A Risk-Neutral Stochastic Volatility Model

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We construct a risk-neutral stochastic volatility model using no-arbitrage pricing principles. We then study the behavior of the implied volatility of options that are deep in and out of the money according to this model. The motivation of this study is to show the difference in the asymptotic behavior of the distribution tails between the usual Black-Scholes log-normal distribution and the risk-neutral stochastic volatility distribution.

In the second part of the paper, we further explore this risk-neutral stochastic volatility model by a Monte-Carlo study on the implied volatility curve (implied volatility as a function of the option strikes) for near-the-money options. We study the behavior of this "smile" curve under different choices of parameter for the model, and determine how the shape and skewness of the "smile" curve is affected by the volatility of volatility ("V-vol") and the correlation between the underlying asset and its volatility.

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1 Introduction

The Black-Scholes (BS) formula is widely used by traders because it is easy to use and understand. An important characteristic of the model is the assumption that the volatility of the underlying security is constant. However, practitioners have observed, especially after the crash of 1987, the so called volatility "smile" effect. Namely, options written on the same underlying asset usually trade, in Black-Scholes term, with different implied volatilities[3]. Deep-in-the-money and deep-out-of-the-money options are traded at higher implied volatility than at-the-money options. There is also a time effect. For example, in Foreign Exchange market, options with longer maturities are traded at higher implied volatility than shorter maturities. This evidence is not consistent with the constant volatility assumption made in Black-Scholes (Black & Scholes, 1973).

This is due to the presence of "fat tails": extreme values for the price are more likely in the real probability measure than in the lognormal model. There are several ways to address this empirical issue. Merton (R. Merton, 1990) points out that a jump-diffusion process for the underlying asset could cause such an effect. A more explored direction is the stochastic volatility assumption. Hull and White (Hull and White, 1987) proposed a log-normal stochastic volatility model, namely, the volatility of the underlying asset follows another Geometric Brownian Motion. However, these models have a drawback: since the volatility is not a traded asset, the option price in the stochastic volatility context actually depends on investors risk preferences, that is, the pricing formula is not risk-neutral.

In this paper, we study a risk-neutral pricing model in the context of log-normally distributed stochastic volatility. In order to find a risk-neutral probability measure suitable for pricing options and OTC derivatives, we have to analyze hedging strategies involving a traded asset which is perfectly correlated with volatility of the underlying security. For this purpose, we propose to use short term options on the underlying asset to hedge the volatility risk. The key assumption made here is that the maturities of these options are short enough that the options are reasonably marked-to-market (priced) by Black-Scholes formula.

This paper is organized as follows. In Section 2 we derive a risk-neutral stochastic process for the underlying asset and its volatility. For this, we assume in particular a general correlation (negative, zero, positive) between the asset price and its volatility. Section 3 is devoted to the asymptotic estimation of the

[3]Implied volatility is the volatility value at which the option is traded if the Black-Scholes formula is used. Given an option price, there is a corresponding volatility value by inverting the Black-Scholes formula.

implied volatility of the derived model, i.e., how the implied volatility behaves for options that are deep-out-of-the-money. In Section 4 we study the implied volatility behavior of near-money options using Monte Carlo simulation. One of the interesting features of the model is the different behavior among positive, zero, and negative correlations. Calls with positive correlations correspond to puts with negative correlations, and vice versa. The analysis therefore suggests an asymmetry between puts and calls.[4] The three appendices in the last section contain mathematical details for the asymptotic analysis in section 3.

2 The Risk-Neutral Measure

In this section, the risk-neutral probability measure of the log-normal stochastic volatility model is derived. Specifically, we consider an underlying security S_t, and its volatility σ_t, which follow the stochastic processes:

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma_t S_t dZ_t \\ d\sigma_t &= \gamma \sigma_t dt + V \sigma_t dW_t \end{aligned}$$

where Z_t and W_t are two standard Brownian motions with correlation coefficient ρ. Formally, $E(dZ_t dW_t) = \rho dt$. Let f be the price of a derivative security contingent on the price S of an underlying asset. Specifically,

$$f = f(S, \sigma, t)$$

By Ito's lemma, the price process of the derivative security satisfies,

$$df(S, \sigma, t) = f_S dS + f_\sigma d\sigma + \mathcal{L} f dt$$

where $\mathcal{L}$ is the infinitesimal generator:

$$\mathcal{L} \equiv \partial_t + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \frac{1}{2} V^2 \sigma^2 \frac{\partial^2}{\partial \sigma^2} + \rho \sigma^2 S V \frac{\partial^2}{\partial S \partial \sigma} \tag{1}$$

In the same spirit of the original derivation of the Black-Scholes formula, we give a hedging strategy using the underlying asset and a short-term call option on it, along with a money market account. The riskless portfolio will include the contingent claim with price f, underlying asset with price S and a short term call[5] on asset S with price $C(S, K, \sigma, \Delta t)$, where Δt is the maturity of the

[4] For example, for the options on Standard & Poor's 500 Index of Chicago Mercantile Exchange, deep out-of-money puts trade at approximately $\sigma = 30\%$, and at-the-money options trade at 17%.

[5] The strike of this short term option is to be determined.

short-term call option.[6] We short[7] 1 unit of derivative security with price f, go long Δ units of the underlying asset with price S, and μ units of the short term call option on the asset S with price C. The major approximation we make in order to derive a risk-neutral probability measure is to assume the short term call price $C(S, K, \sigma, \Delta t)$ to be the Black-Scholes price. This assumption is the essence of our model. If one doesn't make such an identification, one can only achieve, in the general framework, a no-arbitrage pricing relationship between the short-term call and the general derivative security f.[8]

In addition to the underlying security and the short-term call, we consider a money market account with riskless interest rate r For an infinitesimal time interval dt, the value change of the portfolio is given as:

$$\begin{aligned} df - \Delta dS - \mu dC &= (f_S dS + f_\sigma d\sigma + \mathcal{L} f dt) - \Delta dS - \mu(C_S dS + C_\sigma d\sigma + \mathcal{L} C dt) \\ &= (f_S - \Delta - \mu C_S) dS + (f_\sigma - \mu C_\sigma) d\sigma + (\mathcal{L} f - \mu \mathcal{L} C) dt \end{aligned}$$

where $\mathcal{L}$ is the infinitesimal generator defined in (1).

A riskless portfolio is obtained by setting :

$$\begin{aligned} \Delta + \mu C_S &= f_S \\ \mu C_\sigma &= f_\sigma \end{aligned}$$

specifically,

$$\begin{aligned} \mu &= \frac{f_\sigma}{C_\sigma} \\ \Delta &= f_S - \frac{C_S}{C_\sigma} f_\sigma \end{aligned}$$

Therefore, the value change of the riskless portfolio is:

$$df - \Delta dS - \mu dC = (\mathcal{L} f - \mu \mathcal{L} C) dt$$

According to no-arbitrage pricing principle, the return of the riskless portfolio must be identified with the riskless interest rate, i.e.,

$$\begin{aligned} (\mathcal{L} f - \mu \mathcal{L} C) dt &= r(f - \Delta S - \mu C) dt \\ &= r(f - f_S S + \mu C_S S - \mu C) dt \end{aligned} \tag{2}$$

[6] Δt is small compared with the maturity of the contingent claim in consideration, while large compared with hedging period dt.

[7] In financial terminology, short means sell, and long means buy.

