BINOMIAL MODELS FOR INTEREST-RATE DERIVATIVES

We shall discuss the construction of simple binomial models for pricing interest-rate derivatives. The main feature of these models is that the "risk-free" interest rate can vary stochastically from one period to the next. Stochastic rate models can be used to price interest-rate options (caps and floors), options on bonds, callable bonds, etc. They can also be incorporated, at the expense of more complexity, into a model that prices derivatives contingent on one or more traded assets (*e.g.* equity indices, currencies) in an environment with changing interest rates. This point of view is important, for instance, for pricing long-term options or other instruments in which changes in the costs of funding may have significant impact on pricing.

1. Binomial Tree for interest rates: statement of the problem

In the lecture "Refinements of the Binomial Model" we assumed that rates would change according to the current yield curve, i.e. that the spot rate for a given period in the future would be equal to the corresponding implied forward rate. This assumed that interest rates had zero volatility. Here, we we will consider a binomial model with N periods to model interest rate fluctuations. The duration of each period is a specified fraction of year dt. (In practice, dt will correspond to 3 or 6 months intervals, i.e. dt=.25 or dt=.50). At the beginning of each successive trading period, an interest rate for the period is revealed to investors. A **realization** of the interest rate process consists therefore of a sequence of rates

$$r_0 \ , \ r_1 \ , \ r_2 \ , \ \dots \ , \ r_{N-1} \ . \tag{1}$$

Interest rates are assumed to be annualized and continuously compounded. We assume that rates follow recombining binomial process, as depicted below:

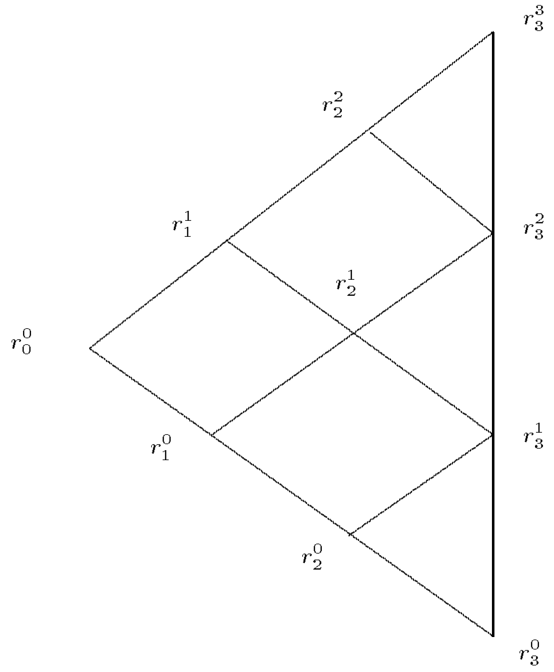

FIGURE 1. Binomial tree for interest rates corresponding to four lending periods.

To determine an arbitrage-free model for interest rates we must find risk-neutral probabilities for the different realizations of sequences (1). These probabilities must be such that the risk-neutral prices match the market prices of liquidly traded instruments such as

- Short-term and long-term bonds
- Interest rate futures
- Interest rate options.

The most important set of data used to calibrate interest rate models is the zero-coupon yield curve or, equivalently, the curve of forward interest rates

$$f_1 \ , \ f_2 \ , \ \dots \ , \ f_{N-1} \ . \tag{2}$$

The problem which we face is the construction of suitable "up-down" moves (or increments of the short-term interest rate) and probabilities for different paths which will be consistent with the data.

2. Binomial tree for discount factors

It is convenient to consider the **discount factors**

$$\mathcal{D}_n \;=\; e^{-\, r_n\, dt} \quad , \quad \mathcal{D}_n^j \;=\; e^{-\, r_n^j\, dt} \quad , \qquad 0 \leqq \; j \; \leqq \; n \; \leqq \; N-1 \tag{3}$$

as auxilliary variables to construct the tree. We postulate that the discount factors evolve according to a binomial rule

$$\begin{aligned} \mathcal{D}_{n+1}^{j+1} \;&=\; \mathcal{D}_n^j \cdot \; U_n \\ \mathcal{D}_{n+1}^{j} \;&=\; \mathcal{D}_n^j \cdot \; D_n \; . \end{aligned} \tag{4}$$

In order to have a recombining tree, we must impose that, for all n, we have the relation

$$U_n \cdot D_{n+1} \;=\; D_n \cdot U_{n+1} \; ,$$

which is equivalent to

$$\frac{U_n}{D_n} \;=\; \text{constant independent of } n \;=\; \lambda \; . \tag{5}$$

