DYNAMIC HEDGING WITH TRANSACTION COSTS: FROM LATTICE MODELS TO NONLINEAR VOLATILITY AND FREE-BOUNDARY PROBLEMS

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Abstract. We study the dynamic hedging of portfolios of options and other derivative securities in the presence of transaction costs. Following Bensaid, Lesne, Pagés & Scheinkman (1992), we examine hedging strategies which are risk-averse and have the least initial cost, in the framework of a multiperiod binomial model. This paper considers the asymptotic limit of the model as the number of trading periods becomes large. This limit is characterized in terms of nonlinear diffusion equations. If $A = k/(\sigma\sqrt{dt}) < 1$ (k is the roundrip transaction cost, σ is the volatility and dt is the lag between trading dates), the optimal cost approaches the solution of a nonlinear Black-Scholes-type equation in which the volatility is dynamically adjusted upward to $\sigma\sqrt{1+A}$ or downward to $\sigma\sqrt{1-A}$ according to the local convexity of the solution. For $A \geqq 1$, the upward adjustment is similar but the downward adjustment assigns zero nominal volatility to the underlying asset for long-Gamma positions. In the latter case, the optimal cost function is the solution of a free-boundary problem. We also characterize the associated hedging strategies. We shown that if $A < 1$, it is optimal to replicate the final payoff via "nonlinear Delta hedging". On the other hand, if $A \geq 1$, the optimal strategies are path-dependent, non-unique and typically super-replicate the final payoff.

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1. Introduction and Statement of the Main Results

The problem of accounting for transaction costs in the dynamic hedging of derivative securities has attracted considerable attention from theoreticians and practitioners. S-trategies which account for transaction costs and pricing models that incorporate the costs into the premium have considerable practical interest, especially for trading derivatives in markets with moderate or low liquidity.

The problem can be formulated in terms of an agent that buys and sells options on the stock of a company. At some point in time, he or she decides to hedge the "book", or options portfolio, against future price fluctuations. The agent would like to determine the least costly strategy, taking into account the projected transaction costs due to dynamic hedging. The initial cost of such strategy can be interpreted as the minimal capital reserve needed to protect the portfolio against future market moves.[1]

We shall make the assumption that the agent is *totally risk-averse*: only strategies which ensure non-negative cash-flows after closing all positions are deemed admissible. This assumption is important in the framework of this paper but by no means necessary. Several theorists have considered hedging strategies which allow for losses but maximize a utility function assigned to the agent (Constantinides (1979,1986), Hodges and Neuberger (1989), Davis, Panas and Zariphopolou (1993)). In principle, a utility-based approach may present greater flexibility[2] but has the disadvantage of producing utility-dependent results. For this and other reasons, total risk-aversion plays a central role in the assessment of transaction costs in derivative strategies.

Risk-averse hedging in the presence of transaction costs was first considered by Leland (1985) for the log-normal model and later by Boyle and Vorst (1992) for the binomial lattice. Both papers consider only the problem of hedging a single option. A non-trivial generalization of the Leland model applicable to option portfolios and exotic options was proposed later by Hoggard, Whalley and Wilmott (1993), based on dynamic replication of the payoff. However, Boyle and Vorst and Hoggard *et al.* both observed that replication may not always be feasible since it can lead to infinite option prices when transaction costs exceed a critical value and the agent is long Gamma (as in the case of a long call position).

Avellaneda and Parás (1994) examined this issue, which is related to the mathematical ill-posedness of the replication equation. They proposed an explanation for the ill-posedness, which points to a fundamental difference between long and short options positions in markets with large bid-offer spreads. The risk-averse *seller* of an option is obliged to dynamically hedge his or her exposure in the cash market, regardless of transaction

[1] Throughout this paper, we consider only dynamic hedging with shares of stock and a money-market account. Of course, in practice, traders also hedge their books with other derivative securities to (among other things) diminish the transaction costs. We assume that the agent has already taken a definite position in the derivatives market and seeks to hedge the "residual" exposure with a position in the cash market.

[2] The utility framework contains risk-aversion as a special case in which losses are assigned "infinitely negative utility".

costs. On the other hand, the *buyer* of the option risks only the initial premium: hedging is done primarily to offset the time-decay in the option's value. It is intuitively clear (and can be proved mathematically) that if transaction costs in the cash market are sufficiently large, Delta-hedging to offset time-decay is impossible. This key observation applies also to option portfolios: if transaction costs are high and volatility is low, Delta-hedging a position which is long-Gamma is counterproductive. In such situations, a temporary holding strategy is preferable, since it does not carry immediate market risk as long as capital reserves are sufficiently high. Based on this observation, Avellaneda and Parás proposed a new scheme for pricing and hedging option portfolios which is based on solving an *obstacle problem* for Leland's volatility-adjusted PDE. The corresponding dynamic hedges are non-Markovian (path-dependent) and *dominate* – or super-replicate – the final payoff.

The aforementioned strategies are based on replication or super-replication. This raises the question of characterizing strategies which *minimize the initial cost of hedging.* How are the two concepts related ? The least-initial-cost formulation was first considered by Bensaid, Lesne, Pagès and Scheinkman (1992) (BLPS) in the framework of a finite-period binomial tree, using a a dynamic programming algorithm (see Section 2, equation 2.10). Due to dependency on the previous stock holdings, the BLPS optimal hedging strategies are path-dependent, with some notable exceptions. For instance, in the case of a *short* option settled in shares, Bensaid *et al.* showed that it is always optimal to replicate the payoff. This result has an interesting consequence: the BLPS least-initial-cost strategy for a short option corresponds, in the asymptotic limit of many trading periods, to a Black-Scholes pricing formula with volatility adjusted upwards to reflect transaction costs, analogous to the Boyle and Vorst (1992) formula.[3]

The least-initial-cost pricing and hedging of complex options portfolios is the main subject of this paper. Our approach consists in characterizing the solutions of the BLPS algorithm in the limit of infinitely many trading periods for general contingent claims. We show that the algorithm admits a simple interpretation in this asymptotic limit. In fact, the optimal initial costs satisfy nonlinear partial differential equations analogous to those proposed by Hoggard *et al.* and by Avellaneda and Parás, with minor changes in the values of the adjusted volatilities that reflect the difference between the normal and binomial statistics. As a consequence of this result, we determine precisely in which instances the BLPS algorithm gives rise to replicating strategies and when the optimal strategies are path-dependent and dominating. We also show that whenever path-dependency holds for BLPS, the hedging strategies are essentially analogous to those proposed by Avellaneda and Parás (1994) for the log-normal model. Thus, path-dependency occurs when (i) transaction costs exceed a critical level and (ii) the agent is long-Gamma for some level of the spot price.

1.1 Overview of the results

Following Bensaid, Lesne, Pagès and Scheinkman (1992), we formulate the problem in

[3] This is also the analogue of the Leland (1985) for the binomial tree.

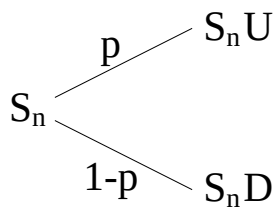
terms of the binomial probability model. In this framework, there are $N+1$ trading dates, the last one being the expiration date. The lag between successive trading dates is fixed. The impact of imperfect liquidity is modeled through a bid-offer spread in the cash market, assuming that the agent will buy stock at the offer and sell at the bid. The bid/offer prices for one share of stock are defined as

$$S_n^{bid} \;=\; S_n \left(1 - \frac{k}{2}\right) \qquad \text{and} \qquad S_n^{ask} \;=\; S_n \left(1 + \frac{k}{2}\right) \;,$$

where S_n is the average between the bid and offer and k represents the (percentage) round-trip transaction cost for buying/selling stock, i.e.,[4]

$$k \;=\; \frac{S_n^{ask} \;-\; S_n^{bid}}{S_n} \;.$$

For simplicity, we assume that k is constant. The change in the average stock price over a single period is modeled by a two-state random variable



with probabilities p and $1-p$ for upward and downward moves. We also assume that the dollar return for lending/borrowing over one trading period is R (constant), and that U, D and R satisfy the "pure" no-arbitrage condition

$$D \;<\; 1 + R \;<\; U \;.$$

A key feature of this model is given in

Definition 1.1. *The stock is said to be* risk-feasible *if there is a positive probability of posting a profit by following either one of the following strategies*:

(i) *Borrow money to purchase a share of stock; hold the stock for one trading period; sell the stock and pay back the loan.*

(ii) *Short-sell a share of stock; deposit the proceeds in a money market account for one period; close the account and unwind the short position after the period.*

[4] Thus, an agent who buys and immediately sells one share of stock assumes a loss of $k \cdot S_n$ dollars.

Mathematically, conditions (i) and (ii) are satisfied if and only if

$$D\left(\frac{1+k/2}{1-k/2}\right) \;<\; 1+R \;<\; U\left(\frac{1-k/2}{1+k/2}\right)\,.$$

The risk-feasibility of an asset depends on how much its price is expected to oscillate over a single trading period in relation to the round-trip transaction costs. As we shall see, the BLPS optimal strategies are very different according to whether risk-feasibility holds or not. If the stock is risk-feasible, replicating strategies are always optimal for arbitrary contingent claims. Otherwise, replication may not be optimal.

To illustrate the sub-optimality of replicating strategies, we consider a simple example.

Example 1.2 Assume there is a single trading period. An agent wishes to hedge a short position in a hypothetical claim contingent on the value of a *non-risk-feasible* stock. Assume that this claim pays \$1 in the "up" state and \$0 in the "down" state. Consider first the case in which the agent has no endowment in shares. Opening and subsequently closing any position in shares after one period will result in an overall loss under both states of the world. Hence, it follows that the initial hedging cost of any replicating portfolio using shares and an interest-bearing money-market account must exceed the present value of \$1 (the maximum liability). On the other hand, simply holding the present value of \$1 (i.e., $\$(1+R)^{-1}$) in cash constitutes a (cheaper) dominating riskless strategy. Notice, however, that the situation is different if either (i) the agent has an endowment in shares or (ii) if the contingent claim is settled in shares rather than in cash. In such cases, a replicating strategy may be cheaper than $\$(1+R)^{-1}$ because transaction costs are lower for the agent. As a matter of fact, the reader can verify that it is optimal to replicate when the agent's endowment in shares exceeds $1/[S_0(1-k/2)(U-D)]$ shares in Case (i) and $1/[S_0(U-D)]$ in Case (ii) (See Section 3). This suggests that, aside from the risk-feasibility of the underlying cash instrument, the agent's endowment and the form of payment may determine whether the it is optimal to replicate or not.

For a binomial model with a large number N of trading periods per year — and hence with small duration between successive trading dates $dt = T/N \ll 1$ — risk-feasibility is equivalent to

$$A \;\equiv\; \frac{k}{\sigma\sqrt{dt}} \;<\; 1\,, \tag{1.1}$$

where σ is the annualized stock volatility. We shall refer to A as the *Leland number*.

Table 1.1 shows that the Leland number takes moderate values (between 0 and 10) for standard values of volatility, time-lag and transaction costs (volatility of 20%, time-lags ranging from one week to four times a day and round-trip costs of up to 6%). From this table, we draw the heuristic conclusion that if the time-to-expiration is sufficiently long, then (i) the minimum lag between trades can be taken to be small enough to warrant a

dt \ k/2	0.01	0.02	0.03
one week $1.9*10^{-2}$	0.72	1.44	2.17
half week $9.6*10^{-3}$	1.02	2.04	3.06
one day $3.8*10^{-3}$	1.61	3.23	4.84
half day $1.9*10^{-3}$	2.28	4.57	6.85
quarter day $9.6*10^{-4}$	3.23	6.46	9.69

TABLE 1.1. Leland number A for different values of the minimum lag between trading dates dt (from one week to $1/4$ day) with round-trip transaction costs between 2% and 6% and volatility of 20%.

continuum description of the system, and (ii) at the same time, A can be considered to be finite. The interesting asymptotic regime of parameters for the asymptotic analysis of the BLPS algorithm is, in fact, $dt \ll 1$ and $A = O(1)$.[5]

Under these assumptions, the asymptotic analysis of the BLPS algorithm in the limit $N \nearrow \infty$ yields the following results:

(i) *If the underlying asset is risk-feasible, the minimal initial cost of risk-averse hedging a European-style contingent claim with payoff value $F(S)$ and expiration date T approaches uniformly to the solution of the nonlinear diffusion equation*

$$\frac{\partial P}{\partial t} + \frac{1}{2} S^2 \widehat{\sigma}^2 \left[\frac{\partial^2 P}{\partial S^2}\right] \cdot \frac{\partial^2 P}{\partial S^2} + r\left(S \frac{\partial P}{\partial S} - P\right) = 0 , \tag{1.2}$$

with

$$P(S,T) = F(S) ,$$

[5] The regime $dt \ll 1$, $A \gg 1$, which might be considered relevant when dt is small, gives rise to trivial buy/hold strategies. The BLPS cost function for this regime is recovered by first letting $dt \nearrow \infty$ and then taking the limit $A \gg 1$ in the asymptotic equations (1.4) derived hereafter (cf. Avellaneda and Parás (1994)).

where

$$\widehat{\sigma}^2 \left[\frac{\partial^2 P}{\partial S^2}\right] \;=\; \sigma^2 \left(1 + A \,\mathrm{sign}\left[\frac{\partial^2 P}{\partial S^2}\right]\right) \;. \tag{1.3}$$

Here, r represents the annualized interest rate. Notice that, since the Leland number is less than unity, the nonlinear parabolic equation (1.2) is well-posed. The optimal hedging strategy corresponds to maintaining a hedge-ratio of $\frac{\partial P}{\partial S}$ shares of stock at each trading date.