[8] A similar situation occurs in interest rate models. For instance, the Vascicek model (Vascicek, 1979) is a no-arbitrage model of interest rate derivatives. This is not a risk-neutral model because there is a non-determined parameter, the market price of risk, which depends on investor's risk preference. This risk premium, however, doesn't depend on the particular choice of a derivative security. Therefore, it serves as a price relation between different derivative securities.

where in the second equation, we substitute $\Delta = f_S - \mu C_S$ into the formula.

Notice that

$$\begin{aligned} \mathcal{L}C &= (\partial_t + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2})C + \rho\sigma^2 SVC_{S\sigma} + \frac{1}{2}V^2\sigma^2 C_{\sigma\sigma} \\ &= rSC_S - rC + \rho\sigma^2 SVC_{S\sigma} + \frac{1}{2}V^2\sigma^2 C_{\sigma\sigma} \end{aligned} \tag{3}$$

where for (3) we use the Black-Scholes PDE for function C.

Substitute (3) into (2), we obtain

$$\mathcal{L}f - \rho\sigma^2 VS\frac{C_{S\sigma}}{C_\sigma}f_\sigma - \frac{1}{2}\sigma^2 V^2 \frac{C_{\sigma\sigma}}{C_\sigma}f_\sigma = rf - rf_S S \tag{4}$$

Now use Black-Scholes formula for C,

$$C(S, K, \sigma, \Delta t) = SN(d_1) - Ke^{-r\Delta t}N(d_2)$$

where

$$d_{1,2} = \frac{\ln \frac{Se^{r\Delta t}}{K}}{\sigma\sqrt{\Delta t}} \pm \frac{1}{2}\sigma\sqrt{\Delta t}$$

we have the following Greeks:

$$\begin{aligned} C_\sigma &= SN'(d_1)\sqrt{\Delta t} \\ C_{\sigma\sigma} &= SN''(d_1)\sqrt{\Delta t}(-\frac{\ln \frac{Se^{r\Delta t}}{K}}{\sigma^2\sqrt{\Delta t}} + \frac{1}{2}\sqrt{\Delta t}) \\ C_S &= N(d_1) \\ C_{S\sigma} &= N'(d_1)(-\frac{\ln \frac{Se^{r\Delta t}}{K}}{\sigma^2\sqrt{\Delta t}} + \frac{1}{2}\sqrt{\Delta t}) \end{aligned}$$

Now, in order to have a manageable drift term for σ_t, we set the strike of the short term call to be ATMF (at the money forward), i.e.,

$$K = S\exp(r\Delta t).$$

Therefore we have

$$\begin{aligned} C_\sigma &= SN'(d_1)\sqrt{\Delta t} \\ C_{\sigma\sigma} &= SN''(d_1)\frac{\Delta t}{2} = \frac{1}{\sqrt{2\pi}}S\exp(-\frac{d_1^2}{2}) \cdot \frac{1}{4}(\Delta t)^{\frac{3}{2}} \\ C_{S\sigma} &= N'(d_1) \cdot \frac{1}{2}\sqrt{\Delta t} \end{aligned}$$

Substitute the above formula into (4), and neglect the higher order term of Δt, we have

$$\mathcal{L}f - \frac{1}{2}\rho\sigma^2 V f_\sigma = rf - rSf_S$$

$\mathcal{L}$ is defined as in (1). In terms of SDEs, the corresponding risk-neutral process can be written as,

$$\begin{aligned} \frac{dS}{S} &= rdt + \sigma_t dZ_t \\ d\sigma_t &= -\frac{1}{2}\rho V \sigma_t^2 dt + V\sigma_t dW_t \end{aligned} \tag{5}$$

This is the risk-neutral probability for the stochastic volatility model. We see that in the risk-neutral world, the drift term of the underlying security is the short-term interest rate, while the drift of the stochastic volatility also becomes independent of that in "real" probability measure.

Notice that the behavior of the volatility process is different according to the sign of the correlation coefficient. In the case of *positive* correlation, the local volatility tends to zero when time goes to infinity. In the case of *negative* correlation, the local volatility blows up in finite time.[9] This means that when the correlation is positive, the hedging procedure works rather nicely; while the correlation is negative, there is a contradiction between buying the underlying and buying the ATM call option as a hedge. In other words, in the case of negative correlation, when market goes down, volatility goes up, one needs to buy the short-term ATM call to "delta" hedge the volatility . But the call price (therefore its sensitivity to volatility) drops as the market goes down, the result is there is not enough "volatility" to buy.[10] [11]

The rest of the paper is devoted to classifying the asymptotic behavior of the implied volatility curve as a function of strike in this risk-neutral volatility model. One possible application of this analysis is to find a suitable function space in which one could fit the "smile" curve observed in the market.

[9]The blow-up time is typically very large compared with the maturity of options. The blow-up time can be approximated by $-\frac{2}{\sigma_0 \rho V}$. For a typical volatility $\sigma_0 = 10\%$ per annum, correlation $\rho = -0.5$ and "V-vol" $V = 100\%$ per annum, the blow-up time is 40 years.

[10]At first sight, the reader may think that the situation could be resolved by using short term *puts* to hedge the volatility risk. However, the ATM puts' prices also drop when the market goes down. So it doesn't make difference using puts or calls, as long as they are at-the-money.

[11]Negative correlation is present in general in equity market.

3 Asymptotics

This section discusses the asymptotic behavior of the implied volatility of out-of-the-money calls for very large strikes. The main tool used is Large Deviation theory. We classify the behaviors for positive, zero, and negative correlations. The main result is in Theorem 7 at the end of this section. *Lemma* 1 through 6 are steps towards the derivation. Tedious and technical mathematical details are presented in appendix.

To fix the notation, let $BS(S, \sigma, K, T)$ be the Black-Scholes formula for an European call option price, with spot price S, strike K, volatility σ, and maturity T. Without loss of generality, we assume $r = 0$.

Lemma 1. In the limit where the strike is large compared with spot price of the underlying asset, the Black-Scholes option price satisfies asymptotically

$$\lim_{K\to\infty} BS(S,\sigma,K,T) = \frac{1}{\sqrt{2\pi}X^2}\exp\left(-\frac{(X-\frac{1}{2}\sigma^2T)^2}{2\sigma^2T}\right), \tag{6}$$

where $X \equiv \ln(\frac{K}{S})$.

Proof. The key point is the following inequality, which can be found in, e.g., Mckean (1969):

$$\frac{x}{1+x^2}e^{-\frac{x^2}{2}} \leq \int_x^\infty e^{-\frac{u^2}{2}}\,du \leq \frac{1}{x}e^{-\frac{x^2}{2}}$$

or equivalently,

$$\int_x^\infty e^{-\frac{u^2}{2}}du \approx \frac{1}{x}e^{-\frac{x^2}{2}} \qquad \text{for } x \text{ large}$$

We also have

$$\int_x^\infty \frac{1}{u}e^{-\frac{u^2}{2}}du \approx \frac{1}{x^2}e^{-\frac{x^2}{2}} \qquad \text{for } x \text{ large}$$

These approximations can be proved by using change the order of integration.