Set

$$H_n \;=\; \begin{cases} U_n & , \text{ with probability } \; P_{U,n} \\ & \\ D_n & , \text{ with probability } \; P_{D,n} \end{cases} \tag{6}$$

where $P_{U,n}$ and $P_{D,n}$ are conditional probabilities which may depend the current interest rate level and past information. (These will be determined later). Then, we have,

$$\frac{1}{dt} \ln H_n \;=\; r_{n+1} \;-\; r_n \; . \tag{7}$$

To calculate U_n and D_n, recall that if interest rates had zero volatility, then the future interest rates r_n would be equal to today's forward rates f_n , $n = \; 1, 2 \ldots N-1$. Hence, it is natural to postulate for the general case that

$$U_n \;=\; e^{-\,(f_{n+1} \;-\; f_n\,)\, dt} \cdot \hat{U}_n \tag{8a}$$

and

$$D_n \;=\; e^{-\,(\,f_{n+1} \;-\; f_n\,)\,dt} \cdot \hat{D}_n \;, \tag{8b}$$

where $\hat{U}_n$ and $\hat{D}_n$ are, in a sense, the essential fluctuations due to interest rate volatility.[1] Let us define a random variable associated with the interest rate fluctuations by

$$\hat{H}_n \;=\; \begin{cases} \hat{U}_n & \text{, with probability } \; P_{U,n} \\ \\ \hat{D}_n & \text{, with probability } \; P_{D,n} \;. \end{cases} \tag{9}$$

From the above considerations, we conclude that

$$\mathcal{D}_n^j \;=\; e^{-\,f_{n+1}\,dt}\; \hat{H}_1 \cdot \; \hat{H}_2 \cdot \; ...\hat{H}_n \qquad , \; n \;\geqq\; 1. \tag{10}$$

The no-arbitrage condition for matching the prices of zero-coupon bonds results from the equations

$$\mathbf{E}\,\left\{\mathcal{D}_0^0 \cdot \;\mathcal{D}_1 \cdot ...\mathcal{D}_{n-1}\;\right\} \;=\; e^{\;-\;\sum\limits_{1}^{n} f_k\,dt} \quad .$$

It follows that our model will be consistent with no-arbitrage if and only if

$$\mathbf{E}\,\left\{\;\hat{H}_1^{n-1} \cdot \hat{H}_2^{n-2} \cdot \;\cdot\;\cdot\;\; \hat{H}_{n-1}\right\} \;=\; 1\;, \quad \forall\, n \;\leqq\; N\;. \tag{11}$$

To proceed further, we assume that the increments of the spot rate are independent from one another and depend only on the period of interest.[2] Under this assumption, we find that the variance of the increment

$$dr_n \;=\; r_{n+1} \;-\; r_n$$

is given by

[1] On can also view these variables as the primitive variables in the case of a flat forward curve.

[2] This is a simplifying assumption which may not be appropriate for pricing derivative securities when interest rates are extremely high or extremely low. The reason is that that in these cases, the volatility may correlate strongly with the interest rate.

$$\sigma_n^2\, dt \;=\; \frac{1}{(dt)^2} \left(\ln \left(\frac{\hat{U}_n}{\hat{D}_n} \right) \right)^2 P_{U,n}\, P_{D,n}$$

$$= \frac{(\ln \lambda)^2}{(dt)^2}\, P_{U,n}\, P_{D,n} \;. \qquad (12)$$

Now set $\lambda \;=\; 2\rho\,(dt)^{3/2}$ (notice that the power $3/2$ is the correct one, given the above relations). This leads to the relation between the volatility at each node and the corresponding probabilities:

$$\sigma_n^2 \;=\; 4\,\rho^2\, P_{U,n}\, P_{D,n} \;. \qquad (13)$$

In particular, the stability condition

$$\rho \;\geqq\; \max_n \; \sigma_n \qquad (14)$$

must hold.

