MATHEMATICS OF FINANCE I and II: SYLLABUS

This is a proposal for a two-semester course in Mathematical Finance. The course presents a theoretical perspective on the valuation of financial instruments, focusing on derivative securities, their risk-management and, more generally, on derivatives-related problems in quantitative Finance. The goal is to introduce students to the ample spectrum of techniques constituting the current "toolkit" of derivative valuation. This will be done progressively, in two semesters. In the first part, we introduce the main issues using discrete, tree-type models and elementary probability concepts. The second semester will deal with topics that require greater mathematical depth. This latter part covers continuous- time Finance and, in particular, an extensive discussion of the Black-Scholes-Merton ideas. Also,we will use stochastic differential equations to introduce forward-rate models for pricing and hedging interest-rate options. The implementation of continuous-time models through discretization and the use of Monte-Carlo techniques for pricing path-dependent derivatives will be one of the main focuses.

(Note: Students will be required to program valuation models in both courses)

Mathematics of Finance I

Prerequisite: Elementary probability and discrete time Markov chains.
(Each item approximately one lecture)

1. *Options, futures and derivative securities.* Overview of fundamental concepts in securities markets: equities, currencies, commodities and debt markets. Introduction to derivative products: forward contracts, futures, warrants and options. Exchange-traded and OTC products. Hedging using forwards and futures. Swaps. Discussion of cost-of-carry, marking-to-market, financing forward positions. Common hedging strategies using options and futures.

2. *Basic arbitrage Ideas.* Futures and forwards arbitrage. Put-call parity. American versus European-style contracts. Variants of the put-call parity formula for options on stock-indices and futures. Application to interest-rate arbitrage using listed options.

3. *Introduction to arbitrage pricing theory.* The Arrow-Debreu model for a securities market with finite state-space. Hedged portfolios and arbitrage. The Fundamental Theorem: equivalence between the absence of arbitrage opportunities and the existence of state-contingent prices, or martingale measures. Market completeness and the existence of replicating portfolios for derivative securities. Example: one period binomial and multinomial models.

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4. *The binomial option pricing model.* Basic assumptions. Pricing European-style contingent claims in a multi-period setting. Calibration of the binomial model: volatility. Coarse-graining the binomial tree: the Cox-Ross- Rubinstein derivation of the Black-Scholes formula.

5. *Refinements of the binomial model.* Term-structure of interest rates. Term-structure of volatility. Derivatives securities on dividend-paying assets. Options on futures. Derivation of the Black 76 formula by coarse-graining the binomial model. Examples of valuation of derivative securities with multiple, index-contingent cash-flows: equity- and commodity- linked notes.

6. *The Black Scholes analysis.* Systematic study of the Black-Scholes formula and its sensitivity on the model's parameters. Delta, Gamma, Vega, Rho. Applications to hedging strategies. Implied and historical volatilities. Discrepancies between market prices and Black-Scholes: the "smile" curve for implied volatility.

7. *American options and other instruments with early-exercise Features.* Rational exercise policy. Non-linear difference equations for instruments with early-exercise features. American put. American options on indices and futures. Introduction to pricing callable and convertible bonds.

8. *Exotic options I.* Taxonomy of exotic options and some of their main applications. Financial engineering of cash-flow structures. Static hedges. Baskets, spreads and range-forwards. Capped options. Out-performance options (discrete analysis). Binary options. Barrier options: knock-ins and knock-outs. Hedging derivative securities with options portfolios (work of Derman & Kani).

9. *Exotic options II.* Chooser options, forward start options. Lookback options. Asian Options. Introduction to Monte-Carlo evaluation of exotics on the binomial tree.

10. *Stochastic volatility.* The concept of volatility risk. The trinomial tree. Implementation of local volatilities and interest rates depending on the spot price and time-to-expiration. Random volatility models. Discrete Ornstein-Uhlembeck model. Hedging volatility risk. Dominating strategies on the trinomial tree and non-linear difference equations. Diversification of volatility risk using derivative securities.

11. *Implied martingale measures.* The "inverse problem" of Mathematical Finance: incorporating derivative asset prices to calculate the forward pricing measure. Rubinstein's implied tree for S&P 500 options.

12. *Introduction to interest-rate sensitive securities.* Pricing the term-structures of interest rates. Adjusting the local volatility in interest-rate models. The Ho-Lee model and applications. Mean reversion of forward rates. Arbitrage-free motions of forward curves. Analysis of other discrete binomial models and applications to pricing interest rate caps, floors and swaptions.

Mathematics of Finance II

Prerequisite: (possibly corequisite) A more advanced course in probability that covers Wiener Measure, quadratic variation, Donsker's invariance principle (functional Central Limit Theorem), the stochastic integral and its financial interpretation, Itô's Lemma for Brownian Motion, Gauss-Wiener measure and the Ornstein-Uhlembeck process, diffusion processes, the generalized Itô Lemma (diffusions), the Feynman-Kac formula, and Girsanov's Theorem.

(Each item approximately one lecture)

1. *From finite-difference equations to PDEs.* Analysis of finite-difference equations in the limit of many trading periods. Derivation of the Black-Scholes PDE for European-style derivatives and the PDE for American-style options.

2. *Arbitrage pricing theory in continuous-time finance.* Martingale measures. The case of Geometric Brownian Motion. The Harrison-Pliska theory; exponential martingales.

3. *The Black-Scholes analysis in continuous time.* Riskless portfolios, replication. Risk-neutral valuation. Option formulas revisited. Analysis of the optimal exercise boundary for American-Style options. Applications to options on indices and index futures. Early-exercise premium and approximations. Portfolio insurance.

