{\partial S^2}(MS, t) \right) - \sigma_{\text{max}}^2 \right] (MS)^2 \frac{\partial^2 W}{\partial S^2}(MS, t) \Delta t + o(\Delta t). \end{aligned} \quad (\text{A.4})$$

A similar analysis done for the operator L_{conc} , would yield

$$\begin{aligned} W(S, t + \Delta t) - L_{\text{conc}}[W](S, n\Delta t) \\ = \frac{1}{2} \left[\sigma^2 \left(\frac{\partial^2 W}{\partial S^2}(MS, t) \right) - \sigma_{\text{min}}^2 \right] (MS)^2 \frac{\partial^2 W}{\partial S^2}(MS, t) \Delta t + o(\Delta t). \end{aligned} \quad (\text{A.5})$$

Note that by the definition of σ (see (17)) and since $\sigma_{\text{min}}^2 \leq \sigma_{\text{max}}^2$, we have that

$$\begin{aligned} \min \left\{ \left[\sigma^2 \left(\frac{\partial^2 W}{\partial S^2}(MS, t) \right) - \sigma_{\text{max}}^2 \right] (MS)^2 \frac{\partial^2 W}{\partial S^2}(MS, t), \right. \\ \left. \left[\sigma^2 \left(\frac{\partial^2 W}{\partial S^2}(MS, t) \right) - \sigma_{\text{min}}^2 \right] (MS)^2 \frac{\partial^2 W}{\partial S^2}(MS, t) \right\} = 0. \end{aligned}$$

From this and equations (A.4) and (A.5) the Lemma follows.