By change the order of integration, we get

$$\begin{aligned}
& E[(S-K)^+] \\
= \;& \frac{1}{\sqrt{2\pi}}\int_{\frac{X+\frac{1}{2}\sigma^2T}{\sqrt{\sigma^2T}}}^\infty \left(\exp(\sqrt{\sigma^2T}y - \frac{1}{2}\sigma^2T) - \exp(X)\right)\exp(-\frac{y^2}{2})dy \\
= \;& \frac{1}{\sqrt{2\pi}}\int_{\frac{X+\frac{1}{2}\sigma^2T}{\sqrt{\sigma^2T}}}^\infty \exp(-\frac{y^2}{2})\;dy\int_{\frac{X+\frac{1}{2}\sigma^2T}{\sqrt{\sigma^2T}}}^y \exp\left(\sqrt{\sigma^2T}z - \frac{1}{2}\sigma^2T\right)\times\sqrt{\sigma^2T}\;dz \\
= \;& \frac{\sqrt{\sigma^2T}}{\sqrt{2\pi}}\int_{\frac{X+\frac{1}{2}\sigma^2T}{\sqrt{\sigma^2T}}}^\infty \exp\left(\sqrt{\sigma^2T}z - \frac{1}{2}\sigma^2T\right)\;dz\int_z^\infty \exp(-\frac{y^2}{2})\;dy
\end{aligned}$$

$$
\begin{aligned}
&\approx \frac{\sqrt{\sigma^2 T}}{\sqrt{2\pi}} \int_{\frac{X+\frac{1}{2}\sigma^2 T}{\sqrt{\sigma^2 T}}}^{\infty} \exp\left(\sqrt{\sigma^2 T} z - \frac{1}{2}\sigma^2 T\right)\left(\frac{1}{z}\right)\exp\left(-\frac{z^2}{2}\right)dz \\
&\approx \frac{1}{\sqrt{2\pi}X^2}\exp\left(-\frac{(X-\frac{1}{2}\sigma^2 T)^2}{2\sigma^2 T}\right).
\end{aligned}
$$

□**QED**

Comment. It can be easily verified that the deep out-of-the-money puts also satisfies this asymptotic formula. Therefore, the discussions that follow are also true for puts.

Lemma 2. For the above derived risk-neutral stochastic volatility model, the call option price is

$$
\mathbf{E}\{\mathbf{BS}(\xi, K, (1-\rho^2)\int_0^T \sigma_s^2 ds)\}
$$

where $\mathbf{E}$ is the expectation with respect to the stochastic process σ_t, and

$$
\xi \equiv \exp[\frac{\rho}{V}(\sigma_t - \sigma_0)]
$$

$\mathbf{BS}$(S, K, $\sigma^2 T$) is defined as above.

Proof. From the last section, the risk-neutral measure of the stochastic volatility model (5) can be written as

$$
\frac{dS}{S} = \rho\sigma_t dW_t + \sqrt{1-\rho^2}\sigma_t dZ_t \tag{7}
$$

$$
d\sigma_t = -\frac{1}{2}\rho V \sigma_t^2 dt + V\sigma_t dW_t \tag{8}
$$

Z_t and W_t are two independent Wiener processes. Substitute (8) into (7), we get

$$
\frac{dS}{S} = \frac{\rho}{V}d\sigma_t + \frac{1}{2}\rho^2\sigma_t^2 dt + \sqrt{1-\rho^2}\sigma_t dZ_t
$$

Formally integrating this SDE, we get

$$
\begin{aligned}
S &= \exp\{\frac{\rho}{V}(\sigma_t - \sigma_0) + \frac{1}{2}\rho^2\int_0^t \sigma_s^2 ds - \frac{\rho^2}{2V^2} < \sigma_t, \sigma_t >\} \\
&\quad \cdot \exp\{\int_0^t \sqrt{1-\rho^2}\sigma_s dZ_s - \frac{1-\rho^2}{2}\int_0^t \sigma_s^2 ds\}
\end{aligned}
$$

where $< \sigma_t, \sigma_t >$ is the quadratic variation of σ_t, $< \sigma_t, \sigma_t >= V^2\int_0^t \sigma_s^2 ds$. Therefore,

$$
S = \exp\{\frac{\rho}{V}(\sigma_t - \sigma_0)\} \cdot \exp\{\int_0^t \sqrt{1-\rho^2}\sigma_s dZ_s - \frac{1-\rho^2}{2}\int_0^t \sigma_s^2 ds\}
$$

Since Z_t and σ_t are independent, one can think that, for each realization of σ_t, the distribution of the underlying asset price at maturity is equivalent to a geometrical Brownian Motion, starting from

$$\xi \equiv \exp[\frac{\rho}{V}(\sigma_t - \sigma_0)],$$

with total variance $(1-\rho^2)\int_0^t \sigma_s ds$. The lemma is proved. $\Box$ **QED**
Using the asymptotic Black-Scholes formula (6), as derived in *Lemma 1*, we can formally write down the asymptotic formula for the call price under the risk-neutral stochastic volatility measure, namely,

$$\frac{1}{\sqrt{2\pi}} \int\int \frac{1}{X^2} \exp[-\frac{(X - \frac{1-\rho^2}{2}A_t - \frac{\rho}{V}(\sigma_t - \sigma_0))^2}{2(1-\rho^2)A_t}] f(\sigma_t, A_t) d\sigma_t dA_t \qquad (9)$$
$$\text{while} \quad X \to \infty$$

where $A_t \equiv \int_0^t \sigma_s^2 ds$. We suppose $S_0 = 1$, X is defined as in *Lemma* 1. $f(\sigma_t, A_t)$ is the joint probability density of σ_t and A_t.

$$f(\sigma_t, A_t) = g(\sigma_t | A_t) h(A_t)$$

$g(\sigma_t|A_t)$ is the probability density of σ_t conditional on A_t.

Next we are going to characterize the probability distribution $g(\sigma_t|A_t)$. Observe that

$$\sigma_t = -\frac{1}{2}\rho V \int_0^t \sigma_s^2 ds + V \int_0^t \sigma_s dZ_s$$

Let τ_∞ be the first time $\sigma_s = 0$. Let

$$A_\infty = \int_0^{\tau_\infty} \sigma_s^2 ds$$

Use the random time change formula, we have

$$\begin{aligned} \sigma_t &= -\frac{1}{2}\rho V A_t + V B(A_t) \qquad (A_t < A_\infty) \\ &= 0 \qquad\qquad\qquad\qquad\quad (A_t \geq A_\infty) \end{aligned}$$

where B_t is another Brownian motion. Therefore, the above formula is equivalent to saying that the distribution of σ_t conditional on A_t is equivalent in law to the distribution of a drifting Brownian motion conditional on not hitting upon 0.

Based on the above observation, we can derive an approximate formula for the conditional distribution $g(\sigma_t|A_t)$.

In what follows, we treat the cases of $\rho > 0$ and $\rho < 0$ differently.

Lemma 3. When $\rho > 0$ and $A_t \gg 1$, the conditional distribution $g(\sigma_t|A_t)$ satisfies

$$g(\frac{\sigma_t}{V} \in d\eta | A_t) \approx \frac{\rho'^2}{\sqrt{2\pi}} \eta \exp\big(-\frac{(\eta + \rho' A_t)^2}{2A_t}\big) \exp(\frac{1}{2}\rho'^2 A_t) d\eta.$$

where we let $\rho' = \frac{1}{2}\rho$.
Proof. Let $\eta \equiv \frac{\sigma_t}{V} = -\rho' A_t + B_{A_t}$. The distribution of $-\rho' t + B_t$ conditional on not hitting 0 is

$$(P^{(-\rho')}(t; \eta, \frac{\sigma_0}{V}) - P^{(-\rho')}(t; \eta, -\frac{\sigma_0}{V}))/\bar{N}(t) \qquad \text{(Mckean, 1969)}$$

where

$$P^{(b)}(t; x, y) \equiv \frac{1}{\sqrt{2\pi t}} \exp\big(-\frac{(x-y)^2}{2t}\big) \exp(bx - \frac{1}{2}b^2 t); \tag{10}$$

