REFINEMENTS OF THE BINOMIAL MODEL AND APPLICATIONS

The binomial model discussed in Chapter 2 used two input parameters: the interest rate and the volatility. Until now, we assumed implicitly that these parameters were constant. In this lecture, we remove these assumptions, allowing for variations of these parameters with time.

A *term-structure of interest rates* is introduced to model a (more realistic) economy in which deposit rates can vary with the duration of loans. We will also study time-inhomogeneity of the volatility process by introducing a *term-structure of volatilities*. Time-dependent volatilities are useful to incorporate into the pricing model the market's expectations about risk across time. Information about the temporal behavior of volatility is contained in the prices of liquid option instruments with different maturities written on a given underlying asset.

We will also discuss refinements of the binomial model that will permit us to extend the theory to several derivative securities of practical interest. These include derivatives contingent on underlying assets that pay dividends, options on futures and "structured" derivative instruments providing a stream of uncertain cash-flows across time.

1. Term-structure of interest rates

We incorporate into the model different interest lending rates for different trading periods. Usually, interest rates are not constant in time. For example, the following table gives market for interbank dollar deposits on August 23, 1995:[1]

maturity	bid	offer
1 month	5.8700	6.0000
2 months	5.7800	5.9000
3 months	5.7800	5.9000
6 months	5.8100	5.9500
9 months	5.8100	5.9300

Implied forward interest rates can be obtained from such a "strip" of deposit rates, or from the markets in Eurodollar futures or Treasury bill futures. For instance, the December 1997 Eurodollar 90-day futures contract gives the expected

[1] Unless otherwise specified, interest rates are quoted in "bond-equivalent", or continuously compounded form.

London Interbank Offered Rate (LIBOR) for the period of January 1998 through March 1998, the March 1998 contract gives the expected 90-day LIBOR from April 1998 through June 1998, and so forth. These values can then be input in the model at different time periods[2].

Example 1. The table given above can be used to obtain a 1-month interest rate, a 1-month forward interest rate for a loan starting in one month, a 1-month forward interest rate for a loan starting in two months, a 3-month interest rate for a loan starting in 3 months and a 3-month interest rate for a loan starting in 6 months. For instance, to compute the three-month interest rate for *lending* 6 months from now, $r^l_{6,9}$, we note that

- borrowing \$ 1 for 9 months
- lending $e^{-r^l_{6,9}\,0.25}$ dollars from month 6 to month 9
- lending $e^{-r^l_{0,6}\,0.50}\,e^{-r^l_{6,9}\,0.25}$ dollars from month 0 to month 6

results in a cash-flow of zero dollars 9 months from today. For borrowing for 9 months we take the 9 months offer rate, 5.93 and for lending for 6 months we take the bid rate 5.81. Therefore, the effective rate for lending from 6 to 9 months satisfies

$$5.93 \times 0.75 = 5.81 \times 0.50 + r^l_{6,9} \times 0.25$$

which gives an *offer* rate of 6.17 %. To calculate the effective rate for borrowing over the same period, we observe that

- lending \$ 1 for 9 months
- borrowing $e^{-r^b_{6,9}\,0.25}$ from month 6 to month 9
- borrowing $e^{-r^b_{0,6}\,0.50}\,e^{-r^b_{6,9}\,0.25}$

results in a cash-flow of zero dollars 9 months from today. Hence,

$$5.81 \times 0.75 = 5.95 \times 0.50 + r^b_{6,9} \times 0.25\,,$$

which gives an effective *bid* rate of 5.53 %.

In the simple models considered hereafter, differences between bid and offer prices will not be taken into account. Instead, we will consider the "riskless rate" to be the average between bid and offer rates. If we follow this rule, the equation for the effective 6-to-9 rate is

$$5.87 \times 0.75 = 5.88 \times 0.50 + r_{6,9} \times 0.25\,,$$

[2] Modeling the future interest rates as the forward rates implied by a strip of interest rate futures or deposit rates is not entirely correct for pricing and hedging derivative securities. The reason is that forward rates are just *expected* future rates whereas the future short-term rates are not predictable in advance. However, this " forward rate approximation" is extremely useful to obtain a first-order approximation to varying interest-rate environments.

which gives an effective *bid* rate of 5.85 %. This is just the average between the bid and offer rates obtained above.

Suppose that we have, as in §II.1, a binomial model with N trading periods. We consider a sequence of interest rates for the N periods quoted on a continuously-compounded (bond-equivalent) basis:

$$r_0 \ , \ r_1 \ , \ r_2 \ , \ \ldots . , \ r_{N-1} \ . \tag{1}$$

(This sequence sequence could have been obtained from the procedure outlined above or otherwise.) The interest rates $R_n \ , \ 0 \leq n \leq N-1$ for the different periods are[3]

$$R_n = e^{r_n \, dt} - 1 \ , \tag{2}$$

where $\Delta t = T \ / \ N$ represents the duration of each period. We wish to incorporate the term-structure of interest rates (1) assuming, for simplicity, that the *local volatility* – the standard deviation of the yield over a single period – remains constant through time. This can be done by defining, for each n, the parameters

$$U_n = e^{r_n \, dt} \cdot U'$$

$$D_n = e^{r_n \, dt} \cdot D' \tag{3}$$

An arbitrage-free measure on the space of price paths is defined by setting

$$S_{n+1} = H_{n+1} \, S_n \quad , \quad \text{for } 0 \leq n \leq N-1 \ ,$$

where the random variables H_n are independent and satisfy

$$\text{Prob.}\{ \, H_{n+1} = U_n \, \} = P_U \quad , \quad \text{Prob.}\{ \, H_{n+1} = D_n \, \} = P_D \ .$$

Here, P_U and P_D are the probabilities

$$P_U = \frac{1 + R_n - D_n}{U_n - D_n} = \frac{1 - D'}{U' - D'}$$

and

$$P_D = \frac{U_n - 1 - R_n}{U_n - D_n} = \frac{U' - 1}{U' - D'} \tag{4}$$

[3] We make the convention that r_n is the interest rate that applies to the $(n+1)^{st}$ period.