From equations (12) and (13) we obtain the formulas

$$P_{U,n} \;=\; \frac{1}{2} \left[\; 1 \;\pm\; \sqrt{1 \;-\; \frac{\sigma_n^2}{\rho^2}} \;\right] \qquad (15a)$$

$$P_{D,n} \;=\; \frac{1}{2} \left[\; 1 \;\mp\; \sqrt{1 \;-\; \frac{\sigma_n^2}{\rho^2}} \;\right] \;, \qquad (15b)$$

which determine the probabilities. We must now determine the parameters U_n and D_n. In order to do this, we observe that that equation (11) reduces to the system of equations

$$\mathbf{E}\left\{ \hat{H}_1^{n-1} \right\} \cdot \mathbf{E}\left\{ \hat{H}_2^{n-2} \right\} \cdot \ldots\, \mathbf{E}\left\{ \hat{H}_{n-1} \right\} \;=\; 1 \qquad (16)$$

or

$$\mathbf{E}\left\{ \hat{H}_{n-1} \right\} \;=\; \left[\; \mathbf{E}\left\{ \hat{H}_1^{n-1} \right\} \cdot \mathbf{E}\left\{ \hat{H}_2^{n-2} \right\} \cdot \ldots\, \mathbf{E}\left\{ \hat{H}_{n-2}^{2} \right\} \right]^{-1} \quad \forall n \;. \qquad (17)$$

This system of equations can be solved recursively to find U_n, D_n. For instance, for $n = 1$ we have

$$1 \;=\; \hat{U}_1 \cdot P_{U,1} \;+\; \hat{D}_1 \cdot P_{D,1} \;.$$

From equation (17) with $n = 1$ we obtain the formulas

$$\hat{U}_1 \;=\; \frac{e^{\rho\,(dt)^{3/2}}}{P_{U,1}\, e^{\rho\,(dt)^{3/2}} \;+\; P_{D,1}\, e^{-\rho\,(dt)^{3/2}}}$$

$$\hat{D}_1 \;=\; \frac{e^{-\rho\,(dt)^{3/2}}}{P_{U,1}\, e^{\rho\,(dt)^{3/2}} \;+\; P_{D,1}\, e^{-\rho\,(dt)^{3/2}}} \;.$$

Plugging in these values into the equation for $n = 2$ we find U_2 and D_2 and so forth. The general form of the solution takes the form

$$\hat{U}_n \;=\; \frac{e^{\rho\,(dt)^{3/2}}}{P_{U,n}\, e^{\rho\,(dt)^{3/2}} \;+\; P_{D,n}\, e^{-\rho\,(dt)^{3/2}}} \cdot C_n \tag{18a}$$

$$\hat{D}_n \;=\; \frac{e^{-\rho\,(dt)^{3/2}}}{P_{U,n}\, e^{\rho\,(dt)^{3/2}} \;+\; P_{D,n}\, e^{-\rho\,(dt)^{3/2}}} \cdot C_n \tag{18b}$$

where $C_n \;=\; \mathbf{E}\left\{\hat{H}_n\right\}$.

To find numerical values for the parameters, we can choose $\rho \;=\; \max_n \sigma_n$, consistently with the stability condition (13). Notice that the largest volatility corresponds then to equiprobable jumps ($P_{U,n} \;=\; P_{D,n} \;=\; 0.5$) whereas small volatility corresponds to most of the weight carried by a single branch.

3. Arbitrage-free dynamics for interest rates: the limit $dt \ll 1$

From equation (7), and taking logarithms in (18a), (18b), we conclude that the interest rate increments satisfy

$$dr_n \;=\; df_n \;\mp\; \rho\sqrt{dt} \;+\; \frac{1}{dt}\ln\left[\; P_{U,n}\, e^{\rho\,(dt)^{3/2}} \;+\; P_{D,n}\, e^{-\rho\,(dt)^{3/2}} \;\right] \;-\; \frac{1}{dt}\ln C_n \qquad (19)$$

where the - ocurrs with probability $P_{U,n}$ and the + with probability $P_{D,n}$. It is easy to verify directly from this formula that the variance of the interest rate increment is indeed given by $\sigma_n^2\,dt$.

We shall focus on the problem of calculating the drift of the interest rate process in closed form in the limit $dt \ll 1$. Notice that the mean of the interest rate increment is df_n, — the increment in forward rate — plus an additional increment which vanishes when the interest rate volatility is zero and is independent of the forward rates. The latter term is necessary to maintain the no-arbitrage requirement with respect to the yield curve.