$\bar{N}(t)$ is the normalization factor given by

$$\bar{N}(t) \equiv \frac{2\sigma_0}{\rho'^2 A_t^{\frac{3}{2}}} \exp\big(-\frac{\rho'^2 A_t}{2}\big). \qquad (\text{see } \textit{Appendix } 1)$$

When $A_t \gg 1$, we can make the following approximation

$$\begin{aligned} g(\frac{\sigma_t}{V} \in d\eta | A_t) &= (P^{(-\rho')}(A_t; \eta, \frac{\sigma_0}{V}) - P^{(-\rho')}(A_t; \eta, -\frac{\sigma_0}{V}))/\bar{N}(A_t) \\ &\approx \frac{dP^{(-\rho')}(A_t; \eta, 0)}{d\eta} \cdot 2\frac{\sigma_0}{V} \end{aligned}$$

Substituting the formula of $P^{(b)}$ given by (10) into the above approximation, we get the formula as stated in the *Lemma*.
□ **QED**.
Lemma 4. When $\rho > 0$, the call price satisfies

$$\frac{\rho^2}{8\pi} \int \frac{1}{X^2} \xi(A_t) h(A_t) dA_t, \;\; X \gg 1 \tag{11}$$

where

$$\xi(A_t) \equiv (1-\rho^2) A_t \exp\big(-\frac{1}{2}\frac{(X - (\frac{1}{2} - \rho^2)A_t)^2}{\rho^2(1-\frac{\rho}{2})A_t}\big) \exp(\frac{1}{8}\rho^2 A_t) \tag{12}$$

$$\text{as } A_t \le \frac{X}{\frac{1-\rho^2}{2}(1+\frac{1}{\rho})}$$

$$\xi(A_t) \equiv (1-\rho^2) A_t \exp\big(-\frac{1}{2}\frac{(X - \frac{1-\rho^2}{2}A_t)^2}{(1-\rho^2)/\rho^2 A_t}\big) \tag{13}$$

$$\text{as } A_t \ge \frac{X}{\frac{1-\rho^2}{2}(1+\frac{1}{\rho})}$$

Proof. From the asymptotic call price (9) discussed after *Lemma 2*, under the condition $\rho > 0$ and $X \to \infty$, the call price is

$$\int\int \frac{1}{\sqrt{2\pi}}\frac{1}{X^2}\exp\big(-\frac{(X-\frac{1-\rho^2}{2}A_t-\frac{\rho}{V}(\sigma_t-\sigma_0))^2}{2(1-\rho^2)A_t}\big)g(\sigma_t|A_t)h(A_t)d\sigma_t dA_t$$

Substituting the expression (3) of the conditional distribution $g(\sigma_t|A_t)$ from *Lemma 3*, after some calculations (details are in *Appendix 2*), we prove the *Lemma*.

□ **QED**.

The following two lemmas deal with the case $\rho < 0$. The analysis is mostly the same as the case of positive ρ, so we only emphasize the part which is different between positive and negative ρ, omitting the similar parts.

Lemma 5. When $\rho < 0$ and $A_t \gg 1$,

$$g(\frac{\sigma_t}{V} \in d\eta|A_t) \simeq \Gamma(A_t)\exp\big(-\frac{(\eta+\frac{\rho}{2}A_t)^2}{2A_t}\big)$$

where

$$\Gamma(A_t) = \frac{2\sigma_0\eta}{\sqrt{2\pi A_t}VA_t}/\bar{N}(t) \tag{14}$$

with the normalization factor being

$$\bar{N}(t) = 2\sinh(-\frac{\sigma_0\rho}{2V}).$$

Proof. This result follows from the same calculation as in *Lemma 3* but with different normalization factor $\bar{N}(t)$, The reader should refer to *Appendix 1* for the calculation of the normalization factor $\bar{N}(t)$. □ **QED**.

Lemma 6. When $\rho < 0$, the call price satisfies

$$\frac{1}{\sqrt{2\pi}}\int\frac{1}{X^2}\xi(A_t)h(A_t)dA_t,\ X \gg 1 \tag{15}$$

where

$$\xi(A_t) \equiv (1-\rho^2)\Gamma(A_t)\exp\big(-\frac{1}{2}\frac{(X-(\frac{1}{2}-\rho^2)A_t)^2}{\rho^2(1-\frac{\rho}{2})A_t}\big) \tag{16}$$

$\Gamma(A_t)$ is defined in (14).

Proof. According to *Lemmas 1,2,5*, the call price is

$$C\int\int\frac{1}{X^2}\exp(-\frac{(X-\frac{1-\rho^2}{2}A_t-\rho(\eta-\eta_0))^2}{2(1-\rho^2)A_t})\Gamma(A_t)\exp(-(\frac{\eta+\frac{\rho}{2}A_t}{2A_t})^2)d\eta dA_t$$

Use the formula in *Appendix 2* to integrate $d\eta$ part, we obtain the result.
$\square$**QED**.
Now, we are in the position to prove the following result:
Theorem 7 The implied volatility of deep out-of-the-money calls has the following asymptotic properties:
a)When $\rho > 0$, and $X \to \infty$, $\sigma_{imp}\sqrt{T} \simeq \sqrt{2X}$.
b)When $\rho < 0$, and $\rho^2 \leq \frac{1}{2}$, $X \to \infty$, $\sigma_{imp}\sqrt{T} \simeq \sqrt{2X}$.
c)When $\rho < 0$, and $\rho^2 > \frac{1}{2}$, $X \to \infty$, $\sigma_{imp} \simeq Const \cdot \sqrt{X}$, where $Const = \sqrt{2C+2} - \sqrt{2C}$ with $C = \frac{4}{\rho^2}(\frac{\rho^2-\frac{1}{2}}{1-\frac{\rho}{2}})$. Notice that $Const < \sqrt{2}$.
Proof.
a) It is shown in Appendix 3 that the tail distribution of

$$A_t \equiv \int_0^t \sigma_s^2 ds$$

is log-normal, i.e.,

$$h(A_t) \approx \exp(-C \cdot (\ln(A_t))^2) \qquad \text{as} \qquad A_t \to \infty$$

From *lemma 4* we know that the call price asymptotically satisfies

$$\frac{\rho^2}{8\pi} \int \frac{1}{X^2} \xi(A_t) h(A_t) dA_t$$

where ξ is defined in (12) and (13). Use "steepest descent" technique (Appendix 2) to integrate the above call price. Notice that there is a total square in the integrand ξ defined in (13), and when calculating the "saddle point", the contribution from $h(A_t)$ is small as A_t is large, (note the exponent of $h(A_t)$ is $-(\ln(A_t))^2$.) Therefore, the saddle point is $A_t = \frac{2}{(1-\rho^2)}X$, and the resulted call price is (to leading order of X)

$$\text{Call price} \approx C_2 \exp(-C_1 \cdot (\ln X)^2)$$

where C_1, C_2 are constant depending on T and ρ, V. Compare it with the Black-Scholes asymptotic formula 6 (to leading order of X)

$$\exp(-\frac{(X - \frac{1}{2}\sigma_{imp}^2 T)^2}{2\sigma_{imp}^2 T}) \approx \exp(-C_1 \cdot (\ln X)^2)$$

we get

$$\sigma_{imp} \approx \sqrt{2X}.$$

b) When $\rho < 0$, we have $E[A_t] = \infty$. This is because, from Appendix 3, we have the formula for σ_t:

$$\sigma_t = \frac{M_t}{\frac{1}{\sigma_0} + \frac{\rho}{2} V \int_0^t M_s ds}$$

When $\rho < 0$, and for finite t, the probability that

$$\int_0^t M_s ds \geq \frac{2}{\sigma_0(-\rho)V}$$

is always positive, i.e., σ_t goes to infinity with positive (but small) probability. Therefore, we have $E[A_t] = \infty$.