The reader will recognize here the usual arbitrage-free probabilities for the one-period model. Note that P_U and P_D are independent of n. In particular, the **single-period** or **local** volatility is also independent of n and is given by

$$\sigma_{loc}^2 \;=\; \ln\left(\frac{U'}{D'}\right)\; P_U\; P_D\;.$$

The annual volatility is therefore

$$\sigma^2 \;=\; \frac{1}{dt}\,\ln\left(\frac{U'}{D'}\right)\; P_U\; P_D\;.$$

The analysis of the model is very similar to the case treated in S II. In particular, the probabilities P_U and P_D are given by

$$P_U \;=\; \frac{1}{2}\;[\;1\;\pm\;\sqrt{1\;-\;\frac{\sigma^2}{\rho^2}}\;]$$

and

$$P_D \;=\; \frac{1}{2}\;[\;1\;\mp\;\sqrt{1\;-\;\frac{\sigma^2}{\rho^2}}\;]\;,$$

where $\rho \;=\; \frac{1}{2\sqrt{dt}}\,\ln\,(U'/D')$. The parameters U' and D' are given by

$$U' \;=\; \frac{e^{\rho\,\sqrt{dt}}}{P_D\; e^{-\rho\,\sqrt{dt}}\;+\;P_U\; e^{\rho\,\sqrt{dt}}}$$

and

$$D' \;=\; \frac{e^{-\;\rho\,\sqrt{dt}}}{P_D\; e^{-\rho\,\sqrt{dt}}\;+\;P_U\; e^{\rho\,\sqrt{dt}}}\;.$$

Thus P_U , P_D , U' and D' are independent of n.

Using equation (25) in §II we find that the mean of the (annualized) yield for the $(n+1)^{st}$ period is given by

$$\mu_n \;=\; r_n \;+\; \frac{\rho}{\sqrt{dt}}\cdot\;(\;P_U\;-\;P_D\;)\;-\;\frac{1}{dt}\,\ln\left[\;P_D\; e^{-\rho\,\sqrt{dt}}\;+\;P_U\; e^{\rho\,\sqrt{dt}}\;\right] \qquad (6)$$

The average yield over the entire time period of T years, annualized, is therefore

$$\mu \;=\; \frac{1}{N}\sum_{n=0}^{N-1} r_n \;+\; \frac{\rho}{\sqrt{dt}}\cdot\;(\;P_U\;-\;P_D\;)\;-\;\frac{1}{dt}\,\ln\left[\;P_D\; e^{-\rho\,\sqrt{dt}}\;+\;P_U\; e^{\rho\,\sqrt{dt}}\;\right]\;.$$

Remark. Notice that the trajectories followed by the price of the underlying asset form a *recombining* tree, in the sense that there are only n nodes at time n, just like with the tree with constant U , D. Thus, a path that originates from a given point and goes first "up" and then "down" arrives at the same location after two time steps than a path that goes first "down" and then "up". In fact, we have

$$U_n \, D_{n+1} \;=\; e^{r_n \, dt} \, U' \cdot \, e^{r_{n+1} \, dt} \, D' \;=\; D_n \, U_{n+1} \; .$$

The only difference between the variable and the constant interest-rate models is that in the former the slopes of the trajectories vary according to the time period.

From this remark, it follows that the value of the underlying asset at node $(n\ ,\ j)$ is given by

$$S_n^j \;=\; S_0^0 \; e^{\sum\limits_{k=0}^{n-1} r_k \; dt} \cdot (U')^j \; (D')^{N-j} \; . \tag{7}$$

The recursion relation for the price of any derivative security contingent on the risky asset is

$$\begin{aligned} V_n^j \;&=\; \frac{1}{1 \;+\; R_n} \left[\, P_U \; V_{n+1}^{j+1} \;+\; P_D \; V_{n+1}^j \, \right] \\ &=\; e^{-r_n \; dt} \left[\, P_U \; V_{n+1}^{j+1} \;+\; P_D \; V_{n+1}^j \, \right] \end{aligned}$$

Since the risk-neutral probabilities P_U , P_D are independent of the term-structure of interest rates, the valuation of derivative products using this model is particularly simple.

We observe from (7) that

$$\begin{aligned} S_N^j \;&=\; S_0^0 \; e^{\sum\limits_{k=0}^{N-1} r_k \; dt} \cdot (U')^j \; (D')^{N-j} \\ &=\; S_0^0 \; e^{\left(\frac{1}{N} \sum\limits_{k=0}^{N-1} r_k \right) \; T} \cdot (U')^j \; (D')^{N-j} \end{aligned} \tag{8}$$

We conclude that the model prices European-style options and derivative securities expiring after N periods exactly like the binomial model of §II with an effective, constant interest rate given by

$$\overline{r} \;=\; \frac{\sum\limits_{n=0}^{N-1} r_n}{N} \quad , \tag{9}$$

which represents the yield for riskless lending over the duration of the contract.