(ii) *If the underlying asset is not risk-feasible, the minimum initial cost for hedging a European-style contingent claim with payoff value $F(S)$ expiring at time T can be approximated as $N \nearrow \infty$ by the solution of the "obstacle problem":*

$$P(S,t) \;\geq\; e^{-r(T-t)}\, F(e^{r(T-t)}\, S) \;,$$

$$\frac{\partial P}{\partial t} \;+\; \frac{1}{2}\, S^2\, \sigma^2 \;(1+A) \cdot \frac{\partial^2 P}{\partial S^2} \;+\; r\left(S\,\frac{\partial P}{\partial S} \;-\; P\right) \;\leqq\; 0$$

for all (S,t) and

$$\frac{\partial P}{\partial t} \;+\; \frac{1}{2}\, S^2\, \sigma^2 \;(1+A) \cdot \frac{\partial^2 P}{\partial S^2} \;+\; r\left(S\,\frac{\partial P}{\partial S} \;-\; P\right) \;=\; 0$$

$$if \quad P(S,t) \;>\; e^{-r(T-t)}\, F(e^{r(T-t)}\, S),$$

with final condition

$$P(S,T) \;=\; F(S) \;. \tag{1.4}$$

This obstacle problem reduces to a linear PDE analogous to Leland's equation if F_{SS} is positive for all S. Otherwise, the problem splits the (S,t)- plane into two regions: one where the PDE is satisfied, and another, "the contact set", in which the value function coincides with the "obstacle" $e^{-r(T-t)}\, F(e^{r(T-t)}\, S)$ and the nominal volatility is zero. The corresponding dynamic hedging strategies alternate between delta-hedging and holding periods without hedge adjustments. The times at which the agent switches from delta-hedging to static hedging are determined by the position of the spot price with respect to the contact set and by the Delta of the agent's portfolio.

In simple words, the asymptotic analysis of the BLPS algorithm shows that the impact on the total hedging cost arising from bid-offer spreads can be taken into account by adjusting the volatility upward to $\sigma\sqrt{1+A}$ when the agent is short "nonlinear Gamma" ($\partial^2 P/\partial S^2 \;>\; 0$) and adjusting the volatility downward to $\sigma\sqrt{1-A}$ or to zero when the

Short Call Position.

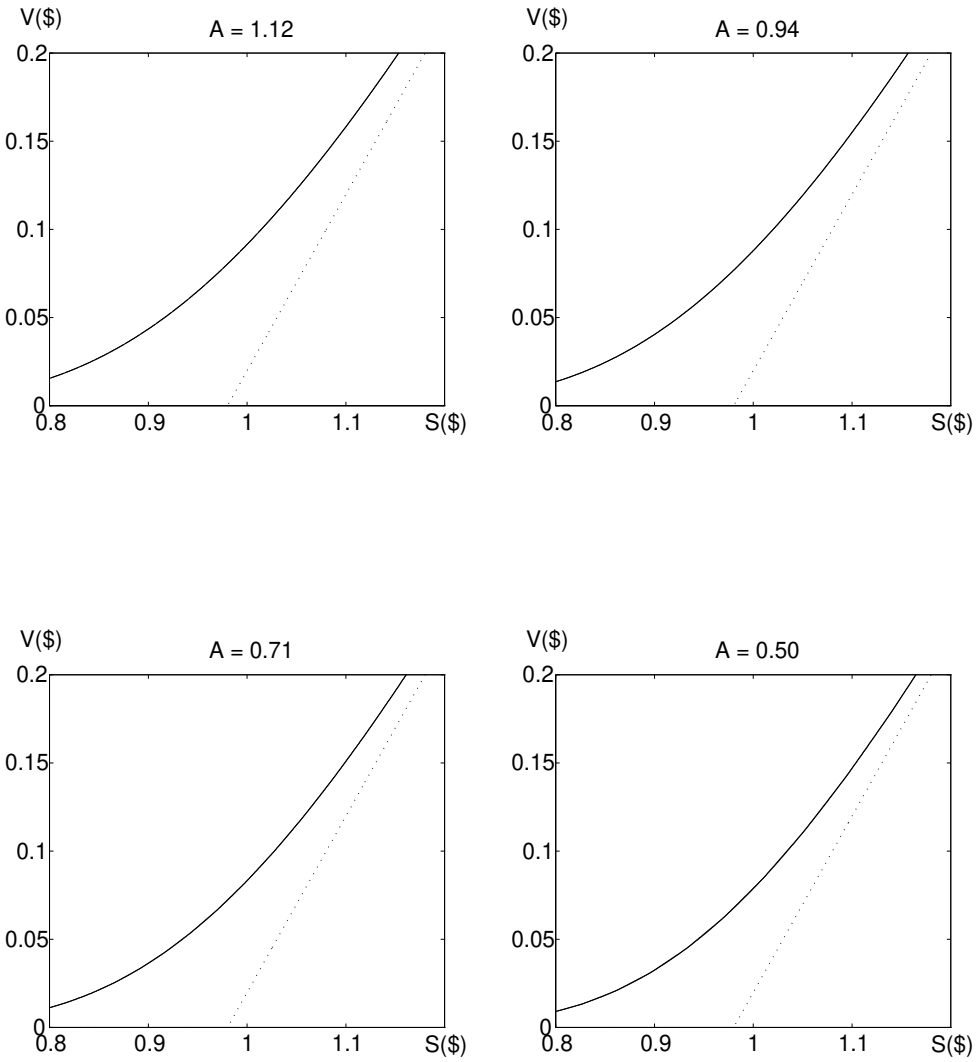


FIGURE 1.1. Value of a short call position with strike price $K = 1$ with 6 months to expiration. The parameter values are $k = 1\%$, $\sigma = 20\%$ and $dt = 0.01\,(A = 0.5)$, $0.005\,(A = 0.71)$, $0.003(A = 0.94)$ and $0.02\,(A = 1.12)$. The curves corresponding to the PDEs and the BLPS algorithm (thick line) are visually indistinguishable. The dotted lines represent the intrinsic value of the call.

agent is long "nonlinear Gamma" ($\partial^2 P/\partial S^2 \leqq 0$).[6] When $A \geqq 1$, an adjustment to zero volatility should be made in the latter case.

In addition to providing a characterization of least-initial-cost dynamic hedging, the asymptotic PDEs provide an efficient computational procedure for solving approximately the BLPS algorithm. In fact, BLPS involves two state-variables — price and stock

[6] Since P represents costs the agent's cost (liabilities), the profit-loss is $-P$. Hence our definition of Gamma.

Long Call Position.

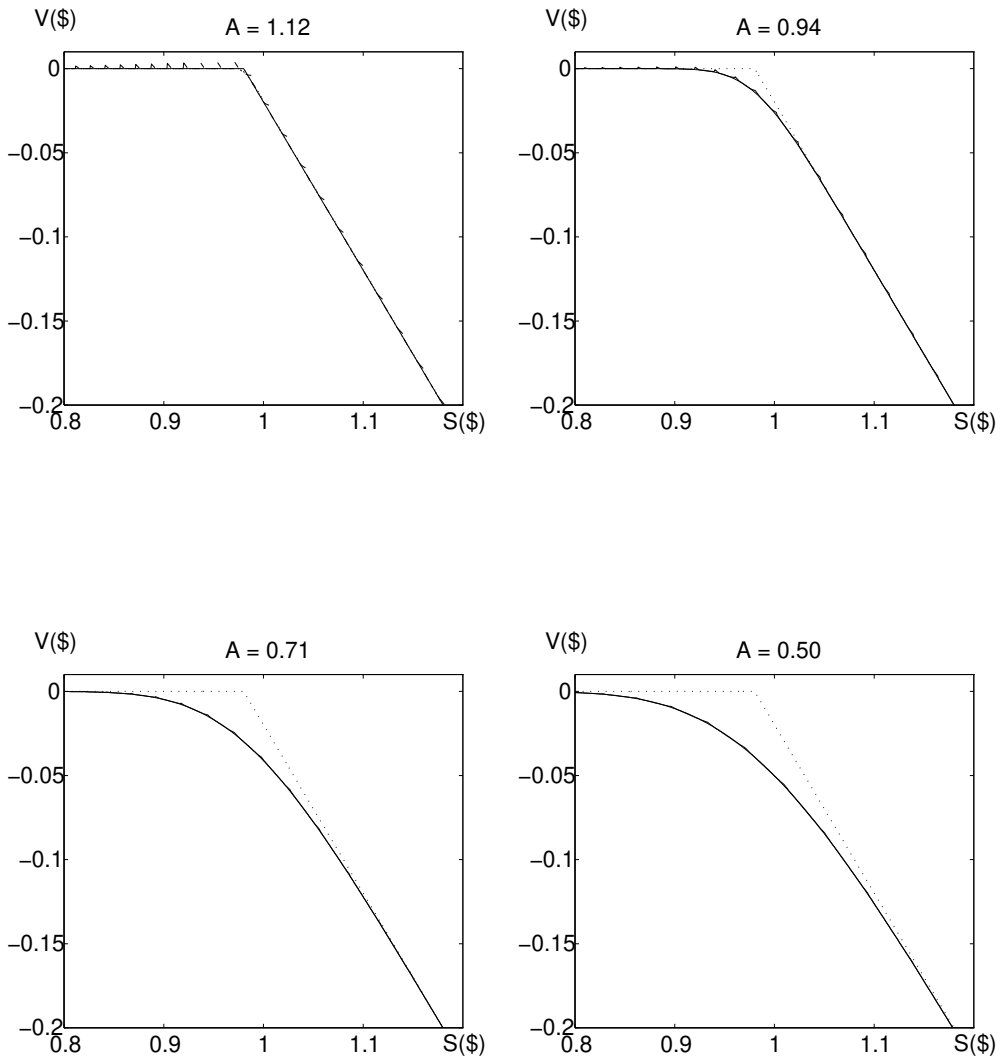


FIGURE 1.2. Same as Figure 1.1 for a long call position. Again, the agreement between BLPS and the nonlinear PDE s is remarkable. Notice that the long call position is priced at intrinsic value (as if the volatility were zero) for $A = 1.12$.

holdings — (see equation (2.11) in Section 2), while the PDEs involve only the spot price. More importantly, the BLPS scheme has exponential complexity in N when the underlying asset is not risk-feasible, unlike the finite-difference PDE solvers which are $O(N^2)$. The error which arises from the use of a finite-difference scheme for the PDEs (1.2) or (1.4) to approximate the BLPS algorithm is essentially proportional to $\sqrt{dt}$, and hence small. The quality of the approximation is exhibited in Figures 1.1, 1.2, 1.3 and 1.4 for various contingent claims under realistic values of the model parameters for equity markets.

To conclude this Introduction, we address an important issue regarding risk-feasibility

Short Butterfly Position.

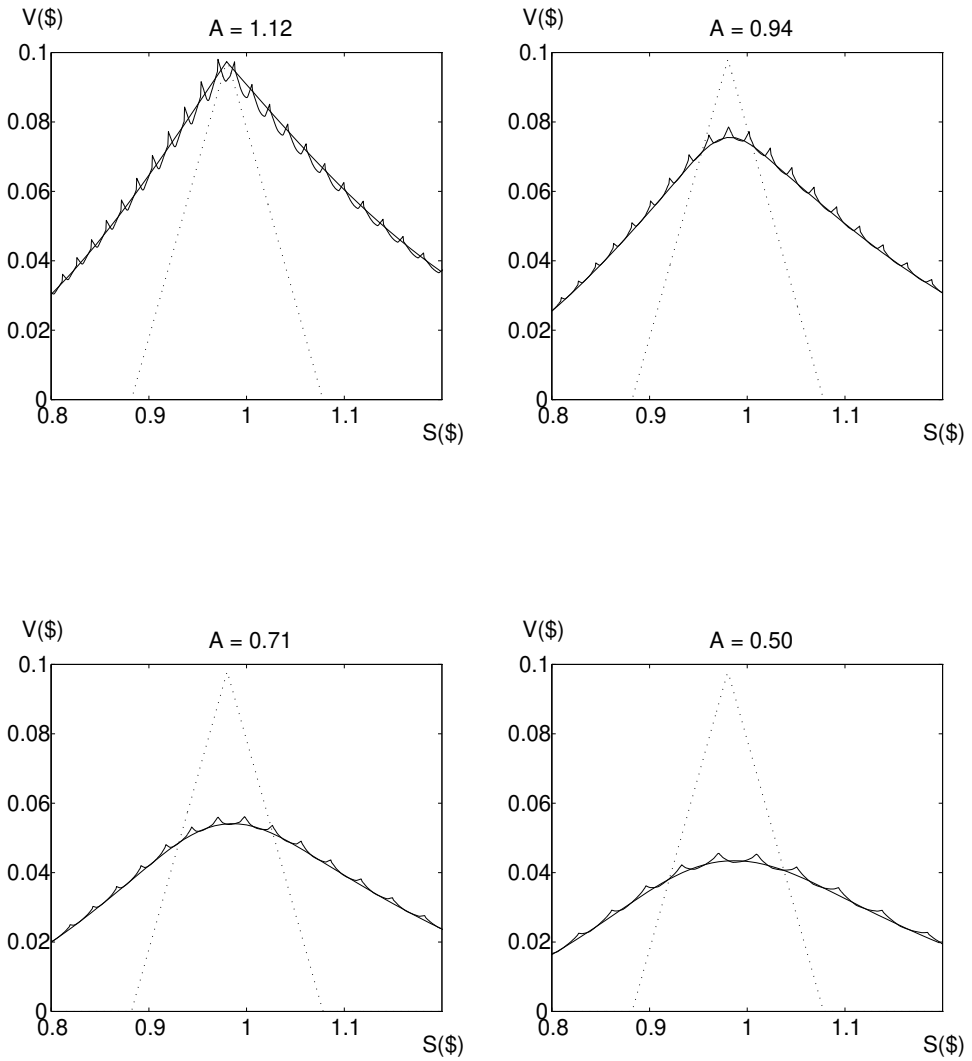


FIGURE 1.3. Comparison between the value functions obtained from the differential equations and the Bensaid-Lesne-Pages-Scheinkman algorithm for a short $0.9-1.0-1.1$ butterfly spread with six months to expiration. The smooth curves are the solutions of the PDEs (1.2) or (1.4) and the oscillating curves are solutions of the BLPS algorithm. The dotted lines represents the intrinsic value of the spread in present dollars. The parameter values are as in Figure 1.1.

and risk-aversion. As shown above, the initial cost and the strategy to follow are determined by the Leland number A or equivalently, by the minimum lag between adjustments, dt. The question of how to choose the Leland number in practice poses itself naturally. Clearly, the BLPS model does not address this issue, requiring an endogenous specification of dt. In practice, a choice of the mesh-size cannot be made without taking into account the agent's aversion to *hedge slippage* between trading dates. This remark also applies to

Short Digital Position.