When we use "steepest descent" technique to integrate (15) the factor $h(A_t)$ doesn't contribute (refer to Appendix 2).

Moreover, when $\rho^2 < \frac{1}{2}$, $A_t = \frac{X}{\frac{1}{2} - \rho^2}$ is the "saddle point", and the exponential part of X disappears. When $\rho^2 = \frac{1}{2}$, the exponential part disappears too. Therefore, we have

$$\frac{(X - \frac{1}{2}\sigma_{imp}^2 T)^2}{2\sigma_{imp}^2 T} \sim Const$$

Therefore,

$$\sigma_{imp}\sqrt{T} \simeq \sqrt{2X}$$

c) Refer to *Lemma 6*, when

$$\rho < 0, \text{ and } \rho^2 > \frac{1}{2}$$

there is not a complete square, the "steepest descent" technique applied to the integral of the call price (15) results in an exponential of the form of

$$\exp(-C \cdot X)$$

where $C = \frac{4}{\rho^2}(\frac{\rho^2 - \frac{1}{2}}{1 - \frac{\rho}{2}})$. Therefore,

$$\frac{(X - \frac{1}{2}\sigma_{imp}^2 T)^2}{2\sigma_{imp}^2 T} \approx C \cdot X$$

We conclude that

$$\sigma_{imp}\sqrt{T} \simeq (\sqrt{2C + 2} - \sqrt{2C}) \cdot \sqrt{X}.$$

□**QED**

Comment: Notice that the asymptotic behaviors are different for the case $\rho > 0$ and $\rho < 0$. This actually suggests the skewness of the "smile" curve when ρ is not equal to 0, i.e., the "smile" curve goes to infinity with different exponents. The argument is the following. When

$$X \equiv \ln(\frac{K}{S}) \to -\infty$$

the same large deviation results we get above is valid for put options. While puts are equivalent to calls with ρ becomes $-\rho$. In fact, in Foreign Exchange market, a put on currency 1/currency 2 is a call on currency 2/currency 1. Therefore, the behavior of "smile" curve when $X \to -\infty$ is the same as that when $X \to \infty$ with opposite sign of ρ. The main result is therefore:

Corollary: Asymptotic behaviors for the implied volatility smile:

1. If $\rho > 0$,
   for deep-out-of-money calls, $\sigma_{imp}\sqrt{T} \to \sqrt{2X}$.
   for deep-out-of-money puts, $\sigma_{imp}\sqrt{T} \to \sqrt{2X}$. when $\rho^2 \leq \frac{1}{2}$
   and $\sigma_{imp}\sqrt{T} \to C \cdot \sqrt{X}$ $(C < \sqrt{2})$ when $\rho^2 > \frac{1}{2}$.

2. If $\rho < 0$,
   for deep-out-of-money calls, $\sigma_{imp}\sqrt{T} \to \sqrt{2X}$. when $\rho^2 \leq \frac{1}{2}$
   and $\sigma_{imp}\sqrt{T} \to C \cdot \sqrt{X}$ $(C < \sqrt{2})$ when $\rho^2 > \frac{1}{2}$.
   for deep-out-of-money puts, $\sigma_{imp}\sqrt{T} \to \sqrt{2X}$.

4 Monte Carlo Study for Near-Money Options

In this section, we use the same risk-neutral stochastic volatility model derived before to study the implied volatility "smile" curve for the near-the-money options. We show how the shape of the "smile" curve changes in terms of the model parameters such as correlation ρ and volatility of volatility V. We also present some results on the term structure of implied volatility.[12]

The methodology we use in this section is Monte Carlo simulation, due to the fact that there is no close-form solution for the risk-neutral stochastic volatility model. The results show that, for zero correlation between the underlying asset and its volatility, one obtains a symmetric "smile" curve, i.e., approximately a centered parabola[13]. For positive correlation, the center of the parabola moves to the left; for negative correlation, the center moves to the right. With other parameters fixed, the bigger the absolute value of ρ, the further the center is moved. Since the observable options are those near-the-money, when ρ is close to 1 or -1, the center of the parabola is further away from at-the-money region, effectively the "smile" curve resembles a line more than a parabola, but is actually the part of a parabola that is far away from the center.

On the other hand, the volatility of volatility , V, has a "centering" effect on the "smile" curve. In other words, with other parameters held fixed, the larger the V, the larger the curvature of the implied volatility, and the center of the parabola returns to the near-the-money region. So for the same correlation ρ, the implied volatility curve with larger V looks like a "smile", while with smaller V the "smile" curve is degenerated to a line.

We use two techniques to reduce the standard deviation of the simulation. One is the antithetic variate, the other is control variate (Hammersley, J., 1964). The antithetic variate is to use one Brownian path along with its mirror path in simulation. The resulting estimate is still unbiased, but with their perfect correlation, the standard deviation is largely reduced.

Control variate is another widely used error deduction technique in Monte Carlo simulations. The idea is to find a variable with which the unknown variable is highly correlated, and which has explicit evaluation formula. One can simulate this variable using the same sample paths as those used for simulating the unknown variable. Effectively, one simulates the difference between two positively correlated random variables. The difference is smaller than the

[12] A more detailed study of the term structure of implied volatility is presented in the second essay.

[13] From the analysis of last section, the "smile" curve is not strictly a parabola out-of-the-money or in-the-money. Nevertheless, in the near-money region, the curve can be well approximated by a parabola.

original variable that is being calculated, accordingly, the standard deviation is smaller. The original simulation is obtained by adding the simulated difference and the theoretical evaluation of the auxiliary variable.

In our simulation, we use the Black-Scholes option price as the control variate. Namely, we simulate the Black-Scholes price with a constant volatility which can be chosen as the initial value of stochastic volatility , using the same Brownian sample path. This reduces the standard deviation greatly.

Figure 1 and 2 exhibit the near-money smile curves corresponding to different model parameters.

Figure 1 shows how the curve changes for different correlation ρ. We observe that, in general, negative correlations correspond to negative skewness, ie, out-of-money puts are more expensive than out-of-money calls; and positive correlations correspond to positive skewness, ie, out-of-money calls are more expensive than out-of-money puts. In particular, strong correlations (positive or negative) corresponds to strong skewness. This can be seen in Figure 1, which compares small correlation coefficient $\rho = \pm 0.2$ with that as large as $\rho = \pm 0.9$.

Figure 2, however, exhibits the different V effect. One can see that the volatility of volatility has the effect of changing the convexity as well as the center of the "smile" curve. In the left panel, where V is 1, what we see is mostly a line (skewness) rather than a "smile" for non-zero correlations. In the right panel, with V equals 3, we see "smiles" with different center rather than a line (skewness).

In Figure 3 and 4, we show the "smile" effect across different time horizons. One can see that for $\rho = \pm 0.2$, the longer the maturity, the higher level and the more convexity for the implied volatility curve. This effect is less for $\rho = \pm 0.9$. This is because for $\rho = \pm 0.9$, the volatility goes to equilibrium or blows up fast.