It is important to notice that the values of derivative assets and the corresponding hedge-ratios predicted by the recursion for *intermediate* time periods, i.e. for n between 1 and N, are different than the ones that would be obtained using a binomial tree with constant rate $\overline{r}$. This is because the recursion relation takes into account the fact that interest rates vary over the duration of the contract. Using the same average rate $\overline{r}$ of (9) in the recursion relation instead of r_n will give incorrect prices and hedge-ratios for $1 \leq n < N$.

For $dt \ll 1$, we can use the lognormal approximation to the binomial model. For this purpose, it is convenient to introduce a *function* $r(t)$ to model variability of interest rates. This function is connected to the discrete rates r_n through the formula

$$r_n = r(n\,dt) \ .$$

The effective average rate is then

$$\overline{r} \approx \frac{1}{T} \int_0^T r(s)\,ds \ .$$

It is easy to see from (7) that the mean yield satisfies

$$\lim_{dt \to 0} \mu = \overline{r} - \frac{1}{2}\sigma^2$$

Hence, under the risk-neutral probability, the price of the underlying asset satisfies

$$\begin{aligned} S_T &= S_0\, e^{\sigma\sqrt{T}\,Z + \left(\frac{1}{T}\int_0^T r(s)\,ds - \frac{1}{2}\sigma^2 \right) T} \\ &= S_0\, e^{\sigma\sqrt{T}\,Z + \left(\overline{r} - \frac{1}{2}\sigma^2 \right) T} , \end{aligned}$$

where Z is Normal with mean 0 and variance 1. In particular, the Black-Scholes formula for option pricing holds, with r replaced by $\overline{r}$.

2. Term-structure of volatility

First, we need to state precisely what we mean by a term-structure of volatilities. In §II, we defined volatility as the standard deviation of the annual yield of the underlying asset under the no-arbitrage pricing measure. Since we also assumed that the statistics for price shocks were essentially the same at all nodes (up to a multiplicative factor associated with changing interest rates), the volatility is completely determined from the variance of the price shock over a single period. The latter quantity is what we call the *local volatility.* We can generalize the model by assuming that the local volatility is time-dependent. A *term-structure of volatilities* then consists in a specification of the local volatilities of the underlying asset over the different trading periods.

To illustrate how term-structures of volatilities arise consider the following example, drawn from the Dollar/Deutschemark options market on August 23, 1995. On that date, option market-makers (usually banks dealing over-the-counter) were trading options using the following volatility table:

maturity	bid	offer
1 month	.1390	.1425
2 months	.1365	.1390
3 months	.1330	.1360
6 months	.1300	.1320
9 months	.1290	.1310

The meaning of this table is the following: the bid/offer prices for at-the-money options on USD/DEM on that day were computed using the Black-Scholes formula with the above volatilities. One way to derive a local volatility structure is to "strip" this data (commonly referred as a "curve of implied volatilities"), similarly to what we did earlier for interest rates. The procedure will be explained in detail later.

A term-structure of volatilities can be specified as a sequence of parameters

$$\sigma_0 \ , \ \sigma_1 \ , \ \sigma_2 \ , \ \ldots \ \sigma_{N-1}$$

corresponding to the annualized standard deviation of the yield for each period. Thus,

$$\sigma_n^2 \ = \ \frac{1}{dt} \ \mathrm{Var}\left[\ \ln\left(\frac{S_{n+1}}{S_n}\right) \ \right] \tag{10}$$

for $0 \ \leq \ n \ \leq N-1$.

We want to construct a simple binomial model which is arbitrage-free and consistent with given term-structure of volatilities and interest rates. It is also convenient that the resulting trajectories for the price of the underlying asset form a "recombining" tree, so that derivative asset prices can be obtained by solving simple recursive relations. The problem consists in specifying parameters U_n , D_n , $P_U^{(n)}$, $P_D^{(n)}$, $n = 0$, 1, , ... $N-1$ so that

$$1 \ + \ R_n \ = \ P_U^{(n)} \ U_n \ + \ P_D^{(n)} \ D_n \ , \tag{11}$$

with

$$P_U^{(n)} \;=\; \frac{1 \;+\; R_n \;-\; D_n}{U_n \;-\; D_n}$$

$$P_D^{(n)} \;=\; \frac{U_n \;-\; 1 \;-\; R_n}{U_n \;-\; D_n}\;, \tag{12}$$

and, in addition,

$$dt\;\sigma_n^2 \;=\; \Big[\;\ln\Big(\frac{U_n}{D_n}\Big)\;\Big]^2 \cdot P_U^{(n)} \cdot\; P_D^{(n)} \tag{13}$$

for all n. Finally, we need to impose the conditions

$$U_n \cdot\; D_{n+1} \;=\; D_n \cdot\; U_{n+1} \tag{14}$$

so as to have a recombining tree.