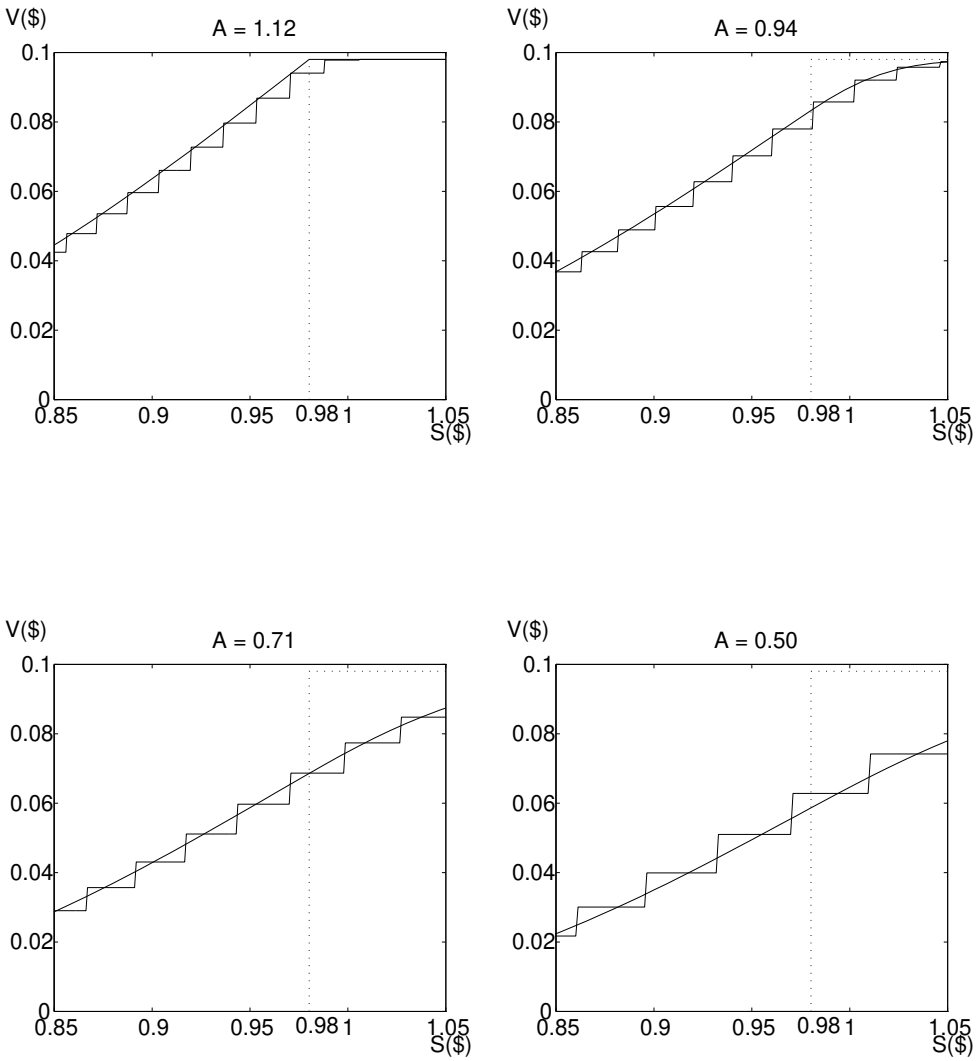


FIGURE 1.4. Same as Figure 1.3 for a digital option with 6 months to expiration. The payoff is \$1 if the stock is worth more than \$1 at expiry and \$0 otherwise.

the asymptotic PDEs. This dependency on the time-lag is a conceptual disadvantage of the model *vis a vis* the utility-dependent approaches such as Davis, Panas and Zariphopolou (1993) which determine dt endogenously. It is clear however that small Leland numbers correspond to large intervals between adjustments, with greater slippage risk, and large Leland numbers correspond to smaller lags between adjustments and thus to less risk but more transaction costs. A practical way to specify dt would be to view the slippage risk as being determined by the magnitudes of the nonlinear Delta ($\partial P/\partial S$) and nonlinear Gamma ($-\partial^2 P/\partial S^2$). Thus, a choice of dt can be made using either a "worst-case scenario" time-step, or a time-step that varies according to the magnitudes of $\frac{\partial P}{\partial S}$, $\frac{\partial^2 P}{\partial S^2}$ and the time to expiration. In practice, specification of dt can be done by analyzing profit/loss his-

tograms obtained by Monte Carlo simulation (Avellaneda and Parás (1994)). The overall result would be to have a variable Leland number A and a PDE that would combine the features of (1.2)-(1.3) and (1.4).[7]

The remaining sections of this of the paper contain a detailed mathematical analysis of the passage from the binomial lattice model to the asymptotic PDEs. In Section 2, we review the BLPS algorithm. In Section 3, we study two special conditions under which the optimal hedging strategy is to replicate the payoff and thus BLPS reduces to a *backward-induction algorithm* on the binomial lattice. This analysis pertains to the properties of the BLPS algorithm at the discrete level. In Section 4, we study the asymptotic behavior of the algorithm in the cases when replicating strategies are optimal. The technique used in the proofs is to derive approximate solutions of the backward-induction algorithm using the solutions of the PDE (1.2). This requires some regularity properties of the solutions as well as a "comparison principle", adapted to the nonlinear induction relation, to evaluate rigorously the approximation error. Section 5 studies the case $A \geqq 1$, when optimal dynamic hedging is path-dependent. To characterize this regime, we construct super-replicating strategies using the solution of the obstacle problem following Avellaneda and Parás (1994). This leads to an upper bound on the BLPS solution in terms of (1.4). A lower bound in terms of $P(S,t)$ is obtained by comparing the BLPS optimal cost with the values of certain "barrier options" that knock out on the boundary of the contact set of the obstacle problem. The main asymptotic results are contained in Theorems 4.4, 4.5 and 5.4 in Sections 4 and 5. For the reader's convenience, the regularity properties of equations (1.2) and (1.4) used in the proofs are presented in an Appendix.

[7] The use of variable-dt strategies was suggested in Whalley and Wilmott (1993) for small values of A for which equation (1.3) remains well-posed. More recently, Grannan and Swindle (1995) investigated strategies for hedging standard options using Leland's framework with variable time-steps. Other recent papers which investigate option replication with transaction costs and the time-lag issue include Hodges and Clelow (1993,1994), Flesaker and Houghston (1994), Albanese and Tompaidis (1995) and Cvitanic and Karatzas (1995).

2. The Bensaid-Lesne-Pages-Scheinkman Algorithm

Consider a binomial tree with N periods, starting at the initial date t_0 and ending at the expiration date t_N. A *trading strategy* is defined as a sequence of portfolios (Δ_n, B_n), $n = 0, \ldots, N$, to be held during each period. Here, Δ_n represents the number of shares of the underlying security held long or short and B_n the balance of a money-market account. Adjustments of the portfolio are done at the beginning of each trading period: at time t_n an amount $\Delta_n - \Delta_{n-1}$ of shares is traded and the balance of the operation (transaction cost included) is added to the money-market account. A *self-financed* strategy is one for which the balance in the money-market account after adjusting the share holdings is exactly equal to B_n . We will only consider self-financed strategies.

At time t_n, but before readjusting the position, the portfolio is composed of Δ_{n-1} shares and $B_{n-1}(1+R)$ in cash. The cost of trading $\Delta_n - \Delta_{n-1}$ shares is

$$(\Delta_n - \Delta_{n-1})S_n + \frac{k}{2}|\Delta_n - \Delta_{n-1}|S_n \equiv \Psi(\Delta_n - \Delta_{n-1})S_n,$$

where

$$\Psi(y) = \begin{cases} (1+\frac{k}{2})\ y\ , & y \geqq 0 \\ \\ (1-\frac{k}{2})\ y\ , & y < 0. \end{cases}$$

Therefore, a self-financed strategy satisfies

$$B_n = B_{n-1}(1+R) - \Psi(\Delta_n - \Delta_{n-1})S_n\ , \qquad n = 1, 2, ..., N. \tag{2.1}$$

Applying this relation recursively, we obtain

$$B_n = B_0(1+R)^n - \sum_{j=1}^{n} \Psi(\Delta_j - \Delta_{j-1})S_j(1+R)^{n-j}\ , \tag{2.2}$$

so that at time t_N

$$B_N = B_0(1+R)^N - \sum_{j=1}^{N} \Psi(\Delta_j - \Delta_{j-1})S_j(1+R)^{N-j}. \tag{2.3}$$

This formula shows that a self-financed strategy is uniquely determined by the initial money-market balance B_0 and the (projected) sequence of stock holdings $\{\Delta_0, \ldots, \Delta_N\}$.[8]

A European-style derivative security is a contingent claim with payoff

[8] It is assumed throughout the paper that the sequence $\{\Delta_n\}$ defining a trading strategy is non-anticipating with respect to the flow of information, i.e., $\Delta_n = \Delta_n(S_0, S_1, ..., S_n)$.

$$\Delta_N^* = \Delta_N^*(S_N), \qquad B_N^* = B_N^*(S_N)$$

in shares and cash, respectively, at time t_N. A *dominating (or risk-averse) strategy* of a short position in this claim is such that

$$\Delta_N \geqq \Delta_N^* \quad \text{and} \quad B_N \geqq B_N^* \tag{2.4}$$

for all final states. Note that if a dominating strategy satisfies $\Delta_N > \Delta_N^*$ for some final state, the excess of shares can be sold and the profit credited to the money-market account. Therefore, we can restrict our attention to risk-averse hedging strategies for which

$$\Delta_N = \Delta_N^*, \qquad B_N \geqq B_N^*. \tag{2.5}$$

If a hedging strategy satisfies the more stringent conditions

$$\Delta_N = \Delta_N^*, \qquad B_N = B_N^* \tag{2.6}$$

for all final states, we say that it is a *replicating strategy.*

Given an adapted sequence of stock holdings $\{\Delta_0, \ldots, \Delta_N = \Delta_N^*\}$, it follows from (2.3) that

$$B_0 = \max_{\{S_j\}_{j=1}^N} \left\{ B_N^*(1+R)^{-N} + \sum_{j=1}^{N} \Psi(\Delta_j - \Delta_{j-1}) S_j (1+R)^{-j} \right\} \tag{2.7}$$

is the minimum initial cash reserve required to constitute a dominating strategy of the contingent claim (Δ_N^*, B_N^*).[9]

Assume that at time t_0 an agent is short the contingent claim and has an inventory of Δ_{-1} shares (long or short). The minimum investment required to implement $\{\Delta_0, \ldots, \Delta_N = \Delta_N^*\}$ as a risk-averse hedging strategy is given by

$$\Delta_{-1} S_0 + \Psi(\Delta_0 - \Delta_{-1}) S_0 + B_0 \ ,$$

which represents the combined sum of the value of the initial endowment in shares, the cost of trading $\Delta_0 - \Delta_{-1}$ shares, and the minimum initial cash reserves necessary to ensure that the strategy $\{\Delta_j\}_{j=0}^N$ dominates the final payoff. An *optimal hedging strategy* is one that, given the initial endowment Δ_{-1}, minimizes the sum of the last two terms, $\Psi(\Delta_0 - \Delta_{-1}) S_0 + B_0$. By definition, this is the *effective cost* of the strategy.

[9] Here, the maximum is taken over all forward paths $\{ S_j \}_{j=0}^N$ followed by the stock price.