Let us expand formula (19) with respect to the small parameter $\sqrt{dt}$. The result, is after some calculation,

$$dr_n \;=\; \mp\rho\sqrt{dt} \;+\; (\,P_{U,n} \;-\; P_{D,n}\,)\;\rho\sqrt{dt} \;+\; df_n \;+\; R_n\,dt \;+\; \text{terms or order}\,(dt)^{3/2}\;. \quad (20)$$

Here, R_n represents the **excess drift** due to the no-arbitrage condition. This drift can be computed directly from the formulas of the previous section[3] or can be obtained directly as follows: from the Central Limit Theorem, the continuous version of the interest rate process must be, according to (20),

$$r_t \;=\; N(t) \;+\; f(t) \;+\; \int_0^t R(s)\,ds\;, \qquad (21)$$

Here, $f(t)$ an $R(t)$ are continuous functions describing the term-structure of forward rates f_n and the drift correction R_n, respectively. The function $N(t)$ is a random process with independent, normally distributed increments. The mean of the increments is zero and the variance is

[3] This is a slightly tedious calculation because of the the recursive way in which the constants C_n are derived from equation (17). It is left as an exercise for the interested reader.

$$\mathbf{E}\left\{(N(t)-N(s))^2\right\} = \int_s^t \sigma_u^2\,du\ , \tag{22}$$

where σ_t is the term-structure of volatilities.

To characterize $R(t)$, it suffices to observe that the pricing formula for zero-coupon bonds

$$e^{-\int_0^t f(s)\,ds} = \mathbf{E}\left\{ e^{-\int_0^t r_s\,ds} \right\}\ .$$

must hold. This implies, from (21), that

$$\mathbf{E}\left\{ e^{-\int_0^t N_s\,ds} \right\} = e^{\int_0^t ds \int_0^s R(u)\,du}\ .$$

We conclude, from the explicit form of the moment-generating function of the Gaussian distribution[4], that

$$\int_0^t ds \int_0^s R(u)\,du = \frac{1}{2}\,\mathbf{E}\left\{ \left[\int_0^t N(s)\,ds\right]^2 \right\}$$

$$= \int_0^t \int_0^s \mathbf{E}\left\{ N(s)\,N(u) \right\} du\,ds$$

Differentiating both sides of the last equation with respect to t, we obtain

$$\int_0^t R(s)\,ds = \int_0^t \mathbf{E}\left\{N(t)\,N(s)\right\} ds$$

$$= \int_0^t \mathbf{E}\left\{N(s)^2\right\} ds\ ,$$

[4] Namely, if X is a Gaussian with mean zero and variance σ^2, then $\mathbf{E}\left\{e^{\theta X}\right\} = e^{\frac{1}{2}\theta^2\sigma^2}$.

where we used the independence on the increments of $N(t)$. Therefore, using (22) and differentiating again with respect to t we obtain finally

$$R(t) \;=\; \int_0^t \sigma_s^2\, ds \;. \tag{23}$$

The continuous-time approximation for the the interest rate process is therefore

$$r_t \;=\; N(t) \;+\; f(t) \;+\; \int\limits_0^t \int_0^s \sigma_u^2\, du\, ds \;. \tag{24}$$

This Gaussian model which is consistent with the structure or interest rates is known in the Mathematcal Finance literature as the **Ho-Lee model**. One interesting consequence of the closed-from solution is that the binomial tree can be considerlaby simplified if dt is small. Taking a hint from equation (24), we may chose to implement the following discretization of Ho-Lee:

$$dr_n \;=\; \pm\,\sigma_n \sqrt{dt} \;+\; df_n \;+\; \left(\sum j = 1^n\, \sigma_j^2\, dt \right)\, dt \;, \tag{25}$$

where the + and - shocks ocurr with probability 0.5.

4. A multiplicative random walk model

The previous model led to an interest rate process satisfying a simple random walk with a drift associated to the curve of forward rates and the term-structure of volatilities. An alternative model consits of assuming that the interest rates, and not the discount factors, are modified multiplicatively at each node of the tree.

Consider for instance a binomial tree such that

$$r_{n+1}^{j+1} \;=\; r_n^j\, U_n \quad \text{with prob.}\;\; P_{U,n} \tag{26a}$$

and

$$r_{n+1}^{j} \;=\; r_n^j\, D_n \quad \text{with prob.}\;\; P_{U,n} \;. \tag{26b}$$

Notice that this model is structurally similar to the models used to price stock options. The difference with the latter lies in how the no-arbitrage condition enters. How should we choose the parameters to make the model arbitrage-free with repect to the term-structure of interest rates?

In this section, we shall see that the answer is particularly simple if we adjust the model to the term-structure of **interest-rate futures** rather than to the zero-coupon rates. By a term structure of interest rate futures, we mean that there are $N-1$ futures contracts on the interest rates r_n for each period period in the future. As shown in a previous lecture, let R_n represent the interest rate implied by the price of the futures contract.[5] We will neglect issues about the difference between simple and compounded rates: although the ED futures are for simply compounded rates, it is easy to convert the implied rate into a continuously compounded rate over the period of interest.

