Additional exercise for Homework # 1:

Effect of a term-structure of volatility on digital barrier options.

This problem studies a "digital knockout option" — a derivative security that pays some notional amount if the underlying index remains under a specified level (the knockout barrier) over a period of time (the term) and pays nothing if the index crosses the barrier.

1. Let S_t represent the index at time t, let H be the knockout barrier and let T be the term. Assume first that

$$S_t \;=\; S_0\, e^{\sigma Z_t \;-\; \frac{\sigma^2 t}{2} \;+ (r-q)\,t}\;, \quad , \; t \;\leq\; T\;,$$

where S_0 represents the current value of the index, and r, q and σ are constants representing, respectively, the interest rate, the dividend yield and the volatility. The fair value of the digital knockout as a percentage of the notional is

$$V \;=\; e^{-r\,T}\,N\,(\,d_1\,) \;-\; e^{-r\,T}\left(\frac{H}{S_0}\right)^{\frac{2(r-q)}{\sigma^2}\;-1} N\,(\,d_2\,)\;, \tag{1}$$

where $N(\cdot)$ is the normal distribution function,

$$d_1 \;=\; \frac{1}{\sigma\,\sqrt{T}}\left\{ln\,\left(\frac{H}{S_0}\right)\;-(r\;-\;q\,)\,T\;+\;\frac{\sigma^2\,T}{2}\;\right\}$$

and

$$d_2 \;=\; \frac{1}{\sigma\,\sqrt{T}}\left\{\;-ln\,\left(\frac{H}{S_0}\right)\;-(r\;-\;q\,)\,T\;+\;\frac{\sigma^2\,T}{2}\;\right\}\;.$$

To derive this formula, you can use the results from Chapter 7 in Math of Finance I course notes. Notice that the first term in (1) represents the fair value of the European digital. The second term can be interpreted as the value of a "knock-in" option with the same terminal payoff.

2. Study the effect of having a **time-dependent spot volatility**, $\sigma(t)$, $t\;\leq T$, assuming that the latter function is independent of S_t. Define a "term volatility" σ by the well-known formula

$$\sigma \;=\; \sqrt{\frac{1}{T}\int_0^T \sigma^2(s)\,ds} \tag{2}$$

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Consider the new "clock", $\tau(t)$, defined by

$$\tau(t) \;=\; \int_0^t \frac{\sigma^2(s)}{\sigma^2}\, ds \;.$$

Show that, with respect to this new clock, the index has constant volatility σ and time-dependent carry (drift)

$$\mu(\tau) \;=\; \frac{\sigma^2\,(r-q)}{\sigma^2(t)} \quad . \tag{3}$$

Conclude that if $r - q = 0$ the value of the digital knockout at time $t = 0$ is given by (1) with the term volatility (2).

3. Use a binomial or trinomial tree to model an asset with spot volatility

$$\sigma(t) \;=\; \begin{cases} .16 \quad \text{if } t \;<\; 90/255 \\ \\ .05656 \quad \text{if } 90/255 \;\leq\; t \;\leq\; 180/255\;. \end{cases} \tag{4}$$

(Assume a 255-day business year.) Price a digital knockout option assuming that

$$S_0 \;=\; 1.50\;, \quad H \;=\; 1.60\;,$$

$$r \;=\; 3.20\% \quad , \quad q \;=\; 5.50\%\;,$$

$$T = \;180 \text{ days} \;=\; 180/255 \text{ yr.}\;.$$

Check that the value of the option is less than the one given by (1) using a constant term volatility of 12% (which is consistent with (4)). Recompute the value changing the order of the two volatilities in (4). Draw some conclusions.