According to this definition, the minimum effective cost over all possible risk-averse hedging strategies (minimum effective cost, for short) is

$$V_0(\Delta_{-1}, S_0) = \min_{\{\Delta_j\}_{j=0}^N} \{\Psi(\Delta_0 - \Delta_{-1})S_0 + B_0\}$$
$$= \min_{\{\Delta_j\}_{j=0}^N} \max_{\{S_j\}_{j=1}^N} \left\{ B_N^*(1+R)^{-N} + \sum_{j=0}^{N} \Psi(\Delta_j - \Delta_{j-1})S_j(1+R)^{-j} \right\}. \quad (2.8)$$

2.1 Dynamic Programming Equation.

A similar argument shows that if an agent who has an endowment of Δ_{n-1} shares at time t_n sells the derivative security with payoff (Δ_N^*, B_N^*) and plans to implement the strategy $\{\Delta_j\}_{j=n}^N$, the minimum cash reserve she requires to generate a dominating strategy is

$$\widehat{B}_n = \max_{\{S_j\}_{j=n+1}^N} \left\{ B_N^*(1+R)^{-(N-n)} + \sum_{j=n+1}^{N} \Psi(\Delta_j - \Delta_{j-1})S_j(1+R)^{-(j-n)} \right\}. \quad (2.9)$$

Hence, the minimum effective cost at time t_n is

$$V_n(\Delta_{n-1}, S_n) = \min_{\{\Delta_j\}_{j=n}^N} \{\widehat{B}_n + \Psi(\Delta_n - \Delta_{n-1})S_n\}$$
$$= \min_{\{\Delta_j\}_{j=n}^N} \max_{\{S_j\}_{j=n+1}^N} \left\{ B_N^*(1+R)^{-(N-n)} + \sum_{j=n}^{N} \Psi(\Delta_j - \Delta_{j-1})S_j(1+R)^{-(j-n)} \right\}, \quad (2.10)$$

for $n = 0, \ldots, N$.

Proposition 2.1 *The function $V_n(\Delta, S)$ satisfies the equation*

$$V_n(\Delta_{n-1}, S_n) =$$
$$\min_{\Delta_n} \left\{ \Psi(\Delta_n - \Delta_{n-1})S_n + \frac{1}{1+R} \max\{V_{n+1}(\Delta_n, S_nU), V_{n+1}(\Delta_n, S_nD)\} \right\} \quad (2.11)$$

with final condition[10]

$$V_N(\Delta_{N-1}, S_N) = B_N^*(S_N) + \Psi(\Delta_N^*(S_N) - \Delta_{N-1})S_N \ . \tag{2.12}$$

Proof: The minimum cash reserve in eq. (2.9) can be rewritten in the form

$$\widehat{B}_n = \max_{S_{n+1}=\{S_n U, S_n D\}} \left\{ \frac{\widehat{B}_{n+1}}{1+R} + \frac{1}{1+R}\Psi(\Delta_{n+1} - \Delta_n)S_{n+1} \right\}$$

$$= \frac{1}{1+R} \max_{\{U,D\}} \left\{ \widehat{B}_{n+1} + \Psi(\Delta_{n+1} - \Delta_n)S_{n+1} \right\} \ .$$

Therefore, we have

$$\min_{\{\Delta_j\}_{j=n+1}^N} \widehat{B}_n = \min_{\{\Delta_j\}_{j=n+1}^N} \frac{1}{1+R} \max_{U,D} \{\widehat{B}_{n+1} + \Psi(\Delta_{n+1} - \Delta_n)S_{n+1}\}$$

$$\geqq \frac{1}{1+R} \max_{U,D} \min_{\{\Delta_j\}_{j=n+1}^N} \{\widehat{B}_{n+1} + \Psi(\Delta_{n+1} - \Delta_n)S_{n+1}\}$$

$$= \frac{1}{1+R} \max\{V_{n+1}(\Delta_n, S_n U), V_{n+1}(\Delta_n, S_n D)\}.$$

Hence, substituting this into (2.10), we conclude that

$$V_n(\Delta_{n-1}, S_n) = \min_{\{\Delta_j\}_{j=n}^N} \{\Psi(\Delta_n - \Delta_{n-1})S_n + \widehat{B}_n\}$$

$$\geqq \min_{\Delta_n} \left\{ \Psi(\Delta_n - \Delta_{n-1})S_n + \frac{1}{1+R} \max\{V_{n+1}(\Delta_n, S_n D), V_{n+1}(\Delta_n, S_n D)\} \right\}. \tag{2.13}$$

We claim that the opposite inequality also holds. In fact, recall that $V_{n+1}(\Delta_n, S_{n+1})$ represents the minimum effective cost at time t_{n+1} given that the endowment is Δ_n and the spot price is S_{n+1}. Because of this, the maximum of the two amounts

[10] The algorithms for solving the BLPS equation can have complexity $O(e^{\alpha N})$ with $\alpha > 0$ for some contingent claims. This is due to increasing complexity with the time-to-maturity of the quantity $\max\{V_{n+1}(\Delta_n, S_n U), V_{n+1}(\Delta_n, S_n D)\}$ as a function of Δ_n. This function must be stored in memory to deduce V_n from V_{n+1}.

$$\Psi(\Delta_n - \Delta_{n-1})S_n + \frac{1}{1+R}V_{n+1}(\Delta_n, S_nU)$$

and

$$\Psi(\Delta_n - \Delta_{n-1})S_n + \frac{1}{1+R}V_{n+1}(\Delta_n, S_nD)$$

is sufficient to cover the value of a riskless hedge at time t_n regardless of whether $S_{n+1} = S_nU$ or S_nD. Consequently, we have

$$V_n(\Delta_{n-1}, S_n) \leqq \min_{\Delta_n}\left\{\Psi(\Delta_n - \Delta_{n-1})S_n + \frac{1}{1+R}\max\{V_{n+1}(\Delta_n, S_nU), V_{n+1}(\Delta_n, S_nD)\}\right\} . \quad (2.14)$$

Combining (2.13) and (2.14), we obtain the nonlinear dynamic programming equation (2.10). The final condition in (2.11) follows from the definition of effective cost. Q.E.D.

Remark 2.2. The argument generalizes to derivative securites which deliver a sequence of payoffs at different dates, contingent on the value of the stock at each date.[11] Any such derivative security can be specified by its sequence of payoffs (Δ_n^*, B_n^*), where

$$\Delta_n^* = \Delta_n^*(S_n)\,,\ B_n^* = B_n^*(S_n). \quad (2.15)$$

The effective cost function and the optimal hedging strategy for this security can be found by solving the modified dynamic programming equation

$$V_n(\Delta_{n-1}, S_n) = \min_{\Delta_n}\left\{B_n^* + \Psi(\Delta_n + \Delta_n^* - \Delta_{n-1})S_n + \frac{1}{1+R}\max\{V_{n+1}(\Delta_n, S_nU), V_{n+1}(\Delta_n, S_nD)\}\right\} \quad (2.16)$$

with final condition (2.12). Also, the same equation applies to derivative securities in which the final date t_N is a random stopping time, like barrier options, since the proof of Proposition 2.1 does not use the fact that N is deterministic.

[11] Examples of such "contingent claims" include portfolios of European options with different maturities and equity-linked debt instruments with coupon payments contingent on the price of a stock or a stock index.

Remark 2.3 In the formalism presented here, forward contracts and bonds correspond to contingent claims such that Δ_N^* and B_N^* are constant. It is easy to verify that for these claims with no optionality the unique optimal strategy is then $\Delta_n^* = \Delta_N^*$ for all n. The minimum effective cost at time t_n is given by

$$\begin{aligned} V_n(\Delta_{n-1}, S_n) &= \Psi(\Delta_N^* - \Delta_{n-1})\, S_n + \frac{1}{(1+R)^{N-n}}\, B_N^* \\ &= \left\{ \Delta_N^*\, S_n + \frac{1}{(1+R)^{N-n}}\, B_N^* \right\} + \frac{k}{2}\, |\Delta_N^* - \Delta_{n-1}|\, S_n - \Delta_{n-1}\, S_n \ . \end{aligned} \tag{2.17}$$

Here, the term in brackets corresponds to the well-known cost-of-carry formula (without transaction costs). The other two terms represent the transaction cost of the initial share purchase/sale minus the value of the agent's endowment.

More generally, assume that the final payoff has the form $\Delta_N^* = \Delta_N(S_N) + \Delta^{(0)}$ and $B_N^* = B_N(S_N) + B^{(0)}$ where $\Delta^{(0)}$ and $B^{(0)}$ are constants. It is then easy to verify that if $[\{\Delta_n^*\}_{n=0}^N, B_0]$ is an optimal strategy for hedging the final payoff $(\Delta_N(S_N), B_N(S_N))$ given initial endowment Δ_{-1}, then $[\{\Delta_n^* + \Delta^{(0)}\}_{n=0}^N, B_0 + B^{(0)}\,(1+R)^{-N}]$ is an optimal strategy for hedging the payoff (Δ_N^*, B_N^*) given an initial endowment $\Delta_{-1} + \Delta^{(0)}$.

2.2 A stability property of the BLPS algorithm.

Replication of the final payoff can lead to mathematical instabilities if transaction costs are large (Whalley and Wilmott (1993), Boyle and Vorst (1992), Avellaneda and Parás (1994)). Even though there are finitely many trading periods, replication may be more expensive than an (already expensive) buy-and-hold strategy. The BLPS algorithm does not have this pathology.

Proposition 2.4: *Suppose that two European-style derivative securities with payoffs* $\left(\Delta_N^{(1)}, B_N^{(1)}\right)$ *and* $\left(\Delta_N^{(2)}, B_N^{(2)}\right)$ *are such that their respective effective costs at expiration satisfy*

$$V_N^{(1)}(\Delta, S_N) \leqq V_N^{(2)}(\Delta, S_N) \tag{2.18}$$

for all Δ *and all* S_N*. Then*

$$V_n^{(1)}(\Delta, S_n) \leqq V_n^{(2)}(\Delta, S_n) \tag{2.19}$$

for all $0 \leqq n \leqq N$, Δ *and* S_n*.*

Proof: Using the inequality (2.18) and equation (2.11), we find that for $n = N - 1$, we have

$$V_{N-1}^{(1)}(\Delta, S_{N-1}) \;\leqq\; V_{N-1}^{(2)}(\Delta, S_{N-1}) \;\; .$$

This argument can be applied recursively to obtain (2.19) for all n. Q.E.D.

3. Replication vs. Super-replication: "Fine Structure " of the BLPS Algorithm

In this section, we show that the risk-feasibility of the underlying asset is sufficient to guarantee that replicating strategies are optimal. The dynamic programming equation (2.11) (or (2.16)) then reduces to a backward-induction relation for the pairs (Δ_n, B_n) (see Boyle and Vorst (1992)). Another condition under which replication is optimal is the *k-convexity of the final payoff*. This assumption can be seen as a discrete version of convexity of the value the final payoff.

Prices and state-variables corresponding to different nodes of the binomial tree are represented using double-index notation; *e.g.* S_n^j represents the price at time t_n at the node (n, j).

3.1 Risk-Feasibility.

Proposition 3.1. *If the underlying asset is risk-feasible, i.e., if*

$$\frac{1+k/2}{1-k/2}\,D \;<\; 1+R \;<\frac{1-k/2}{1+k/2}\,U\ , \tag{3.1}$$

then

(i) *the optimal sequences of stock holdings* $\{\Delta_n\}$ *and of cash balances* $\{B_n\}$ *are independent of the initial endowment* Δ_{-1}*;*

(ii) *the optimal hedging strategy is replicating;*

(iii) *given the pairs* $(\Delta_{n+1}^j, B_{n+1}^j)$ *and* $(\Delta_{n+1}^{j+1}, B_{n+1}^{j+1})$*, the hedge ratio* Δ_n^j *is the unique solution of the equation*

$$\Psi(\Delta_{n+1}^{j+1} - \Delta_n^j)S_{n+1}^{j+1} + B_{n+1}^{j+1} = \Psi(\Delta_{n+1}^j - \Delta_n^j)S_{n+1}^j + B_{n+1}^j, \tag{3.2}$$

and the money-market account balance B_n^j *is given by*

$$\begin{aligned} B_n^j &= \frac{1}{1+R}(\Psi(\Delta_{n+1}^{j+1} - \Delta_n^j)S_{n+1}^{j+1} + B_{n+1}^{j+1}) \\ &= \frac{1}{1+R}(\Psi(\Delta_{n+1}^j - \Delta_n^j)S_{n+1}^j + B_{n+1}^j)\ . \end{aligned} \tag{3.3}$$

Proof. Assume first that the starting date is t_{N-1}, that the starting state is j ,($i.e.\, S = S^j_{N-1}$) and that the initial endowment is Δ_{N-2}. We abbreviate the notation, setting

$$S_{N-1} = S^j_{N-1} \ ,$$

$$S^U_N = S_{N-1}U = S^{j+1}_N \ ,$$

$$S^D_N = S_{N-1}D = S^j_N \ ,$$

$$(\Delta^U_N, B^U_N) = (\Delta^{j+1}_N, B^{j+1}_N) \ ,$$

and

$$(\Delta^D_N, B^D_N) = (\Delta^j_N, B^j_N) \ .$$

According to Equation (2.11), the optimal hedge ratio Δ^*_{N-1} is found by solving the minimization problem

$$\min_{\Delta_{N-1}} \max \{\Upsilon_U(\Delta_{N-1}), \Upsilon_D(\Delta_{N-1})\} \tag{3.4}$$

where the functions Υ_U and Υ_D are defined by

$$\Upsilon_U(\Delta) \equiv \Psi(\Delta - \Delta_{N-2})S_{N-1} + \frac{1}{1+R}\{\Psi(\Delta^U_N - \Delta)S_{N-1}U + B^U_N\} \tag{3.5}$$

and

$$\Upsilon_D(\Delta) = \Psi(\Delta - \Delta_{N-2})S_{N-1} + \frac{1}{1+R}\{\Psi(\Delta^D_N - \Delta)S_{N-1}D + B^D_N\}. \tag{3.6}$$