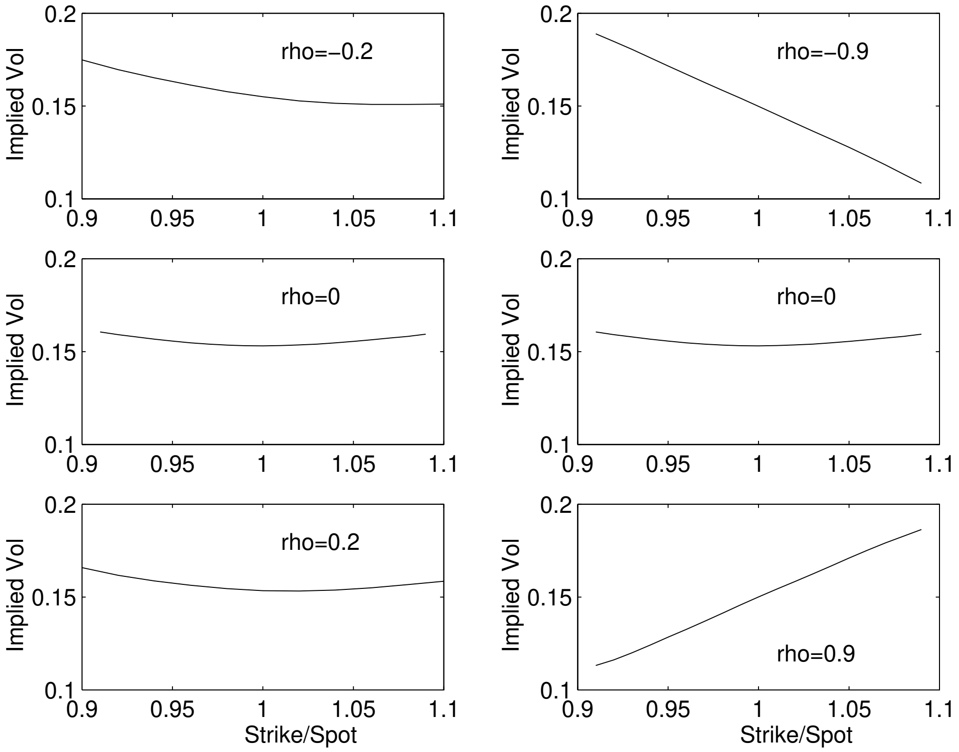

Figure 1: Simulated Smile Curves for Options with Maturity of 120 days. The Volatility of Volatility, V, is Fixed at 1 per annum; the Level of Correlations, ρ, varies from -0.9 to 0.9, as Marked in Each Plot.

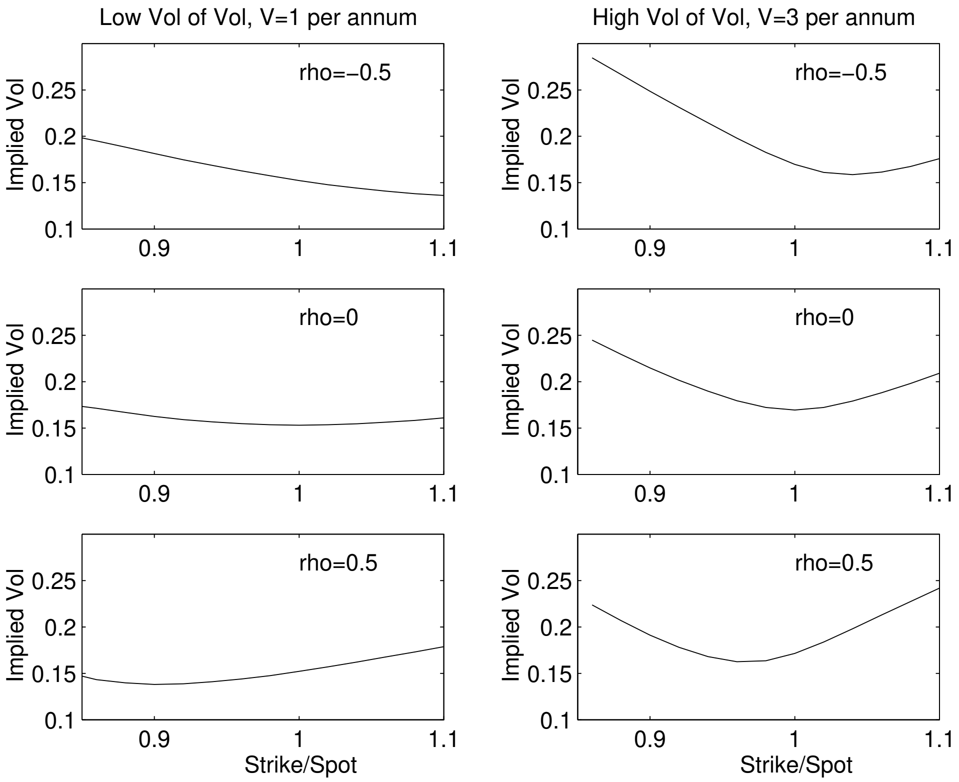

Figure 2: Simulated Smile Curves for Options with Maturity of 120 days. The Volatility of Volatility, V, is Fixed at 1 per annum for the Left Panels, and 3 per annum for the Right Panels; the Level of Correlations, ρ, varies from -0.5 to 0.5, as Marked in Each Plot.

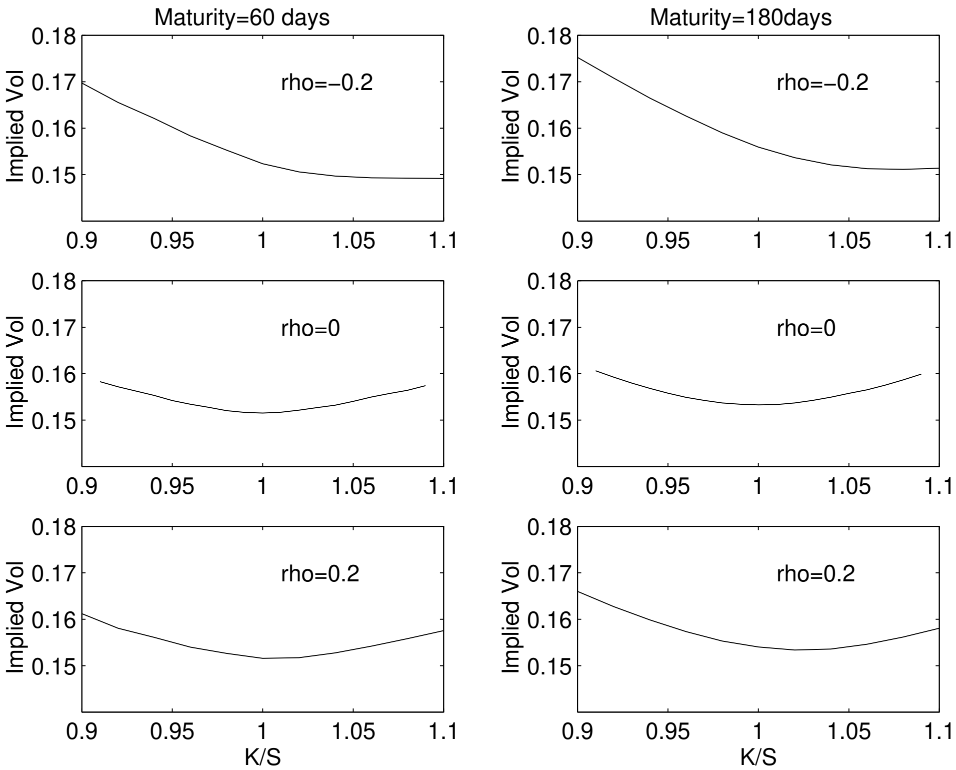

Figure 3: Simulated Smile Curves for Options with Maturity of 60 days and 180 days. The Volatility of Volatility, V, is Fixed at 1 per annum; the Level of Correlations, ρ takes value of ± 0.2 as Marked in Each Plot.