The no-arbitrage condition (11) is immediately satisfied if we set

$$U_n \;=\; e^{r_n\;dt}\;U'_n\;, \qquad D_n \;=\; e^{r_n\;dt}\;D'_n\;,$$

$$P_U^{(n)} \;=\; \frac{1\;-\;D'_n}{U'_n\;-\;D'_n} \qquad \text{and} \qquad P_D^{(n)} \;=\; \frac{U'_n\;-\;1}{U'_n\;-\;D'_n}\;. \tag{15}$$

Furthermore, condition (14) is equivalent to to

$$\frac{U'_n}{D'_n} \;=\; \lambda \tag{16}$$

where $\lambda \;>\; 1$ is a constant. Therefore, matching the term-structure of volatilities requires finding λ , $P_U^{(n)}$ and $P_D^{(n)}$ such that

$$\Delta t\;\sigma_n^2 \;=\; \big(\ln\lambda\big)^2 \cdot P_U^{(n)} \cdot\; P_D^{(n)}\;, \qquad n = 1, ..., N\;. \tag{17}$$

Observe that the right-hand side of (18) reaches a maximum when the probabilities are equal. Thus, a necessary condition for the existence of solutions to the N equations corresponding to (17) is

$$dt\;\sigma_n^2 \;\leq\; \frac{1}{4}\,(\ln\lambda)^2$$

for all n, or

$$2\sqrt{dt}\ \operatorname*{Max}_n\ \sigma_n\ \leq\ \ln\lambda\ .$$

Thus, if we set

$$\sigma_{max}\ =\ \operatorname*{Max}_n\ \sigma_n\ ,$$

an admissible λ should therefore have the form

$$\lambda\ =\ e^{\ 2\rho\ \sqrt{dt}}\ .$$

with $\rho\ \geq\ \sigma_{max}$. Substituting into (16) and solving for the probabilities, we obtain

$$P_U^{(n)}\ =\ \frac{1}{2}\ \left[\ 1\ \pm\ \sqrt{1\ -\ \frac{\sigma_n^2}{\rho^2}}\ \right]$$

$$P_D^{(n)}\ =\ \frac{1}{2}\ \left[\ 1\ \mp\ \sqrt{1\ -\ \frac{\sigma_n^2}{\rho^2}}\ \right] \tag{18}$$

The parameters U'_n and D'_n can then be obtained from (14). They are given by

$$U'_n\ =\ \frac{e^{\rho\sqrt{dt}}}{P_D^{(n)}\ e^{-\rho\sqrt{dt}}\ +\ P_U^{(n)}\ e^{\rho\sqrt{dt}}}$$

and

$$D'_n\ =\ \frac{e^{-\ \rho\sqrt{dt}}}{P_D^{(n)}\ e^{-\rho\sqrt{dt}}\ +\ P_U^{(n)}\ e^{\rho\sqrt{dt}}}\ . \tag{19}$$

Notice that the risk-neutral probabilities are now time-dependent. The prices of the underlying asset at the different nodes in the tree are, according to the model,

$$S_n^j\ =\ S_0^0\cdot\ e^{\sum_{k=0}^{n-1} r_k\ dt}\cdot\frac{(e^{\rho\ \sqrt{dt}}\ )^j\ (\ e^{-\rho\ \sqrt{dt}}\ )^{n-j}}{\prod_{k=0}^{n-1}\ (\ P_D^{(k)}\ e^{-\rho\sqrt{dt}}\ +\ P_U^{(k)}\ e^{\rho\sqrt{dt}}\ )}\ . \tag{20}$$

Finally, the equation that gives the value of a European-style derivative security with payoff $F(S)$ maturing after N periods is

$$\begin{cases} V_n^j\ =\ e^{-r_n\ dt}\ [\ P_U^{(n)}\ V_{n+1}^{j+1}\ +\ P_D^{(n)}\ V_{n+1}^j\ ] & ,\quad n=0,\ldots N-1 \\ \\ V_N^j\ =\ F(\ S_N^j\ )\ . & \end{cases} \tag{21}$$

3. Deriving a volatility term-structure from option market data

Volatility is the most important variable in option pricing. Many methods have been porposed to "calibrate" the volatility variable in pricing models (this is an indication that there is no "correct" way of doing this!). Historically, there are two paradigms for volatility estimation : using *historical volatility* and using *implied volatility.*

Definition 1. *Historical volatility is defined as an estimate of the variance of the logarithm of the price of the underlying asset, obtained from past data.*

Definition 2. *Implied volatility is the numerical value of the volatility parameter that makes the market price of an option equal to the Black-Scholes value.*

The use of historical volatility estimates requires the construction of appropriate statistical estimators. One of the main problems in this regard is to select the sample size, or window of observations, which will be used to estimate σ (6 months of previous data, 3 months, 1 month, etc.). Different time-windows tend to give different volatility values.

The problem with using historical volatility is that it assumes that future market preformance will be reflected in future option prices. Although this may be partially correct, such method will not survive a large "spike" in volatility such as the one which ocurred in October 1987, for example. Another argument against historical voaltility is that it does not incorporate arrivals of new information such as corporate mergers, sudden changes in exchange-rate policy (see Mexico circa December 1994), etc.

Implied volatility, on the other hand, is not a predictor of option prices. It is simply a way of quoting option prices in terms of a risk parameter. However, it is important to notice that implied volatility is a "forward looking" parameter. Therefore, one can say that it incorporates the market's expectations about prices of derivative prices or, more concisely, about *risk.* Measuring risk through the construction of appropriate ddiscounting probability measures is the name of the game in Financial Mathematics.