We claim that $\Upsilon_U(\Delta)$ and $\Upsilon_D(\Delta)$ are, respectively, strictly decreasing and strictly increasing functions of Δ. To see this, we differentiate formally Equations (3.5) and (3.6). Accordingly,

$$\begin{aligned}
\frac{d\Upsilon_U(\Delta)}{d\Delta} &= \left(1 \pm \frac{k}{2}\right) S_{N-1} - \frac{1}{1+R}\left(1 \pm \frac{k}{2}\right) S_{N-1}U \\
&\leqq \left(1 + \frac{k}{2}\right) S_{N-1} - \frac{1}{1+R}\left(1 - \frac{k}{2}\right) S_{N-1}U \\
&= \left(1 + \frac{k}{2}\right) S_{N-1}\left[1 - \frac{(1-k/2)U}{(1+k/2)(1+R)}\right] \\
&< 0 \ ,
\end{aligned}$$

where risk-feasibility was used to obtain the last inequality. A similar calculation shows that

$$\begin{aligned}
\frac{d\Upsilon_D(\Delta)}{d\Delta} &= \left(1 \pm \frac{k}{2}\right) S_{N-1} - \frac{1}{1+R}\left(1 \pm \frac{k}{2}\right) S_{N-1}D \\
&\geqq \left(1 - \frac{k}{2}\right) S_{N-1} - \frac{1}{1+R}\left(1 + \frac{k}{2}\right) S_{N-1}D \\
&= \left(1 - \frac{k}{2}\right) S_{N-1}\left[1 - \frac{(1+k/2)D}{(1+R)(1-k/2)}\right] \\
&> 0 \ .
\end{aligned}$$

Since Υ_U is decreasing and Υ_D increasing, we have

$$\max\{\Upsilon_U(\Delta), \Upsilon_D(\Delta)\} = \begin{cases} \Upsilon_U(\Delta) & \text{for } \Delta < \Delta^*_{N-1} \\ \Upsilon_D(\Delta) & \text{for } \Delta > \Delta^*_{N-1}, \end{cases}$$

where Δ^*_{N-1} is the point where the two graphs intersect. Clearly, Δ^*_{N-1} is also the value of Δ_{N-1} at which the minimum of $\max\{\Upsilon_U(\Delta), \Upsilon_D(\Delta)\}$ is achieved. Hence, the optimal hedge-ratio satisfies

$$\Upsilon_U(\Delta^*_{N-1}) = \Upsilon_D(\Delta^*_{n-1}) \ ,$$

which is equivalent to

$$\Psi(\Delta_N^U - \Delta_{N-1}^*)S_{N-1}U + B_N^U = \Psi(\Delta_N^D - \Delta_{N-1}^*)S_{N-1}D + B_N^D. \tag{3.7}$$

Notice that Equation (3.7) does not involve the initial endowment Δ_{N-2}. In particular, Δ_{N-1}^* is completely determined by the final cash-flows (Δ_N^U, B_N^U) and (Δ_N^D, B_N^D) for the two connecting nodes at time N. Furthermore, according to Equation (2.9), we have

$$\begin{aligned} B_{N-1}^* &= \frac{1}{1+R}\{\Psi(\Delta_N^U - \Delta_{N-1}^*)S_{N-1}U + B_N^U\} \\ &= \frac{1}{1+R}\{\Psi(\Delta_N^D - \Delta_{N-1}^*)S_{N-1}D + B_N^D\}. \end{aligned} \tag{3.8}$$

Substituting the right-hand sides of (3.8) into the self-financing equation (2.1), we see that the strategy $(\Delta_{N-1}^*, B_{N-1}^*)$ replicates the final payoff. Equations (3.7) and (3.8) show that Δ_{N-1}^* and B_{N-1}^* are both functions of the spot price S_{N-1}, independent of the initial endowment Δ_{N-2}.

The procedure outlined here for time t_{N-1} can be repeated at earlier times t_{N-2}, t_{N-3}, etc. Thus, Proposition 2 follows by induction on n.

3.2 k-Convexity.

Another condition which ensures that (Δ_n^*, B_n^*) is path-independent and replicating is *k-convexity.*

Definition 3.2. The payoff $(\Delta_N(S_N), B_N(S_N))$ is said to be *k-convex* if the following conditions are satisfied:

(i) $$\Delta_N^j \leqq \Delta_N^{j+1} \quad \text{for} \quad 0 \leqq j \leqq N-1 \tag{3.9}$$

and

(ii) $$\left(1 - \frac{k}{2}\right) S_N^j(\Delta_N^{j+1} - \Delta_n^j) \leqq B_N^j - B_N^{j+1} \leqq \left(1 + \frac{k}{2}\right) S_N(\Delta_N^{j+1} - \Delta_N^j)\ . \tag{3.10}$$

The definition is motivated by the following important special case.

Proposition 3.3. *Let $F(S)$ be a convex function and set*

$$\Delta_N = F'(S_N) \quad \textit{and} \quad B_N = F(S_N) - S_N F'(S_N). \tag{3.11}$$

Then (Δ_N, B_N) is k-convex for any $k \geqq 0$.

Proof. According to (3.11)

$$\Delta_N^{j+1} - \Delta_N^j = F'(S_N^{j+1}) - F'(S_N^j) = \int_{S_N^j}^{S_N^{j+1}} F''(S)dS,$$

and

$$B_N^j - B_N^{j+1} = \int_{S_N^{j+1}}^{S_N^j} (F(S) - SF'(S))' \, dS = -\int_{S_N^{j+1}}^{S_N^j} SF''(S)ds = \int_{S_N^j}^{S_N^{j+1}} SF''(S)ds.$$

In particular, since $F''(S) \geqq 0$, (3.9) holds. Moreover,

$$\begin{aligned} B_N^j - B_N^{j+1} &\leqq S_N^{j+1} \int_{S_N^j}^{S_N^{j+1}} F''(S)ds \\ &= S_N^{j+1}(\Delta_N^{j+1} - \Delta_N^j) \\ &\leqq \left(1 + \frac{k}{2}\right) S_N^{j+1}(\Delta_N^{j+1} - \Delta_N^j) \end{aligned}$$

for all $k \geqq 0$. Similarly,

$$\begin{aligned} B_N^j - B_N^{j+1} &\geqq S_N^j \int_{S_N^j}^{S_N^{j+1}} F''(S)ds \\ &= S_N^j(\Delta_N^{j+1} - \Delta_N^j) \\ &\geqq \left(1 - \frac{k}{2}\right) S_N^j(\Delta_N^{j+1} - \Delta_N^j), \end{aligned}$$

for all $k \geqq 0$. Q.E.D.

Example 3.4 Stock options with settlement in shares have k-convex payoffs. In fact, the payoff for a call option is

$$\Delta(S) = \begin{cases} 1 & S \geqq K \\ 0 & S < K \end{cases} \qquad \text{and} \qquad B(S) = \begin{cases} -K & S \geqq K \\ 0 & S < K \end{cases}.$$

Thus, the equations in (3.11) are satisfied with

$$F(S) = \Delta(S)\, S + B(S) = \mathrm{Max}(S - K, 0)\ ,$$

which is convex in S. Cash-settled options do not have k-convex payoffs. Clearly, since $\Delta_N^j = 0$ for all j, condition (3.10) would require B_N to be constant, which is not the case for cash-settled options.[12]

Proposition 3.5. *Suppose that (Δ_N, B_N) is k-convex. Then*

(i) *the optimal sequences $\{\Delta_n^*\}$ and $\{B_n^*\}$ are path-independent, independent of the initial endowment Δ_{-1}, and constitute a replicating strategy;*

(ii) *(Δ_n^*, B_n^*) is k-convex for all n;*

(iii) $\Delta_{n+1}^{*j} \leqq \Delta_n^{*j} \leqq \Delta_{n+1}^{*\,j+1}$;

(iv) *the portfolios $(\Delta_n^{*j}, B_n^{*j})$ satisfy the backward-induction equations*

$$\Delta_n^{*j} = \frac{(1+k/2)\, S_{n+1}^{j+1}\Delta_{n+1}^{*\,j+1} - (1-k/2)\, S_{n+1}^{j}\Delta_{n+1}^{*j} + B_{n+1}^{*\,j+1} - B_{n+1}^{*j}}{(1+k/2)S_{n+1}^{j+1} - (1-k/2)S_{n+1}^{j}}, \tag{3.12}$$

$$\begin{aligned} B_n^{*j} &= \frac{1}{1+R}\left\{\left(1+\frac{k}{2}\right)(\Delta_{n+1}^{*\,j+1} - \Delta_n^{*j})S_{n+1}^{j+1} + B_{n+1}^{*\,j+1}\right\} \\ &= \frac{1}{1+R}\left\{\left(1-\frac{k}{2}\right)(\Delta_{n+1}^{*j} - \Delta_n^{*j})S_{n+1}^{j} + B_{n+1}^{*j}\right\}. \end{aligned} \tag{3.13}$$

Proof. Assume that at time t_{N-1} the initial endowment is Δ_{N-2}. The optimal hedge ratio Δ_{N-1}^* is the solution to the minimization problem

$$\min_{\Delta} \max_{U,D} \{\Upsilon_U(\Delta),\, \Upsilon_D(\Delta)\}\,,$$

with Υ_U and Υ_D as in (3.5), (3.6). Since the transcation cost function Ψ is convex, Υ_U and Υ_D are also convex and so is

$$\Upsilon(\Delta) \;=\; \max\{\Upsilon_U(\Delta),\, \Upsilon_D(\Delta)\}\,.$$

Therefore, a necessary and sufficient condition for the minimum of $\Upsilon(\Delta)$ to be achieved at $\Delta = \Delta^*$ is

[12] The argument shows that the only cash-settled contingent claims with k-convex payoffs are bonds. The contingent claim of Example 1.2 can be viewed as a one-period cash-settled option.

$$\lim_{\Delta \uparrow \Delta^*} \frac{d\Upsilon(\Delta)}{d\Delta} \leqq 0 \quad \text{and} \quad \lim_{\Delta \downarrow \Delta^*} \frac{d\Upsilon(\Delta)}{d\Delta} \geqq 0. \tag{3.14}$$

We claim that this minimum is unique, and that it is given by the unique solution of the problem

$$\begin{cases} \Upsilon_U(\Delta^*_{N-1}) = \Upsilon_D(\Delta^*_{N-1}) \\ \\ \Delta^D_N \leqq \Delta^*_{N-1} \leqq \Delta^U_N. \end{cases} \tag{3.15}$$

To show this, we shall prove first prove that if the payoff is k-convex then (3.15) has a unique solution. Notice that (3.15) is equivalent to

$$\begin{cases} \left(1 + \frac{k}{2}\right)(\Delta^U_N - \Delta^*_{N-1}) S_{N-1} U \\ \qquad + B^U_N = \left(1 - \frac{k}{2}\right)(\Delta^D_N - \Delta^*_{N-1}) S_{N-1} D + B^D_N \\ \\ \Delta^D_N < \Delta^*_{N-1} < \Delta^U_N. \end{cases} \tag{3.16}$$

Elementary linear algebra shows that a solution to (3.16) exists if and only if

$$\left(1 - \frac{k}{2}\right)(\Delta^U_N - \Delta^D_N)\, S_{N-1}\, D \;\leqq B^D_N \;-\; B^U_N \;\leqq\; \left(1 + \frac{k}{2}\right)(\Delta^U_N - \Delta^D_N)\, S_{N-1}\, U \;.$$

But this is precisely condition (3.10) of the definition of k-convexity, so the claim follows.

It remains to show that the solution to (3.16) achieves the minimum of Υ. For this purpose, we verify that the first-order optimality condition (3.15) holds. We have

$$\begin{aligned} \frac{d\Upsilon_U(\Delta^*_{N-1})}{d\Delta^*_{N-1}} &= \left(1 \pm \frac{k}{2}\right) S_{N-1} - \frac{1}{1+R}\left(1 + \frac{k}{2}\right) S_{N-1} U \\ \\ &\leqq \left(1 + \frac{k}{2}\right) S_{N-1} - \frac{1}{1+R}\left(1 + \frac{k}{2}\right) S_{N-1} U \\ \\ &= \left(1 + \frac{k}{2}\right) S_{N-1} \left(1 - \frac{U}{1+R}\right) \\ \\ &< 0, \end{aligned} \tag{3.17}$$

and

$$\frac{d\Upsilon_D(\Delta^*_{N-1})}{d\Delta^*_{N-1}} = \left(1 \pm \frac{k}{2}\right) S_{N-1} - \frac{1}{1+R}\left(1 - \frac{k}{2}\right) S_{N-1} D$$

$$\geqq \left(1 - \frac{k}{2}\right) S_{N-1} - \frac{1}{1+R}\left(1 - \frac{k}{2}\right) S_{N-1} D$$

$$= \left(1 - \frac{k}{2}\right) S_{N-1}\left(1 - \frac{D}{1+R}\right)$$

$$> 0, \tag{3.18}$$

where we used the no-arbitrage condition $D < 1 + R < U$. Using (3.17)-(3.18) and the fact that $\Upsilon(\Delta^*_{N-1}) = \Upsilon_U(\Delta^*_{N-1}) = \Upsilon_D(\Delta^*_{N-1})$, we conclude that

$$\lim_{\Delta \uparrow \Delta^*_{N-1}} \frac{d\Upsilon(\Delta)}{d\Delta} = \frac{d\Upsilon_U(\Delta^*_{N-1})}{d\Delta^*_{N-1}} < 0$$

and

$$\lim_{\Delta \downarrow \Delta^*_{N-1}} \frac{d\Upsilon(\Delta)}{d\Delta} = \frac{d\Upsilon_D(\Delta^*_{N-1})}{d\Delta^*_{N-1}} > 0 \ ,$$

as desired.