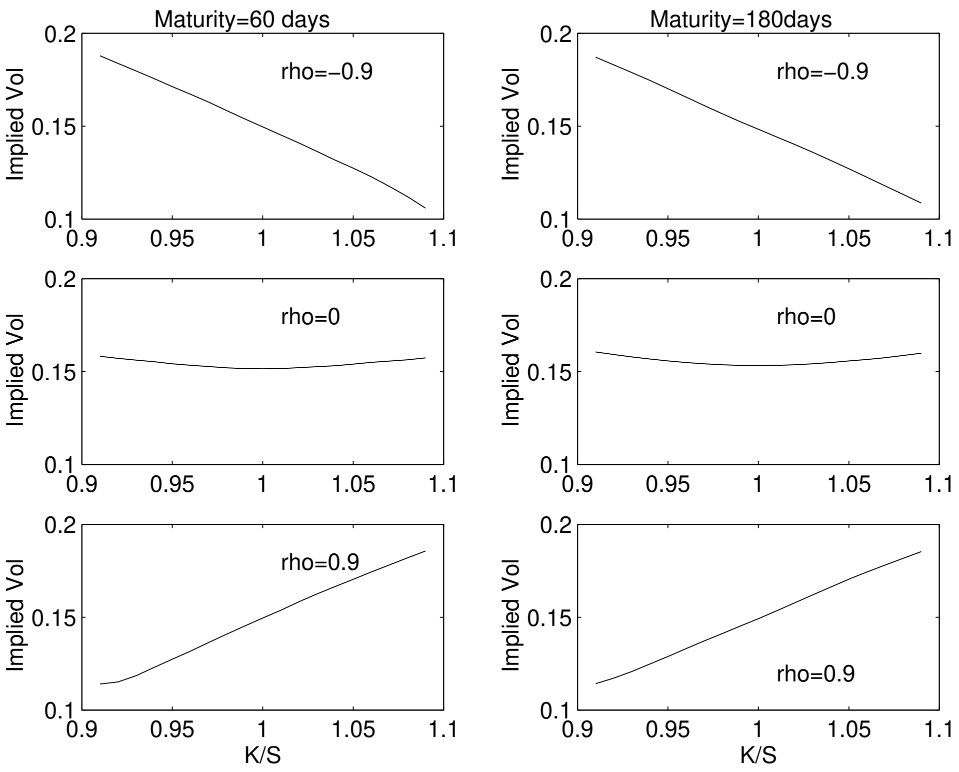

Figure 4: Simulated Smile Curves for Options with Maturity of 60 days and 180 days. The Volatility of Volatility, V, is Fixed at 1 per annum; the Level of Correlations, ρ takes value of ± 0.9, as Marked in Each Plot.

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A Appendix 1

In this *appendix*, we calculate the normalization factor $\bar{N}$ for the conditional probability density of $g(\frac{\sigma_t}{V} \in \eta | A_t)$.

Throughout this appendix, let $\rho' \equiv \frac{\rho}{2}$.

$$\begin{aligned}
& \bar{N}(A_t) \\
= & \int_0^\infty d\eta \frac{1}{\sqrt{2\pi A_t}} \Big(\exp\big(-\frac{(\eta - \frac{\sigma_0}{V})^2}{2A_t}\big) - \exp\big(-\frac{(\eta + \frac{\sigma_0}{V})^2}{2A_t}\big)\Big) \cdot \exp(-\rho'\eta - \frac{1}{2}\rho'^2 A_t) \\
= & \int_0^\infty d\eta \frac{1}{\sqrt{2\pi A_t}} \exp\big(-\frac{(\eta - \frac{\sigma_0}{V} + \rho' A_t)^2}{2A_t}\big) \cdot \exp(-\frac{\sigma_0 \rho'}{V}) \\
& - \int_0^\infty d\eta \frac{1}{\sqrt{2\pi A_t}} \exp\big(-\frac{(\eta + \frac{\sigma_0}{V} + \rho' A_t)^2}{2A_t}\big) \cdot \exp(\frac{\sigma_0 \rho'}{V}) \\
= & \exp(-\frac{\sigma_0 \rho'}{V}) \cdot N\big(-\frac{\sigma_0}{\sqrt{A_t} V} + \rho' \sqrt{A_t}\big) - \exp(\frac{\sigma_0 \rho'}{V}) \cdot N\big(\frac{\sigma_0}{\sqrt{A_t} V} + \rho' \sqrt{A_t}\big)
\end{aligned}$$

From here, it is different for ρ' positive or negative. For $\rho' > 0$,

$$\begin{aligned}
& \bar{N}(A_t) \\
\approx & \exp\big(-\frac{\rho'^2 A_t}{2}\big) \cdot \Big(\frac{1}{\rho' \sqrt{A_t} - \frac{\sigma_0}{\sqrt{A_t} V}} - \frac{1}{\rho' \sqrt{A_t} + \frac{\sigma_0}{\sqrt{A_t} V}}\Big) \\
= & \exp\big(-\frac{\rho'^2 A_t}{2}\big) \cdot \frac{2 \frac{\sigma_0}{\sqrt{A_t} V}}{\rho'^2 A_t - \frac{\sigma_0^2}{A_t V}} \\
= & \frac{2\sigma_0}{\rho'^2 A_t^{\frac{3}{2}}} \exp\big(-\frac{\rho'^2 A_t}{2}\big)
\end{aligned}$$

For $\rho' < 0$, we have

$$\begin{aligned}
& \bar{N}(A_t) \\
\approx & \exp(-\frac{\sigma_0 \rho'}{V}) - \exp(\frac{\sigma_0 \rho'}{V}) \\
= & 2 \sinh(-\frac{\sigma_0 \rho'}{V})
\end{aligned}$$

B Appendix 2

The main tool of our calculation is "steepest descent".

In general, to evaluate integrals of the form

$$I(\alpha) = \int e^{\alpha f(z)} dz \qquad (\alpha \text{ large and positive})$$

observe that for large value of α, the main contribution to the integral comes from the small neighborhood of the maximum points of $f(z)$.
Suppose z_0 is a maximal point of $f(z)$, i.e., $f'(z_0) = 0$. Near the point z_0,

$$f(z) \approx f(z_0) + \frac{1}{2} f''(z_0)(z - z_0)^2$$

The integral is approxmated as

$$\begin{aligned} I(\alpha) &= e^{\alpha f(z_0)} \int e^{\frac{\alpha}{2} f''(z_0)(z-z_0)^2} dz \\ &= C \cdot e^{\alpha f(z_0)} \end{aligned}$$

C is a constant depending on α and $f''(z_0)$.

In our calculation, X is the large parameter. the result is obtained by taking the leading order of X.