Example 2. Consider the following situation: Stock XYZ, is trading at \$ 100.00. A 183-day call option with strike price trades at \$ 9.32 (per share). The interest rate is estimated at 7% annually. The value of σ which makes \$ 9.32 the Black-Scholes price is $\sigma_{implied} = 0.16$ or 16% annual volatility. (Check with your Black-Scholes calculator).

It important to realize that the implied volatilities of options on the same underlying asset is *not* constant across strikes and maturities. At first, this seems like a serious "blow" to the theory, but what really happens is that the market assigns different risk-premia to different strikes and maturities. This does not mean necessarily that there exist arbitrages in the market, but instead that the way in which the market prices risk at different price levels and future dates is different.

One of the simplest ways that this information can be incorporated into a pricing model is through local volatility and a term-structure.

What is the relation between implied volatilities and the term structure? The answer is that, since shock prices are independent in the no-arbitrage world, the variance of the logarithm of the price after N periods is

$$\sigma^2\, T \;=\; \sum_{n=1}^{N} \sigma_n^2\, dt \; .$$

This equation allows us to use market data to calculate local volatilities that can be used in the pricing model. We outline the procedure using the data given on p.7. As a first step, we take the average between bid and offer implied volatilities. The result is

maturity	volatility
1 month	.1407
2 months	.1382
3 months	.1345
6 months	.1310
9 months	.1300

Notice is that the data is not given over time intervals of the same duration. To calculate the “forward- forward” volatility from month 1 to month 2, we can use the above equation. Hence

$$2/12 \times (.1382)^2 \;=\; 1/12 \times (.1407)^2 \;+\; 1/12 \times (\sigma_{1,2})^2 \; .$$

Straightforward arithmetic gives $\sigma_{1,2} \;=\; .1357$. The nest step is to compute the 2-to-3 months forward volatility. The corresponding equation is then

$$3/12 \times (.1345)^2 \;=\; 2/12 \times (.1382)^2 \;+\; 1/12 \times (\sigma_{2,3})^2 \; .$$

This gives $\sigma_{2,3} \;=\; 0.1268$. The 3-to-6 month volatility is found by solving

$$6/12 \times (.1310)^2 \;=\; 3/12 \times (.1345)^2 \;+\; 3/12 \times (\sigma_{3,6})^2 \; .$$

The result is $\sigma_{3,6} \;=\; 0.1274$. Finally, the equation for the 6-to-9 month volatility is

$$9/12 \times (.1300)^2 \;=\; 6/12 \times (.1310)^2 \;+\; 3/12 \times (\sigma_{6,9})^2 \; ,$$

which gives $\sigma_{6,9} = 0.1279$.

This calculation gives an approximate estimate of the annualized "forward-forward" volatilities. We can then set σ_n in the model equal to the appropriate value corresponding to the period under consideration.

4. Underlying assets that pay dividends

We consider the valuation of European-style derivative securities that depend on a dividend-paying asset, such as the stock of a company. The binomial model must be slightly modified to account for this feature.

We assume that there are N trading periods and that dividend payments are made always at the end of a trading period. The values of the stock before dividends are paid are denoted by $\hat{S}_n$ or $\hat{S}_n^j$, for $n = 0, ..., N$. This is the *ex-dividend* value. We denote the value of the stock at the end of the n^{th} period after a dividend payment — the *post-dividend* value — by S_n or S_n^j.

A natural assumption regarding dividend payments is that is that the payment after the n^{th} period is a fraction the ex-dividend value, say

$$D_n = \alpha_{n-1} \hat{S}_n , \tag{22}$$

where $0 \leq \alpha_n < 1$ for all $0 \leq n \leq N-1$. The fractions α_n are assumed to be known in advance and are also allowed to depend on time (to model, for instance, periods without dividend payments)[4]. Equation (22) gives a simple relation between the ex- and post-dividend values: since

$$\hat{S}_n = S_n + D_n = S_n + \alpha_{n-1} \hat{S}_n ,$$

we have

$$\hat{S}_n = \frac{1}{1 - \alpha_{n-1}} S_n \quad \text{or} \quad S_n = (1 - \alpha_{n-1}) \hat{S}_n . \tag{23}$$

We shall take as the basic variable in the model the post-dividend price of the stock, assuming that, given the price history until the en of period n, we have one of two possibilities for the price shock over the next period, namely

$$S_{n+1} = S_n U_n \quad \text{or} \quad S_n D_n .$$

[4]Similarly to the notations for interest rates and volatilities, we make the convention that α_n represents the fraction of the ex-dividend value paid after the $(n+1)^{st}$ period.