We have shown that Δ^*_{N-1} is the optimal hedge-ratio. Since the problem (3.16) does not involve the endowment Δ_{N-2}, the optimal hedge-ratio is independent of the agent's holdings.

The values of Δ^*_{N-1} and B^*_{N-1} can be computed from (3.16). They are

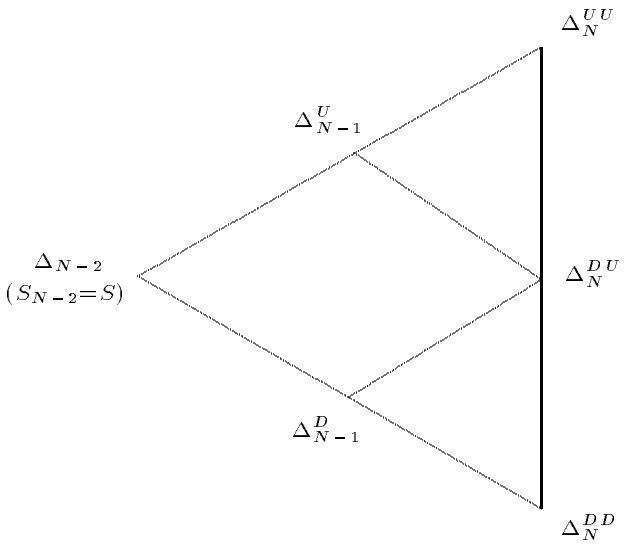
$$\Delta^*_{N-1} = \frac{(1 + k/2)\Delta^U_N S_{N-1} U - (1 - k/2)\Delta^D_N S_{N-1} D + B^U_N - B^D_N}{(1 + k/2) S_{N-1} U - (1 - k/2) S_{N-1} D} \tag{3.19}$$

and

$$B^*_{N-1} = \frac{1}{1+R}\left\{\left(1 + \frac{k}{2}\right)(\Delta^U_N - \Delta^*_{N-1}) S_{N-1} U + B^U_N\right\}$$

$$= \frac{1}{1+R}\left\{\left(1 - \frac{k}{2}\right)(\Delta^D_N - \Delta^*_{N-1}) S_{N-1} D + B^D_N\right\} \ . \tag{3.20}$$

We now show that the optimal portfolio at time t_{N-1}, $(\Delta^{*\,j}_{N-1}, B^{*\,j}_{N-1})_{j=0}^{N-1}$, is a k-convex payoff. For this, we consider the diagram



The inequality in (3.16) implies that

$$\Delta_N^{UD} \leqq \Delta_{N-1}^{U} \leqq \Delta_N^{UU} \quad \text{and} \quad \Delta_N^{DD} \leqq \Delta_{N-1}^{D} \leqq \Delta_N^{DU}, \tag{3.21}$$

and therefore

$$\Delta_{N-1}^{D} \leqq \Delta_{N-1}^{U}. \tag{3.22}$$

Using these inequalities, the formulas for B_n^* in (3.20) and the no-arbitrage condition $D < (1+R) < U$, we find that

$$\begin{aligned}
B_{N-1}^{D} - B_{N-1}^{U} &= \frac{1}{1+R} SUD \left\{ \Delta_N^{UD} + \left(1 - \frac{k}{2}\right) \Delta_{N-1}^{U} - \left(1 + \frac{k}{2}\right) \Delta_{N-1}^{D} \right\} \\
&= \frac{1}{1+R} SUD \left\{ k\Delta_N^{UD} - k\Delta_{N-1}^{U} + \left(1 + \frac{k}{2}\right) (\Delta_{N-1}^{U} - \Delta_{N-1}^{D}) \right\} \\
&\leqq \frac{1}{1+R} \left(1 + \frac{k}{2}\right) SUD(\Delta_{N-1}^{U} - \Delta_{N-1}^{D}) \\
&< \left(1 + \frac{k}{2}\right) SU(\Delta_{N-1}^{U} - \Delta_{N-1}^{D})
\end{aligned} \tag{3.23}$$

and

$$B^D_{N-1} - B^U_{N-1} = \frac{1}{1+R} SUD\{k\Delta^{UD}_N - k\Delta^D_{N-1} + (1-k/2)(\Delta^U_{N-1} - \Delta^D_{N-1})\}$$

$$\geqq \frac{1}{1+R}(1-k/2)SUD(\Delta^U_{N-1} - \Delta^D_{N-1})$$

$$> (1-k/2)SD(\Delta^U_{N-1} - \Delta^D_{N-1}). \tag{3.24}$$

Inequalities (3.22), (3.23) and (3.24) show that (Δ_{N-1}, B_{N-1}) is a k-convex payoff. We have therefore proved Proposition 3.5 for $n = N-1$. The general proof now follows by induction on n.

3.3 The backward-induction equations.

To better describe the minimum effective cost and the optimal hedging strategy, we introduce the state-variables[13]

$$P_n \;\equiv\; \Delta_n\, S_n \;+\; B_n$$

and

$$Q_n \;\equiv\; \Delta_n\, S_n \;.$$

These variables can be interpreted, respectively, as the value of the portfolio and the value of its equity portion at time t_n.[14]

In the case of risk-feasible underlying assets, explicit backward-induction equations for the pairs $(\, P_n \,,\, Q_n \,)$ can be derived by solving equation (3.7).

We shall avoid the use of superscripts by using the notation

$$\begin{cases} P^U_{n+1} = \Delta^U_{n+1} S_n U + B^U_{n+1} \\ \\ Q^U_{n+1} = \Delta^U_{n+1} S_n U \end{cases} \quad , \qquad \begin{cases} P^D_{n+1} = \Delta^D_{n+1} S_n D + B^D_{n+1} \\ \\ Q^D_{n+1} = \Delta^D_{n+1} S_n D \end{cases} \quad . \tag{3.25}$$

[13] In the rest of this section, we drop the superscript $*$ and use $\{\Delta_n\}$ and $\{B_n\}$ to represent the optimal portfolio.

[14] In this interpretation, the stock inventory is valued at the mid-price between the bid and the offer.

Proposition 3.6 *If the underlying asset is risk-feasible, the pairs* (P_n, Q_n) *satisfy*

$$\begin{cases} P_n = \frac{1}{1+R}\left(\pi^U P^U_{n+1} + \pi^D P^D_{n+1}\right) + (-1)^a \frac{k}{2(1+R)}\left(\pi^U Q^U_{n+1} + (-1)^b \pi^D Q^D_{n+1}\right) \\ \\ Q_n = \frac{1}{\gamma}\left(P^U_{n+1} - P^D_{n+1}\right) + (-1)^a \frac{k}{2\gamma}\left(Q^U_{n+1} + (-1)^c Q^D_{n+1}\right) , \end{cases} \tag{3.26}$$

where π^D, π^U, γ, a, b *and* c, *defined hereafter, are functions of* (P_{n+1}, Q_{n+1}) *(or, equivalently, of* (Δ_{n+1}, B_{n+1})*).*
Specifically, we have

Case 1:

$$\Delta^D_{n+1} \leq \Delta^U_{n+1}$$

and

$$\left(1 - \frac{k}{2}\right) S_n D(\Delta^U_{n+1} - \Delta^D_{n+1}) \leqq B^D_{n+1} - B^U_{n+1} \leqq \left(1 + \frac{k}{2}\right) S_n U(\Delta^U_{n+1} - \Delta^D_{n+1}) . \tag{3.27}$$

In this case,

$$\begin{aligned} \gamma &= \left(1 + \frac{k}{2}\right) U - \left(1 - \frac{k}{2}\right) D , \\ \pi^U &= \frac{(1+R) - (1-k/2)D}{\gamma} , \\ \pi^D &= \frac{(1+k/2)U - (1+R)}{\gamma} , \end{aligned}$$

and

$$a = +1 \quad , \; b = -1 \quad c = +1 . \tag{3.28}$$

Case 2:

$$\Delta^U_{n+1} < \Delta^D_{n+1}$$

and

$$\left(1 + \frac{k}{2}\right) S_n D(\Delta^D_{n+1} - \Delta^U_{n+1}) \leqq B^U_{n+1} - B^D_{n+1} \leqq \left(1 - \frac{k}{2}\right) S_n U(\Delta^D_{n+1} - \Delta^U_{n+1}). \tag{3.29}$$

In this case,

$$\gamma \;=\; \left(1-\frac{k}{2}\right)U-\left(1+\frac{k}{2}\right)D\;,$$

$$\pi^U \;=\; \frac{(1+R)-(1+k/2)D}{\gamma}\;,$$

$$\pi^D \;=\; \frac{(1-k/2)U-(1+R)}{\gamma}\;,$$

and

$$a=-1\quad,\; b=-1\quad,c=+1\;. \tag{3.30}$$

Case 3: *Either*

$$\Delta^D_{n+1} \;\leq\; \Delta^U_{n+1}$$

and

$$\left(1-\frac{k}{2}\right)S_nD(\Delta^U_{n+1}-\Delta^D_{n+1})>B^D_{n+1}-B^U_{n+1},$$

or

$$\Delta^U_{n+1} \;<\; \Delta^D_{n+1}$$

and

$$B^U_{n+1}-B^D_{n+1}>\left(1-\frac{k}{2}\right)S_nU(\Delta^D_{n+1}-\Delta^U_{n+1}). \tag{3.31}$$

$$\gamma \;=\; \left(1-\frac{k}{2}\right)(U-D)\;,$$

$$\pi^U \;=\; \frac{(1+R)-(1-k/2)D}{\gamma}\;,$$

$$\pi^D \;=\; \frac{(1-k/2)U-(1+R)}{\gamma}\;,$$

and

$$a=-1\;,\; b=+1\; c=-1\;. \tag{3.32}$$

Case 4: *Either*

$$\Delta^D_{n+1} \leq\; \Delta^U_{n+1}$$

and

$$B^D_{n+1} - B^U_{n+1} > \left(1 + \frac{k}{2}\right) S_n U(\Delta^U_{n+1} - \Delta^D_{n+1}),$$

or

$$\Delta^U_{n+1} \ < \ \Delta^D_{n+1}$$

and

$$\left(1 + \frac{k}{2}\right) S_n D(\Delta^D_{n+1} - \Delta^U_{n+1}) > B^U_{n+1} - B^D_{n+1}. \tag{3.33}$$

In this case,

$$\begin{aligned} \gamma \ &= \ \left(1 + \frac{k}{2}\right)(U - D) \ , \\ \pi^U \ &= \ \frac{(1+R) - (1+k/2)D}{\gamma} \ , \\ \pi^D \ &= \ \frac{(1+k/2)U - (1+R)}{\gamma} \ , \end{aligned}$$

and

$$a = +1 \ , \ b = +1 \ c = -1 \ . \tag{3.34}$$

Proof: The proof follows by solving for Δ^*_{N-1} in Equation (3.7) and generalizing to arbitrary values of n.

Proposition 3.7. *If the final payoff is k-convex then the pairs (P_n, Q_n) satisfy the linear backward-induction relation*

$$\begin{cases} P_n \ = \ \frac{1}{1+R}\left(\pi^U P^U_{n+1} + \pi^D P^D_{n+1}\right) \ + \ \frac{k}{2(1+R)}\left(\pi^U Q^U_{n+1} \ - \ \pi^D Q^D_{n+1}\right) \\ \\ Q_n \ = \ \frac{1}{\gamma}\left(P^U_{n+1} - P^D_{n+1}\right) \ + \ \frac{k}{2\gamma}\left(Q^U_{n+1} \ + \ Q^D_{n+1}\right) \ , \end{cases} \tag{3.35}$$

with

$$\gamma \ = \ \left(1 + \frac{k}{2}\right) U - \left(1 - \frac{k}{2}\right) D \ ,$$

$$\pi^U \;=\; \frac{(1+R)-(1-k/2)D}{\gamma}\;,$$

and

$$\pi^D \;=\; \frac{(1+k/2)U-(1+R)}{\gamma}\;.$$

Proof: The proof is immediate, since the portfolios (Δ_n, B_n) satisfy the conditions in (3.27) for all n.

Remark 3.8. The effective cost of hedging a *short* position in a European option settled in shares can be calculated from (3.35) for arbitrary values of k. On the other hand, a *long* option position corresponds to a linear backward-induction equation defined by Case 2 of Proposition 3.6 — provided that the underlying asset is risk-feasible. In both cases the corresponding values coincide with those obtained by Boyle and Vorst (1992).

Remark 3.9. In general, π^D and π^U satisfy $\pi^D + \pi^U = 1$. For $k > 0$, π^D and π^U are positive (in all four cases) only if the underlying asset is risk-feasible. If this condition holds, π^D and π^U can be interpreted as "risk-neutral" probabilities for no-arbitrage pricing with transaction costs. Otherwise, since at least one of the inequalities in (3.1) does not hold, we have $\pi^D \cdot \pi^U < 0$ in Case 2 (cf. (3.30)). Thus, the scheme (3.26) becomes numerically unstable and replicating strategies may result in negative or infinite prices as $N \nearrow \infty$.