When calculating ξ of *Lemmas* 4 and 6, we encountered the integral of the form

$$\int_0^\infty \exp\left(-\frac{(\xi - m_1)^2}{2\sigma_1^2}\right) \cdot \exp\left(-\frac{(\xi - m_2)^2}{2\sigma_2^2}\right) d\xi$$

In this case, steepest descent method actually gives the exact result. Observe that the maximal point of the integrand is $\xi_0 = \frac{\frac{m_1}{\sigma_1^2} + \frac{m_2}{\sigma_2^2}}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}$. The integral can be written as:

$$\begin{aligned} & \exp\left(-\frac{(m_1 - m_2)^2}{2(\sigma_1^2 + \sigma_2^2)}\right) \cdot \frac{1}{\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}} \cdot N_1\left(-\frac{\frac{m_1}{\sigma_1^2} + \frac{m_2}{\sigma_2^2}}{\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}}\right) \\ \approx\ & \frac{1}{\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}} \cdot \exp\left(-\frac{(m_1 - m_2)^2}{2(\sigma_1^2 + \sigma_2^2)}\right) \qquad \text{as} \qquad \frac{m_1}{\sigma_1^2} + \frac{m_2}{\sigma_2^2} > 0 \\ \approx\ & \frac{1}{\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}} \cdot \exp\left(-\frac{1}{2}\left(\frac{m_1^2}{\sigma_1^2} + \frac{m_2^2}{\sigma_2^2}\right)\right) \qquad \text{as} \qquad \frac{m_1}{\sigma_1^2} + \frac{m_2}{\sigma_2^2} < 0 \end{aligned}$$

while in our context,

$$\begin{aligned} m_1 &= \frac{X - \frac{1-\rho^2}{2} A_t}{\rho} & \qquad \sigma_1^2 &= \frac{1-\rho^2}{\rho^2} A_t \\ m_2 &= -\frac{\rho}{2} A_t & \qquad \sigma_2^2 &= A_t \end{aligned}$$

In general, for integral of the form:

$$\int_0^\infty \exp\left(-\frac{(M - \xi)^2}{2\xi^2}\right) \cdot \exp(-(\ln \xi)^\beta) d\xi$$

when M is large. To find the saddle point, set the derivative of the exponent to zero, we get

$$-\frac{M^2}{2\xi^2}+\frac{1}{2}+\frac{\beta\ln\xi}{\xi}=0.$$

In this case, the saddle point is approximately $\xi = M$, because when M is large, the third term on the left hand side is about zero. Therefore, the integral, to leading order, is

$$C\cdot\exp(-(\ln M)^\beta)$$

More generally, if we have an integral of the form

$$\int_0^\infty \exp\big(-\frac{(M-\xi)^2}{2\xi^2}\big)\cdot\exp(-f(\xi))d\xi$$

when M is large, and $f(\xi)$ is of the form

$$f(\xi)=\xi^\beta$$

with $\beta < 1$, then the saddle point is $\xi = M$. And the integral is

$$C\cdot\exp(-f(M)).$$

If $\beta = 1$, the integral is of the form

$$C\cdot\exp(C'\cdot M).$$

All the integral we used in Section (1.3) can be transformed into the one of the forms of the above.

C Appendix 3

In this appendix, we show that the tail distribution of

$$A_t\equiv\int_0^t\sigma_s^2ds,$$

for positive $\rho'\equiv\frac{\rho}{2}$, is log-normal.

Lemma 1. Let σ_t be the risk-neutral stochastic volatility, i.e., σ_t satisfies the SDE:

$$d\sigma_t=-\rho' V\sigma_t^2dt+V\sigma_t dZ_t,$$

then

$$\sigma_t=\frac{M_t}{\frac{1}{\sigma_0}+\rho' V\int_0^t M_s ds} \tag{17}$$

Proof.
Differentiate (17), and notice that

$$dM_t = V \cdot M_t dZ_t,$$

we get

$$\begin{aligned} d\sigma_t &= \frac{-M_t}{(\frac{1}{\sigma_0} + \rho' V \int_0^t M_s ds)^2} \cdot \rho' V M_t dt + \frac{dM_t}{\frac{1}{\sigma_0} + \rho' V \int_0^t M_s ds} \\ &= -\rho' V \sigma_t^2 + V \sigma_t dZ_t \end{aligned}$$

i.e., σ_t satisfies the original SDE.
Lemma 2. Tail distribution of $\int_0^t M_s ds$ is a log-normal distribution.
Proof.
Let M be a large number, we have the following estimations:

$$P[\int_0^t \exp(Z_s) ds > M] \leq P[C_1 \exp(\int_0^t Z_s ds) > M]$$

according to Jensen's inequality, and we have

$$P[\int_0^t \exp(Z_s) ds > M] \geq P[C_2 \exp(Z_{max})(t) > M]$$

where $Z_{max}(t)$ is the maximum of Brownian motion between 0 to t. Since $\int_0^t Z_s ds$ is normally distributed, the first inequality tells us that the tail of $\int_0^t \exp(Z_s) ds$ is no fatter than a log-normal distribution. Moreover, since distribution of the maximum of Brownian Motion is the same as the absolute Brownian Motion, the tail of which is log-normally distributed. Therefore, the second inequality shows that the the tail of $\int_0^t \exp(Z_s) ds$ is no "thinner" than a log-normal distribution. We conclude that the tail of $\int_0^t \exp(Z_s) ds$ is a log-normal distribution, and so is $\int_0^t M_s ds$, because one can always find two constants D_1 and D_2 (depending on t) such that:

$$D_1 \cdot \int_0^t \exp(Z_s) ds \leq \int_0^t M_s ds \leq D_2 \cdot \int_0^t \exp(Z_s) ds.$$

From Lemmas 1 and 2, we now show that the tail distribution of

$$\sigma_t = \frac{M_t}{\frac{1}{\sigma_0} + \rho' V \int_0^t M_s ds}$$

is log-normal. The argument is the following.

a) Because ρ' and V are positive, and M_t is the positvie exponential Martingale, we have

$$\sigma_t \geq \sigma_0 M_t,$$

so the tail of σ_t is no fatter than a log-normal.
b) Since the tail of $\int_0^t M_s ds$ is also log-normal, (from *lemma 1*), one can construct a random variable ξ such that

$$\xi = M, \qquad \text{when} \int_0^t M_s ds \leq M,$$

$$\xi = \int_0^t M_s ds \qquad \text{when} \int_0^t M_s ds \geq M.$$

where M is a large number. Apparently,

$$\sigma_t \geq \frac{M_t}{\frac{1}{\sigma_0} + \rho' V \xi},$$

and the latter has tail distribution of log-normal. Hence, the tail of σ_t is no thinner than a log-normal. So it is log-normal.

Finally, we claim that the tail of $\int_0^t \sigma_t^2$ is also a log-normal. Since the tail of σ_t is log-normal, so is that of σ_t^2. If we can conclude that a tail of the summation of two log-normal r.v.'s is log-normal, by using induction, we can prove that the tail of $\int_0^t \sigma_t^2$ is log-normal. The next lemma is to show that the sum of log-normal has tail of log-normal.
Lemma 3. Let X and Y be two log-normal r.v.'s. Then the tail of $Z = X + Y$ is log-normal.
Proof. Let $f(Z)$ be the probability density function of Z. Then

$$\begin{aligned} f(Z) &\approx \int \exp(-(\ln X)^2) \cdot \exp(-(\ln(Z-X))^2) dX \\ &= \int \exp(-[(\ln X)^2 + (\ln(Z-X))^2] dX \end{aligned}$$

When Z is large, we use the "steepest descent" technique to integrate it. By differentiate the exponent

$$(\ln X)^2 + (\ln(Z-X))^2$$

we get

$$\frac{2\ln X}{X} - \frac{2\ln(Z-X)}{Z-X}$$

Set it to zero, we find the saddle point $X = Z$. Therefore, the tail distribution of Z behaves as

$$\exp(-2(\ln(\frac{Z}{2}))^2)$$

which is log-normal.