We will also impose the condition $U_n \; D_{n+1} \; = \; D_n \; U_{n+1} \; n \; = \; 0 \; , \; ..., \; N - 1$, so as to have a recombining tree. We must determine a probability on the set of paths that makes the model arbitrage free. We know that such probability be such that the present value of the stock is equal to its discounted future value, including dividends. In particular, from (23), we must have

$$\begin{aligned} S_n &= \frac{1}{1 + R_n} \cdot \mathrm{E}\left\{ S_{n+1} + D_{n+1} \, | S_n \right\} \\ &= \frac{1}{1 + R_n} \cdot \mathrm{E}\left\{ \hat{S}_{n+1} \, | S_n \right\} \\ &= \frac{1}{1 + R_n} \cdot \frac{1}{1 - \alpha_n} \cdot \mathrm{E}\left\{ S_{n+1} \mid S_n \right\} . \end{aligned} \tag{24}$$

Thus, the the post-dividend value is obtained by discounting the expectation of its future (post-dividend) values at a rate that depends on the riskless interest rate and the fraction of dividends paid. The conditional probabilities $P_U^{(n)}$ and $P_D^{(n)}$ corresponding to the expectation in (24) should therefore satisfy

$$\left\{ \begin{array}{l} P_U^{(n)} + P_D^{(n)} = 1 \\ \\ U_n \, P_U^{(n)} + D_n \, P_D^{(n)} = (1 + R_n)\,(1 - \alpha_n) \, . \end{array} \right. \tag{25}$$

To make a parallel with the no-dividend case, we introduce the term-structure of interest rates as in (1) and set

$$1 - \alpha_n = e^{-q_n \; dt} \, . \tag{26}$$

The constants q_n represent the annualized rate at which dividends accrue corresponding to the $(n+1)^{st}$ period. We can then rewrite the second equation in (25) as

$$U_n \, P_U^{(n)} + D_n \, P_D^{(n)} = e^{(r_n \, - \, q_n \,) \, dt} \, . \tag{27}$$

The calculation of the parameters U_n , D_n , $P_U^{(n)}$ and $P_D^{(n)}$ follows a route similar to the one of the two previous sections. We omit unnecessary details and state only the simplest result, corresponding to the case of constant local volatilities. Solving (25) and adjusting for volatility, we find that the post-dividend values of the stock at the different nodes are

$$S_n^j = S_0^0 \cdot e^{\sum\limits_{k=0}^{n-1} (r_k \, - \, q_k) \; dt} \cdot \frac{(e^{\rho \, \sqrt{dt}} \,)^j \; (\, e^{-\rho \, \sqrt{dt}} \,)^{n-j}}{\left(\, P_D \; e^{-\rho \sqrt{dt}} + P_U \; e^{\rho \sqrt{dt}} \, \right)^n} \, , \tag{28a}$$

where $\rho \ \geq \ \sigma$ is a parameter and where

$$P_U \ = \ \frac{1}{2} \left[\, 1 \ \pm \ \sqrt{1 \ - \ \frac{\sigma^2}{\rho^2}} \ \right] , \qquad P_D \ = \ 1 \ - \ P_U \ . \tag{28b}$$

The value of a European-style derivative security with payoff $F(S)$ maturing after N periods is then given by the familiar recursive relation

$$\begin{cases} V_n^j \ = \ e^{-r_n \ dt} \left[\, P_U \ V_{n+1}^{j+1} \ + \ P_D \ V_{n+1}^j \, \right] \\ \\ V_N^j \ = \ F(S_N^j) \ . \end{cases} \tag{29}$$

To get a better feeling for how dividends affect the pricing model, we observe that the expected (post-dividend) value of the stock under the no-arbitrage measure after the N periods is, from (28),

$$\mathrm{E} \ \{ \ S_N \ | \ S_0 \ = \ S_0^0 \ \} \ = \ S_0^0 \cdot e^{\sum\limits_{k=0}^{N-1} \ (r_k \ - \ q_k) \ \Delta t} \ .$$

Thus, the underlying variable (S_n) of the derivative security grows at a rate which is different from the one used to discount the value V_n in (29).

Dividend payments for the underlying asset can be therefore easily incorporated into the binomial pricing model. The pricing equations for European-style derivative securities are very similar to the case without dividend payments. However, if the derivative security can be exercised before its maturity date (as is the case for American-style options) the impact of dividend payments on the pricing equations is more substantial.

As *lognormal approximation* of the binomial model for dividend-paying underlying assets can be derived from the above considerations.[5] As with the case of interest rates, it is convenient to consider a *dividend function* $q(t)$ which interpolates between the discrete values, *viz.*,

$$q_n \ = \ q(n \, dt) \ \ , \ \ n \ = \ 0 \ldots , N - 1 \ .$$

The asymptotic value of the mean annual yield is obtained from equation (28a). The key observation is that $r_k - q_k$ appears as the "effective" interest rate in the post-dividend price (compare with equation (7). Therefore, in the lognormal approximation, the price of the underlying asset satisfies

[5] This approximation assumes however that dividends are pid out continuously. The continuous dividend approximation is used in the case of options on indices such as the S&P 500 and options on foreign currencies. In the latter case, the dividend rate is simply the foreign exchange rate.