If $k = 0$, we have

$$\pi^U \;=\; \frac{1+R-D}{U-D} \qquad\qquad \pi^D \;=\; \frac{U-1-R}{U-D}$$

and the backward-induction equations reduces to the classical result of Cox, Ross and Rubinstein(1982)

$$\begin{cases} P_n = \frac{1}{1+R}(\pi^U P^U_{n+1} + \pi^D P^D_{n+1}) \\ \\ Q_n = \frac{P^U_{n+1} - P^D_{n+1}}{U-D}\;. \end{cases} \tag{3.36}$$

4. k-Convex Payoffs or Risk-Feasible Underlying Assets: Asymptotics for $N \nearrow +\infty$

This section studies the backward-induction equations for $N \nearrow +\infty$. We recall first the standard Cox-Ross-Rubinstein scaling of the binomial tree, in which the parameters U, D, R, and N are related to the annualized volatility, the interest rate, and the maturity of the contingent claim. We then show that the optimal dynamic portfolio can be approximated by the solution of the partial differential equation (1.2)-(1.3) and its derivative with respect to the spot price.

4.1 Adjusting the model parameters.

We denote by T the time-to-maturity of a European-style derivative security and by

$$dt = \frac{T}{N}$$

the time-interval between successive adjustments of the hedging portfolio. Following the classical Cox, Ross, Rubinstein (1982) scaling for the binomial model, we set

$$\begin{cases} U = e^{\sigma\sqrt{dt} \,+\, \mu\, dt} \\ \\ D = e^{-\sigma\sqrt{dt} \,+\, \mu\, dt} \end{cases} \tag{4.1}$$

and

$$R = e^{r dt} - 1 \cong r dt, \qquad dt \ll 1. \tag{4.2}$$

Here, σ and r are the annualized volatility and interest rate, respectively. The parameter μ represents the subjective drift (not adjusted for risk).[15]

The *Leland number*

$$A \equiv \frac{k}{\sigma\sqrt{dt}} \ . \tag{4.3}$$

represents the " percentage transaction costs per standard deviation for a single period".

Lemma 4.1. For dt sufficiently small, the underlying asset is risk-feasible if and only if $A < 1$.

[15] As in the classical CRR theory, the parameter μ will not appear in the asymptotic equations.

Proof. It follows from (3.1) that the underlying asset is risk-feasible if and only if

$$\frac{k}{2} < \min\left[\frac{U-(1+R)}{U+1+R}, \frac{1+R-D}{1+R+D}\right]. \tag{4.4}$$

Using the scaling relations (4.1), we find that, for $dt \ll 1$,

$$\frac{U-(1+R)}{U+(1+R)} = \frac{\sigma\sqrt{dt}+O(dt)}{2+O\left(\sqrt{dt}\right)} \approx \frac{\sigma\sqrt{dt}}{2}$$

and

$$\frac{1+R-D}{1+R+D} = \frac{\sigma\sqrt{dt}+O(dt)}{2+O\left(\sqrt{dt}\right)} \approx \frac{\sigma\sqrt{dt}}{2}\ .$$

Thus, for dt sufficiently small, risk-feasibility is equivalent to $k/2 < \sigma\sqrt{dt}/2$, or just $A < 1$. Q.E.D.

4.2. A Comparison Principle.

Definition 4.2. *A hedging strategy* $\left[\{\widetilde{\Delta}_j\}_{j=n}^N, \widetilde{B}_n\right]$ *is said to be* δ*-optimal at time* t_n *for the payoff* (Δ_N, B_N) *if:*

(i) $\left[\{\widetilde{\Delta}_j\}_{j=n}^N, \widetilde{B}_n + \delta\right]$ *is a dominating strategy for* (Δ_N, B_N);

and

(ii) *the effective cost of* $\left[\{\widetilde{\Delta}_j\}_{j=n}^N, \widetilde{B}_n\right]$ *differs from the optimal effective cost* $V(\Delta, S_n)$ *of* (Δ_N, B_N) *by at most* δ *dollars, i.e.,*

$$\left|V(\Delta, S_n) - \left(\widetilde{B}_n + \Psi(\widetilde{\Delta}_n - \Delta)S_n\right)\right| \leqq \delta.$$

Let us represent formally the non-linear backward-induction equations (3.26) as

$$\begin{pmatrix} P_n \\ Q_n \end{pmatrix} = \Phi\begin{pmatrix} P_{n+1} \\ Q_{n+1} \end{pmatrix} \qquad n = 0, \ldots, N-1\ . \tag{4.5}$$

Proposition 4.3. *Consider a final payoff (Δ_N, B_N). Assume that either this payoff is k-convex or that the underlying asset is risk-feasible. Suppose that there exist sequences $(\widetilde{P}_n, \widetilde{Q}_n)$ and (p_n, q_n) having the following properties:*

(i)
$$\begin{pmatrix} \widetilde{P}_n \\ \widetilde{Q}_n \end{pmatrix} = \Phi \begin{pmatrix} \widetilde{P}_{n+1} \\ \widetilde{Q}_{n+1} \end{pmatrix} + \begin{pmatrix} p_n \\ q_n \end{pmatrix}, \qquad n = 0, \ldots, N-1; \tag{4.6}$$

(ii)
$$\widetilde{P}_N = P_N, \qquad \widetilde{Q}_N = Q_N; \tag{4.7}$$

(iii) *if the underlying asset is not risk-feasible, then for every n the intermediate "portfolios"*
$$\begin{pmatrix} \widetilde{\Delta}_n \\ \widetilde{B}_n \end{pmatrix} \equiv \begin{pmatrix} \widetilde{Q}_n / S_n \\ \widetilde{P}_n - \widetilde{Q}_n \end{pmatrix} \tag{4.8}$$
are k-convex for all n.

Define
$$\delta_n \equiv \left(\frac{1-(1+R)^{-(N-n)}}{(1+R)^{-1} R} \right) \times \max_{n'>n, S} \left\{ |p_{n'}(S)| + \frac{k}{2} |q_{n'}(S)| \right\} .$$

Then, for each n , $0 \leqq n \leqq N-1$, the strategy $\left[\{\widetilde{\Delta}_j\}_{j=n}^{N}, \widetilde{B}_n \right]$ is δ_n – optimal.

Proof. Let us introduce an auxiliary contingent claim with final payoff (Δ_N, B_N) and with intermediate payoffs ("coupons")

$$\begin{pmatrix} d_n \\ b_n \end{pmatrix} = \begin{pmatrix} q_n / S_n \\ p_n - q_n \end{pmatrix}, \tag{4.9}$$

for $n = 0, ..., N-1$. The payoffs of this auxiliary security are represented in Diagram 4.1. Let $\widetilde{V}_n(\Delta, S_n)$ represent its optimal effective cost given an endowment Δ at time t_n .

¿From the assumptions (i) through (iii), we conclude that $(\widetilde{\Delta}_n, \widetilde{B}_n)$ is the optimal risk-averse strategy for hedging this security (see Remark 2.2), and that $(\widetilde{P}_n, \widetilde{Q}_n)$ represents the optimal portfolio in the variables (P, Q).

The minimum effective cost $\widetilde{V}_n(\Delta, S_n)$ can be estimated by stripping the intermediate coupons from the final payoff and pricing them separately. For $n' \geqq n$, the coupon (p'_n , q'_n) can be purchased at

$$p_{n'} + \frac{k}{2} |q_{n'}| \leqq \max_{n''>n, S} \left\{ |p_{n''}(S)| + \frac{k}{2} |q_{n''}(S)| \right\} \equiv \varepsilon_n \tag{4.10}$$

dollars. Its value at time t_n is therefore at most $\varepsilon_n (1+R)^{-(n'-n)}$. Since the effective cost of the auxiliary claim at time t_n is at most equal to the effective cost of the original

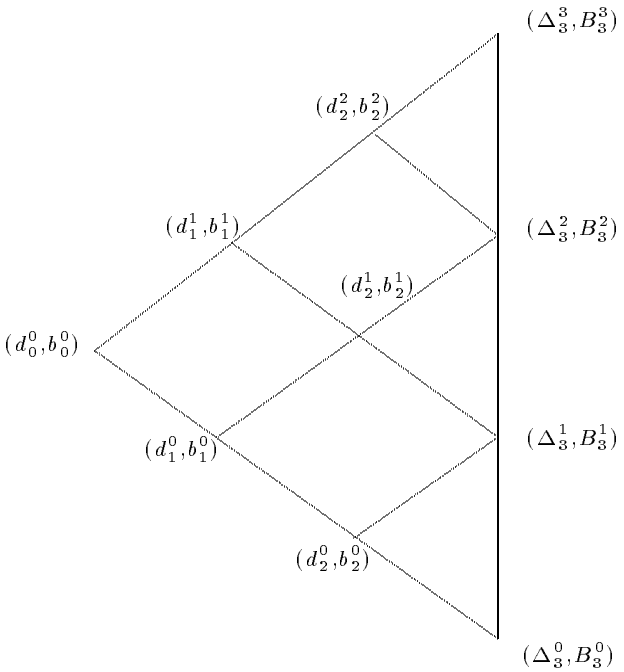


DIAGRAM 4.1.

contingent claim plus the present value of the estimated costs of the intermediate portfolios, we conclude that

$$\widetilde{V}_n(\Delta, S_n) \leqq V_n(\Delta, S_n) + \varepsilon_n \sum_{j=n}^{N-1} (1+R)^{-(j-n)} = V_n(\Delta, S_n) + \delta_n \ . \tag{4.11}$$

Similarly, a lower bound for $\widetilde{V}_n(\Delta, S_n)$ is obtained by considering an auxiliary derivative security that delivers, in addition to the intermediate coupons, ε dollars at each trading date. The strategy $\left[\{\widetilde{\Delta}_j\}_{j=n}^N, \widetilde{B}_n + \delta_n\right]$ is clearly a dominating strategy for such security. But, since the intermediate payments are all positive on account of (4.10), the strategy is also dominating for the European contingent claim with one single payoff (Δ_N, B_N) at time t_N. Thus, we have

$$V_n(\Delta, S_n) \leqq \widetilde{V}_n(\Delta, S_n) + \delta_n \ . \tag{4.12}$$

The proof of Proposition 4.3 is complete.

4.3 Asymptotic analysis.

We consider first payoffs of the form

$$\begin{cases} \Delta_N(S_N) = F'(S_N) \\ \\ B_N(S_N) = F(S_n) - S_N F'(S_N) \quad , \end{cases} \tag{4.13}$$

where $F(S)$ is a four-times continuously differentiable function satisfying

$$\|F\| \equiv \sup_S \left\{ \sum_{j=0}^{4} S^j \left| \frac{d^j F(S)}{dS^j} \right| \right\} < \infty \ . \tag{4.14}$$

The boundedness and regularity assumption (4.14) is mathematically convenient but not essential. In Section 4.4 we extend the results derived for the payoffs (4.13) to option portfolios.

The main result of this section is:

Theorem 4.4. *Let $F(S)$ be a function satisfying the regularity condition (4.14). Assume that $A < 1$ or $F(S)$ is convex. Let $P(S,t)$ be the solution of the final-value problem*

$$\begin{cases} \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 \left[1 + A\, sign\left(\frac{\partial^2 P}{\partial S^2} \right) \right] S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0 \\ \\ P(S,T) = F(S). \end{cases} \tag{4.15}$$

Set

$$\begin{cases} \widetilde{P}_n(S_n) = P(S_n, t_n) \\ \\ \widetilde{Q}_n(S_n) = S_n \frac{\partial P}{\partial S}(S_n, t_n) \ , \end{cases} \tag{4.16}$$

and

$$\begin{pmatrix} \widetilde{\Delta}_n \\ \widetilde{B}_n \end{pmatrix} = \begin{pmatrix} \widetilde{Q}_n / S_n \\ \widetilde{P}_n - \widetilde{Q}_n \end{pmatrix} . \tag{4.17}$$

Then:

(i) *There exists a constant $C = C(r, \mu, \sigma, A, T)$ independent of F and dt, such that $\left(\widetilde{P}_n(S_n), \widetilde{Q}_n(S_n)\right)$ satisfies the backward-induction equation*

$$\begin{pmatrix} \widetilde{P}_n \\ \widetilde{Q}_n \end{pmatrix} = \Phi \begin{pmatrix} \widetilde{P}_{n+1} \\ \widetilde{Q}_{n+1} \end{pmatrix} + \begin{pmatrix} p_n \\ q_n \end{pmatrix} \tag{4.18}$$

for $n = 0, \ldots, N-1$, *with*

$$|p_n| \leqq C\|F\|(dt)^{\frac{3}{2}} \qquad \text{and} \qquad |q_n| \leqq C\|F\|\, dt \ . \tag{4.19}$$

(ii) *In particular, consider a European-style contingent claim with payoff*

$$\begin{cases} \Delta_N(S_N) = F'(S_N) \\ \\ B_N(S_N) = F(S_N) - S_N F'(S_N) \ . \end{cases}$$

Then, $\left[\{\widetilde{\Delta}_j\}_{j=n}^N, \widetilde{B}_n\right]$ *is a* $C'\|F\|(dt)^{1/2}$ *–optimal hedging strategy for* (Δ_N, B_N), *where* $C' = C'(r, \mu, \sigma, A, T)$ *is a constant independent of* F *and* dt. *Furthermore, the minimum effective cost function of this contingent claim,* $V_n(\Delta, S_n)$, *satisfies, for all* Δ *and* S_n,

$$\left| V_n(\Delta, S_n) - \left(\widetilde{P}_n + \frac{k}{2}|\widetilde{Q}_n - \Delta S_n| - \Delta S_n \right) \right| \leqq C'\|F\|(dt)^{1/2} \ . \tag{4.20}$$

Proof. In view of Proposition (4.3), the proof consists in establishing that $\left(\begin{smallmatrix} \widetilde{P}_n \\ \widetilde{Q}_n \end{smallmatrix}\right)$ is an approximate solution to the backward-induction equation (3.26). We shall estimate the remainders p_n and q_n in (4.19) using Taylor series expansions of p_n and q_n in powers of $\sqrt{dt}$.