4. *Exotic options.* Barrier options and Boundary-value problems. Stopping times. Closed-from solutions. Parity relations. Two-asset options, correlations and Margrabe's formula. Barriers depending on a second traded asset. Lookback options.

5. *Arbitrage pricing and the market price of risk.* Assets depending on non-traded factors. Connections with exponential martingales. Introduction to stochastic interest-rate modeling: Vasicek's one factor model. How useful is the notion of market price of risk?

6. *Merton's optimal consumption theory.* Utility-based consumption-investment optimization.

7. *The yield curve.* Zero-Coupon Bonds, Eurodollar futures, Treasuries. Swap Pricing. Yields, forward rates, stochastic spot rate. Expectations hypothesis, term premium and other economic factors determining the shape of the yield-curve. Financial options: options on bonds, caps, floors, swaptions and their basic relations (cap-floor-forward parity, etc.)

8. *One-factor models.* The Hull-White extended Vasicek model. The Cox-Ingersoll-Ross model. Emphasis on yield-curve Cap valuation. Jamshidian's formulas.

9. *Two-factor models.* The Brennan-Schwartz and Longstaff-Schwartz models. The role of the market price of risk.

10. *Heath-Jarrow Morton theory.* Derivation of the Heath-Jarrow Morton arbitrage-free equation for the stochastic interest rate. Admissible models for the movements of forward rates. Examples and comparisons with one factor models.

11. *Implementation of arbitrage-free models on probability trees.* Fitting the term-structure of interest rates and forward-rate volatilities. Practical implementation of arbitrage-free models and derivative security valuation.

12. *Beyond the Black Scholes analysis.* Discrete vs. "continuous" hedging of option portfolios. Hedge-slippage. Transaction costs. Leland's theory for option valuation in the presence of transaction costs. Non-linear PDEs for managing option portfolios in the presence of transaction costs. Study of variable time-step strategies. Hedging according to price movements and Gammas. Hedging volatility risk: Vega-based and synthetic hedges.

Structure of the Course & Requirements

The course will meet once a week for a two-hour period. The students will be required to perform a series of assignments, consisting of valuation problems. Typically, these assignments will involve programming the problem on a workstation, performing simulations of future market scenarios, making risk diagnostics, etc.

Reading List

Books:

J. Hull, *Options, Futures and Other Derivative Securities*, Prentice Hall, 1993.

J. Cox, M. Rubinstein, *Options Markets*

S. R. Marki, *Derivative Financial Products*, Harper-Collins, 1990

Figlewski, W. L. Silber, M. G. Subhramanyam, *Financial Options: From theory to practice*, NYU Solomon Center, Irwin, 1990

D. Duffie, *Dynamical Asset Pricing Theory*, Princeton Univ. Press, 1992.

Papers & Lecture Notes :

F. Black & M. Scholes, "The pricing of options and corporate liabilities", *Journ. Political Economy*, 81 (1973)

J. M. Harrison and D. Kreps, "Martingales and Arbitrage in Multiperiod Securities Markets", *J. Econ. Theory*, 1979, pp. 381-408.

M. Rubinstein, "Exotic Options", Finance Working Paper 220, Haas School of Business, University of California Berkeley, December 1991

M. Rubinstein, "Implied Binomial Trees", *J. of Finance*, July 1994.

M. Avellaneda, course notes, Courant Institute and Buenos Aires Stock Exchange

M. Avellaneda and A. Parás, "Dynamical hedging portfolios for derivative securities in the presence of transaction costs", *Appl. Math. Finance*, Dec. 1994

A. Levy, M. Avellaneda and A. Parás, "A new approach for pricing and hedging derivative securities in markets with uncertain volatility", Courant Institute working paper, 1994

M. Avellaneda, A. Levy and A. Parás, " Pricing and hedging derivative securities in markets with uncertain volatility", Working paper, 1995,

M. Avellaneda, A. Levy and A. Parás, "Non-linear diffusion equations and risk-diversification in derivatives markets", Fields Institute Seminar lecture, Toronto, Feb. ,1995

L. Scott, "Option Pricing when the Variance Changes Randomly: Theory, Estimation and Application", *Journ. of Financial and Quantitative Analysis*, 22, 419-438 (1987)

E. Derman and I. Kani, "Riding on a Smile", *RISK*, Vol.7/No.2, February 1994

B. Dupire, "Pricing with a Smile", *RISK*, Jan.,1994

F. Black, E. Derman and W. Toy, "A one-factor model of interest rates and its application to Treasury bond options", *Financial Analysts Journal*, 46, 33-39, 1990.

F. Black and P. Karasinski, "Bond and Option Pricing when Rates are Lognormal", *Financial Analysts Journal*, 47, 52-59 1991

O. A. Vasicek, "An equilibrium characterization of the term-structure", *Journal of Financial Economics*, 5, Nov 1977, 177-188

J. Hull and A. White, "One-factor interest-rate models and the valuation of interest-rate derivative securities", *Journal of Financial and Quantitative Analysis*, 28 (2) 1993, p. 235

J. C. Cox, J. E. Ingersoll and S.A. Ross, "A theory of the term structure of interest rates", *Econometrica*, 53, 1985, p. 385-407

D. C. Heath, R.A. Jarrow and A. Morton, "Bond pricing and the term structure of interest rates: A new methodology for contingent claim valuation", *Econometrica*, 60, 1992, p.77-105