$$S_T = S_0\, e^{\sigma\sqrt{T}\, Z \,+\, \left(\frac{1}{T}\int\limits_0^T (r(s)-q(s)\, ds -\, \frac{1}{2}\sigma^2\right) T}$$

$$= S_0\, e^{\sigma\sqrt{T}\, Z \,+\, \left(\overline{r}-\overline{q}-\frac{1}{2}\sigma^2\right) T} \,,$$

where

$$\overline{q} = \frac{1}{T}\int\limits_0^T q(s)\, ds \,.$$

In particular, the Black-Scholes formula can be extended to dividend-paying assets. The general valuation formula for European-style derivative securities is

$$V(S,T) = e^{-rT}\,\mathbf{E}\left\{ F\left(S\, e^{\sigma\sqrt{T}\, Z \,+\, (\overline{r}-\overline{q}-\frac{1}{2}\sigma^2)\, T}\right)\right\} \,,$$

where Z is a standardized normal. Notice that the difference with the previous results comes at the level of the risk-neutral average yield. The Black-Scholes formula for the value of a European call option on an asset with continuous dividend yield q is

$$C(S,K;T) = S\, e^{-qT}\, N(d_1) - K\, e^{-rT}\, N(d_2) \,,$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T}}\, ln\left(\frac{S\, e^{(r-q)\, T}}{K}\right) + \frac{1}{2}\sigma\sqrt{T} \,, \qquad d_2 = d_1 - \sigma\sqrt{T} \,.$$

The Delta of the option is now

$$\Delta(S,T) = e^{-qT}\, N(d_1).$$

Notice that the amount of shares of the underlying security is multiplied by the factor e^{-qT}. This is similar to what happens when hedging forward contracts with continuous dividend reinvestment.

5. Futures contracts as the underlying security

Many exchange-traded and OTC derivative securities are based on futures contracts. Examples include options on Eurodollar futures and on Treasury bond futures contracts. In this section, we will consider the problem of pricing a European-style derivative security, assuming that the underlying asset is an "ideal" futures contract.[6] In constructing no-arbitrage models for derivatives based on futures contracts, the cash-flow structure of the futures contract must be taken into account. As we shall see, the situation bears a strong similarity with the case of assets with continuous dividend payments.

Let F_n , $n = 0, 1, ..., N$, represent the sequence of (random) futures prices after the different trading periods. We shall assume that the contract is marked-to-market after each trading period, so that an investor with a long (resp. short) position in one contract at after the n^{th} period will be credited (resp. debited) the amount $F_{n+1} - F_n$ after the next period. We assume, as usual, that at each step the price follows a binary model with

$$F_{n+1} \;=\; F_n\, U_n \qquad \text{or} \qquad F_n\, D_n \ ,$$

where U_n and D_n are parameters such that the trajectories will form a recombining binomial tree.

Opening and closing positions can be done at zero cost (according to our definition of "ideal" contract). Hence, the futures contract can be viewed as a security that has zero market value and obliges its holder to receive or pay the cash-flows $F_{n+1} - F_n$, i.e. to mark-to-market. To determine a possible arbitrage-free probability measure for the random variables F_n, consider an investor which opens a long position in the futures contract after the n^{th} period. Since the position can be closed at any time, we can concentrate on the cash-flows associated with a single trading period. The no-arbitrage probability measure on the set of paths $(F_n)_{n=0}^N$ should be such that the value of the futures contract (zero) is the discounted expectation of its cash-flows. Therefore,

$$0 \;=\; e^{-r_n\, dt}\, \mathrm{E}\left\{\, F_{n+1} \;-\; F_n \mid F_n \,\right\}$$

or simply

$$F_n \;=\; \mathrm{E}\left\{\, F_{n+1} \mid F_n \,\right\} . \tag{30}$$

This last equation states that $(F_n)_{n=0}^N$ must be a *martingale* under the no-arbitrage pricing measure, i.e. that any arbitrage free measure would make today's

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[6] By this we mean a contract which can be opened either going long of short at zero cost and which is marked to market daily. We do not take into account any cashflows resluting from maintenance or posting margins.

futures prices a "fair" bet on later futures prices[7]. As before, the conditional probabilities for the two states (up or down) that can occur in the binomial model can be completely determined from equation (30) and the parameters U_n and D_n. In fact, we have

$$\begin{cases} P_U^{(n)} + P_D^{(n)} = 1 \\ \\ U_n \, P_U^{(n)} + D_n \, P_D^{(n)} = 1 \end{cases}$$

so

$$P_U^{(n)} = \frac{1 - D_n}{U_n - D_n} \quad \text{and} \quad P_D^{(n)} = \frac{U_n - 1}{U_n - D_n} \, .$$

Since, these probabilities are independent of F_n, we conclude that any arbitrage-free probability measure is such that $(F_n)_{n=0}^{N}$ is a multiplicative random walk, i.e. that $\ln F_n$ follows a a standard random walk with independent increments. The model can accommodate if necessary given term-structures of interest rates and/or volatilities using the methods shown above. In the simplest case of constant local volatilities, the (constant) probabilities are given by (28b) and the futures prices at the different nodes of the tree are

$$F_n^j = F_0^0 \cdot \frac{(e^{\rho \sqrt{dt}})^j \, (e^{-\rho \sqrt{dt}})^{n-j}}{\left(P_D \, e^{-\rho \sqrt{dt}} + P_U \, e^{\rho \sqrt{dt}} \right)^n} \, . \tag{31}$$

The pricing equation for a European-style contingent claim with value $G(F_N)$ at expiration is given by

$$\begin{cases} V_n^j = e^{-r \, dt} \left[P_U \, V_{n+1}^{j+1} + P_D \, V_{n+1}^j \right] \\ \\ V_N^j = G(F_N^j) \, . \end{cases} \tag{32}$$