As shown in previous lectures, the futures-implied rate and the spot rate satisfy the following relation:

$$R_n \;=\; \mathbf{E}\left\{ r_n \right\} \; . \tag{27}$$

This puts certain constraints on the aritrage-free measure which we now discuss. We will assume, for simplicity, that the shocks are independent. The volatility of the increments of the **log** of the interest rate are then given by

$$\sigma_n^2 \;=\; \frac{1}{dt}\left(\ln\left(\frac{U_n}{D_n}\right)\right)^2 P_{U,n}\, P_{D,n} \; .$$

Introducing a parameter ρ such that $U_n/D_n \;=\; e^{2\rho\sqrt{dt}}$, we consider the probabilities in (15a) and (15b) and set

$$U_n \;=\; \frac{e^{\rho\,(dt)^{1/2}}}{P_{U,n}\, e^{\rho\,(dt)^{1/2}} \;+\; P_{D,n}\, e^{-\rho\,(dt)^{1/2}}} \cdot e^{\mu_n\, dt} \tag{28a}$$

and

$$D_n \;=\; \frac{e^{-\rho\,(dt)^{1/2}}}{P_{U,n}\, e^{\rho\,(dt)^{1/2}} \;+\; P_{D,n}\, e^{-\rho\,(dt)^{1/2}}} \cdot e^{\mu_n\, dt} \tag{28b}$$

Here, the sequence of parameters μ_n will serve to calibrate the model according to the structure of interest rates R_n. By analogy with the equity model, we can easily conclude that

[5] Recall, for instance, that the Eurodollar futures contract of CME is quoted as $100 - R_n$. The implied interest rate is therefore 100 minus the futures price (see *J. Hull.*

$$r_0 \; = \; e^{-\sum\limits_0^{n-1} \mu_j \, dt} \cdot \mathbf{E}\{r_n\}$$

$$= \; e^{-\sum\limits_0^{n-1} \mu_j \, dt} R_n,$$

so we obtain the relation

$$\frac{R_n}{r_0} \; = \; e^{-\sum\limits_0^{n-1} \mu_j \, dt}, \qquad n \; = \; 1, \, 2, \, \ldots N-1 \; .$$

Inverting this relation, we find rather easily that

$$\mu_n \; = \; \frac{1}{dt} \ln \left(\frac{R_n}{R_{n-1}} \right) \; . \tag{29}$$

Alternatively, we can rewrite the parameters of the model in the form

$$U_n \; = \; \frac{R_n}{R_{n-1}} \cdot \frac{e^{\rho \, (dt)^{1/2}}}{P_{U,n} \, e^{\rho \, (dt)^{1/2}} \; + \; P_{D,n} \, e^{-\rho \, (dt)^{1/2}}} \tag{30a}$$

and

$$D_n \; = \; \frac{R_n}{R_{n-1}} \cdot \frac{e^{-\rho \, (dt)^{1/2}}}{P_{U,n} \, e^{\rho \, (dt)^{1/2}} \; + \; P_{D,n} \, e^{-\rho \, (dt)^{1/2}}} \tag{30b}$$

The interest rate increments for this model satisfy

$$d \ln r_n \; = \; d \ln R_n \; \pm \; \rho \sqrt{dt} \; - \; \frac{1}{dt} \ln \left[\; P_{U,n} \, e^{\rho \, (dt)^{1/2}} \; + \; P_{D,n} \, e^{-\rho \, (dt)^{1/2}} \; \right]$$

which, to leading order in dt gives

$$d \ln r_n \; \approx \; d \ln R_n \; - \frac{1}{2} \sigma_n^2 \, dt \; \pm \; rho \sqrt{dt} \; - \; (P_{U,n} \; - \; P_{D,n} \,) \; \rho \sqrt{dt}$$

with + occurring with probability $P_{U,n}$ and - with probability $P_{D,n}$.

The model presented in this section gives rise to *lognormal* interest rates, rather than normal. The continuous time version is

$$\ln r_t \;=\; N(t) \;+\; \ln R_t \;-\; \int_0^t \sigma_s^2 \, ds \;.$$

Despite the clear theoretical simplicity of the lognormal model, the simple Ho-Lee model is often used because option prices can be computed in closed form using the **bivariate** normal distribution. On the other hand, the model presented here, which is sometimes referred to as the **Black-Karasinski** model does not have simple closed-form solutions fro the prices of options.