We note here that the present scheme is formally equivalent to the one for pricing of a derivative security contingent on the price of a stock with dividend payment rate $q_n = r_n$ (compare with equations (28a) and (30)). This equivalence can be seen more clearly by assuming that the futures contract is based on a traded underlying security with value S_n and that the futures price converges to the price of the underlier at the end of the N^{th} period, i.e. that $F_N = S_N$. (For simplicity, we assume that the traded security pays no dividends.) Since the cash-flow of the futures position after the N^{th} period coincides with that of a forward contract, the cost-of-carry formula implies that

[7] As mentioned earlier, this is not a statement about the statistics of future prices. The no-arbitrage probability measure is just a device for consistently pricing derivatives contingent on F_n.

$$F_n \;=\; S_n \; e^{\sum\limits_{k=n}^{N-1} r_k \;\; \Delta t} \;. \tag{33}$$

Hence, the futures price is equal to the value of a certain number of shares of the traded security, this number changing with time. Now, the value of this "equivalent portfolio" after the next trading period is

$$\hat{F}_{n+1} \;\equiv\; S_{n+1} \; e^{\sum\limits_{k=n}^{N-1} r_k \;\; \Delta t} \;=\; F_{n+1} \; e^{r_n \;\; \Delta t} \;.$$

This shows that the equivalent portfolio appreciates in value to more than F_{n+1}. The excess can be regarded as a dividend that is paid out after each period. The last equation can be viewed as giving the relation between the corresponding ex- and post-dividend values of the equivalent portfolio, as in equation (23) with

$$1 \;-\; \alpha_n \;=\; e^{-r_n \;\; \Delta t} \;.$$

Another point that deserves attention is the construction of equivalent portfolios that replicate derivative securities contingent on a futures contract. Unlike the case of derivatives based on traded assets, in the present situation the hedging strategy consists of a money market account combined with an open position in futures. As in the case of equity, the number of open contracts at any given time depends on the sensitivity of the calculated value of the derivative security on the futures price. More precisely, suppose that the values V_n^j at all the nodes of the tree have been calculated. The replicating strategy corresponding to the node $(n \;,\; j)$ will consist in investing V_n^j in a money-market account and to open Δ_n^j contracts. To find the " hedge-ratio " Δ_n^j, we must match the two possible cash-flows to the calculated values of the derivative security at the following nodes. Accordingly,

$$\begin{cases} \Delta_n^j \;(\; F_{n+1}^{j+1} \;-\; F_n^j \;) \;+\; (1 \;+\; R_n \;)\; V_n^j \;=\; V_{n+1}^{j+1} \\ \\ \Delta_n^j \;(\; F_{n+1}^{j} \;-\; F_n^j \;) \;+\; (1 \;+\; R_n \;)\; V_n^j \;=\; V_{n+1}^{j} \;. \end{cases}$$

It is immediate from these two equations that

$$\Delta_n^j \;=\; \frac{V_{n+1}^{j+1} \;-\; V_{n+1}^{j}}{F_{n+1}^{j+1} \;-\; F_{n+1}^{j}} \;, \tag{34}$$

as expected.

6. Valuation of a stream of uncertain cash-flows

We conclude this chapter by writing down a general valuation equation that prices derivative securities with a given maturity that offer intermediate cash-flows, depending on the value of the underlying asset up to maturity. A simple example of such a security would be a *commodity- or equity-linked* debt instruments. These securities, normally issued by companies, are such that their coupon payments are linked to the value of an index such as the price of copper or oil or the S&P 500 index. This type of security can be often be decomposed or "stripped" into a series of European-style derivatives with different maturities, in the same way that a coupon-paying bond can be viewed as a series of pure discount bonds. Moreover, the payoff for each maturity can be simple enough so that it can be regarded as a collection of simple options each of which could be valued separately. This point of view, which could be called "reverse-engineering", is extremely useful in practice and will be discussed in detail in future chapters. Here we will show in the " binomial world" with one risky asset, all cash flows can be incorporated into a single equation that can be solved recursively to price any stream of uncertain cash-flows.

Consider therefore a derivative security maturing after N trading periods and paying a stream of cash-flows after each period. These cash-flows can be specified by means of N functions of S, namely

$$f_1(\ S\ )\ ,\ f_2(\ S\ )\ ,\ \ldots. \qquad \ldots\ f_N(\ S\ )\ . \tag{35}$$

The value of the cash-flow at each node $(n\ ,\ j)$ in the tree is defined to be equal to

$$f_n^j\ \equiv\ f_n(\ S_n^j\ )\ .$$

Suppose that a no-arbitrage pricing model based on a recombining binomial tree has been determined, consistently with a term-structure of interest rates, a term-structure of volatilities and the dividend payments of the underlying asset. We have seen that such a model can be constructed in terms of a collection of probabilities $P_U^{(n)}$, $P_D^{(n)}$ and "up-down" parameters U_n and D_n, for $n = 0, ..., N-1$. the general recursion relation that we seek follows from the following observation: at any given time, the value of the derivative security is equal to the current "coupon" value plus the discounted expectation of future cash-flows. Therefore we have

$$\begin{cases} V_n^j\ =\ f_n^j\ +\ e^{-r_n\ \Delta t}\cdot\ \left[\ P_U^{(n)}\ V_{n+1}^{j+1}\ +\ P_D^{(n)}\ V_{n+1}^{j}\ \right] \\ \\ V_N^j\ =\ f_N^j\ . \end{cases} \tag{36}$$