Step 1: Reduction to $r = 0$. We can assume without loss of generality that $R = r = 0$. In fact, this amounts simply to measuring values in dollars-at-expiration rather than in present value.

Step 2: $P(S,t)$ is locally convex. If $P_{SS}(S,t) \geqq 0$ in a neighborhood of the node of interest, then the portfolio $\left(\widetilde{\Delta}_{n+1}, \widetilde{B}_{n+1}\right)$ is locally k-convex, *i.e.*, condition (3.27) (Case 1 of Proposition 3.6) holds.

Using the definitions γ, π^D and π^U in (3.28), we find after some calculation that

$$\gamma \ \approx \ (2 + A)\, \sigma \sqrt{dt} \ ,$$

$$\pi^U \;\approx\; \frac{1}{2} \;-\; \frac{\mu \;+\; (1/2)\sigma^2(1+A)}{(2+A)\,\sigma}\;\sqrt{dt}\;,$$

and

$$\pi^D \;\approx\; \frac{1}{2} \;+\; \frac{\mu \;+\; (1/2)\sigma^2(1+A)}{(2+A)\,\sigma}\;\sqrt{dt}\;. \tag{4.21}$$

Expanding $\widetilde{P}_{n+1}$ and $\widetilde{Q}_{n+1}$ in a Taylor series around $(S_n\,, t_{n+1})$, we find that

$$\begin{aligned}\widetilde{P}^U_{n+1} \;&= \widetilde{P}_{n+1} \;+\; (S_nU - S_n)\widetilde{P}_{S,n+1} \;+\; \frac{1}{2}\,(S_nU - S_n)^2\;\widetilde{P}_{SS,n+1} \;+ \\ &\qquad\qquad O\left[\;||S^3\,P_{SSS}||_0\;dt^{3/2}\;\right] \\ =\;\widetilde{P}_{n+1} \;&+\; S_n\widetilde{P}_{S,n+1}\;\sigma\;\sqrt{dt} \;+\; \frac{1}{2}\,(\,S_n\widetilde{P}_{S,n+1} \;+\; S_n^2\widetilde{P}_{SS,n+1}\,)\;\sigma^2\;dt \;+ \\ &\qquad\qquad O\left[\;(||S^2\,P_{SS}||_0 \;+\; ||S^3\,P_{SSS}\,||_0)\,dt^{3/2}\;\right]\;,\end{aligned} \tag{4.22}$$

$$\begin{aligned}\widetilde{P}^D_{n+1} \;&=\; \widetilde{P}_{n+1} \;-\; S_n\widetilde{P}_{S,n+1}\;\sigma\;\sqrt{dt} \\ +\;\frac{1}{2}\,(\,-S_n\widetilde{P}_{S,n+1} \;&+\; S_n^2\widetilde{P}_{SS,n+1}\,)\;\sigma^2\;dt \;+ \\ &\qquad O\left[\;(||S^2\,P_{SS}\,||_0 \;+\; ||S^3\,P_{SSS}||_0)\,dt^{3/2}\;\right]\end{aligned} \tag{4.23}$$

and

$$\begin{aligned}\widetilde{P}_{n+1} \;-\; \widetilde{P}_n \;&=\; \widetilde{P}_{t,n+1}\;dt \;+ \\ &\qquad O\left[\,|||\partial_t P\,||_{t,1/2}\;(dt)^{3/2}\;\right]\end{aligned} \tag{4.24}$$

where we used the abbreviations $\widetilde{P}_{S,n+1} = \frac{\partial P}{\partial S}(S_n, t_{n+1})$, $\widetilde{P}_{t,n+1} \;= \frac{\partial P}{\partial t}(S_n, t_{n+1})$, etc., and the semi-norms

$$||\phi||_0 \;\equiv\; \sup_{S,t}\;|\phi(S,t)|$$

and

$$\|\phi\;\|_{t,1/2} \;\equiv \sup_{S,t}\;|\phi(S,t)| \;+\; \sup_{S,t,t'}\;\frac{|\phi(S,t) \;-\; \phi(S,t')|}{|t-t'|^{1/2}}\;.$$

Substituting these Taylor expansions into (3.26) (with R=0), and using the fact that $P(S,t)$ satisfies the PDE (4.15), we find after a tedious but straightforward calculation that the pairs $\binom{\widetilde{P}_n}{\widetilde{Q}_n}$ and $\binom{\widetilde{P}_{n+1}}{\widetilde{Q}_{n+1}}$ satisfy

$$\widetilde{P}_n \; - \; \left(\pi^U \widetilde{P}^U_{n+1} + \pi^D \widetilde{P}^D_{n+1}\right) \; - \; \frac{k}{2} \; \left(\pi^U \widetilde{Q}^U_{n+1} - \pi^D \widetilde{Q}^D_{n+1}\right) \; = \; p_n \tag{4.25}$$

and

$$\widetilde{Q}_n \; - \; \frac{1}{\gamma} \left(\widetilde{P}^U_{n+1} - \widetilde{P}^D_{n+1}\right) \; - \; \frac{k}{2\gamma} \left(\widetilde{Q}^U_{n+1} + \widetilde{Q}^D_{n+1}\right) \; = \; q_n \; , \tag{4.26}$$

with

$$|p_n| \; \leqq \; C \; \left[\; \left\| S^2 \, P_{SS} \right\|_0 \; + \; \left\| S^3 \, P_{SSS} \right\|_0 \; + \; \left\| P_t \right\|_{t,1/2} \; \right] \; (dt)^{3/2} \tag{4.27}$$

and

$$|q_n| \; \leqq \; C \; \left[\; \left\| S^2 \, P_{SS} \right\|_0 \; + \; \left\| S^3 \, P_{SSS} \right\|_0 \; + \; \left\| P_t \right\|_{t,1/2} \; \right] \; dt \; . \tag{4.28}$$

Here, $C = C(\mu, \sigma, A, T)$ is a constant independent of F and dt. Hence from Proposition A.1 in the Appendix, we conclude that

$$|p_n| \; \leqq \; C(\mu, \sigma, A, T) \; ||F|| \; (dt)^{3/2} \tag{4.29}$$

and

$$|q_n| \; \leqq \; C(\mu, \sigma, A, T) \; ||F|| \; dt \; . \tag{4.30}$$

Step 3: $P(S,t)$ is locally concave. Next, we consider the case where $P_{SS}(S,t) \; < \; 0$ in a neighborhood of the node (S_n, t_n). A calculation completely analogous to the previous one shows that

$$\widetilde{P}_n \; - \; \left(\pi^U \widetilde{P}^U_{n+1} + \pi^D \widetilde{P}^D_{n+1}\right) \; + \; \frac{k}{2} \; \left(\pi^U \widetilde{Q}^U_{n+1} - \pi^D \widetilde{Q}^D_{n+1}\right) \; = \; p_n \tag{4.31}$$

and

$$\widetilde{Q}_n \; - \; \frac{1}{\gamma} \left(\widetilde{P}^U_{n+1} - \widetilde{P}^D_{n+1}\right) \; + \; \frac{k}{2\gamma} \left(\widetilde{Q}^U_{n+1} + \widetilde{Q}^D_{n+1}\right) \; = \; q_n \; , \tag{4.32}$$

where p_n and q_n satisfy (4.29) and (4.30). Thus, if $P(\cdot,t)$ is concave as a function of S in a neighborhood of the node, the recursion relation (3.26) corresponding to Case 2 is satisfied to the desired order of approximation.

Step 4: Local concavity and nondegeneracy imply Case 2. Notice that we have not verified in Step 3 that the portfolios $(\widetilde{\Delta}_{n+1}\,,\widetilde{B}_{n+1})$ satisfy Case 2 (condition (3.29) of Proposition 3.6). This is necessary, since otherwise the estimates (4.31) and (4.32) are not relevant. We now show that if $P(S,t)$ is locally concave, then Case 2 indeed applies, provided that P_{SS} satisfies a non-degeneracy condition.

Setting $P(S) = P(S,t_{n+1})$, we have, as in the proof of Proposition 3.3,

$$\widetilde{\Delta}^D_{n+1} \;-\; \widetilde{\Delta}^U_{n+1} \;=\; \int\limits_{S_n\,D}^{S_n\,U} |P_{SS}(X)|\,dX$$

and

$$\widetilde{B}^U_{n+1} \;-\widetilde{B}^D_{n+1} \;=\; \int\limits_{S_n\,D}^{S_n\,U} X\,|P_{SS}(X)|\,dX \;\;.$$

We estimate the latter integral as follows:

$$\begin{aligned}
\int\limits_{S_n\,D}^{S_n\,U} X\,|P_{SS}(X)|\,dX \;&= S_n\,D \int\limits_{S_n\,D}^{S_n\,U} |P_{SS}(X)|\,dX \;+\; \int\limits_{S_n\,D}^{S_n\,U} (X \;-\; S_n\,D)\,|P_{SS}(X)|\,dX \\
&= S_n\,D\;(\widetilde{\Delta}^D_{n+1} \;-\; \widetilde{\Delta}^U_{n+1}\;) \;+\; \int\limits_{S_n\,D}^{S_n\,U} (X \;-\; S_n\,D)\,|P_{SS}(X)|\,dX \\
&\leqq\; S_n\,D\;(\widetilde{\Delta}^D_{n+1} \;-\; \widetilde{\Delta}^U_{n+1}\;) \;+\; \frac{1}{2}(max\;|P_{SS}|\,)(S_n\,U \;-\; S_n\,D)^2 \\
\leqq\; \left[\, S_n\,D \;+\; \frac{1}{2}\,(\frac{max\;|P_{SS}|}{min\;|P_{SS}|})(S_n\,U \;-\; S_n\,D)\,\right]\;(\widetilde{\Delta}^D_{n+1} \;-\; \widetilde{\Delta}^U_{n+1}\;)\;. \qquad (4.33)
\end{aligned}$$

Here, the maximum and minimum are taken over all $X \;\in\; [\,S_n\,D\,,S_n\,U\,]$. Similarly,

$$\int_{S_n\,D}^{S_n\,U} |P_{SS}(X)|\,dX \;\geqq\; \left[\, S_n\,U \;-\; \frac{1}{2}\,(\frac{max\;|P_{SS}|}{min\;|P_{SS}|})(S_n\,U \;-\; S_n\,D) \,\right]\;(\widetilde{\Delta}^{\,D}_{n+1} \;-\; \widetilde{\Delta}^{\,U}_{n+1}\;)\;. \tag{4.34}$$

We conclude that if

$$D \;+\; \frac{1}{2}\,(\frac{max\;|P_{SS}|}{min\;|P_{SS}|})(U \;-\; D) \;\leqq\; (1 \;-\; \frac{k}{2})\,U \tag{4.35}$$

and

$$U \;-\; \frac{1}{2}\,(\frac{max\;|P_{SS}|}{min\;|P_{SS}|})(U \;-\; D) \;\geqq\; (1+\frac{k}{2})\,D \tag{4.36}$$

then condition (3.29) is satisfied by $(\widetilde{\Delta}_{n+1}\,,\widetilde{B}_{n+1})$. Substituting the values of $U,\;D$, and k into these estimates, we find that both inequalities hold for $dt \;\ll\; 1$ if

$$\frac{max\;|P_{SS}|}{min\;|P_{SS}|} \;<\; 2 \;-\; \frac{A}{2}\;. \tag{4.37}$$

But clearly, since $dt \;\ll 1$,

$$\begin{aligned}\frac{max\;|P_{SS}|}{min\;|P_{SS}|} \;&\approx\; 1 \;+\; \frac{||S^3\,P_{SSS}||_0}{min\;|X^2 P_{SS}(X)|}\;\sigma\;\sqrt{dt} \\ &\leqq\; 1 \;+\; \frac{C\,||F||}{min\;|X^2 P_{SS}(X)|}\;\sigma\;\sqrt{dt}\;\;,\end{aligned} \tag{4.38}$$

from the *a-priori* estimates given in the Appendix. Thus, we conclude that if $P_{SS}(X)$ is negative in a neighborhood of $(S_n\,t_n)$ and

$$\min_{S_n\,D\;\leqq\;X\;\leqq\;S_n\,U} |X^2 P_{SS}(X)| \;\geqq\; \frac{2\,C\,||F||}{2-A}\;\sqrt{dt}\;\;, \tag{4.39}$$

then the pairs $(\widetilde{P}_n\,,\widetilde{Q}_n)$ and $(\widetilde{P}_{n+1}\,,\widetilde{Q}_{n+1})$ satisfy the backward-induction relation (4.18) within the desired order of accuracy (4.19) guaranteed by Step 3. Inequality (4.39) can be viewed as a non-degeneracy condition for the second derivative of $P(S,t)$ that guarantees that the solution of (3.26) is well-approximated locally by $(\widetilde{P}_n\,,\widetilde{Q}_n)$.

Step 5: The marginal cases. To establish that $(\widetilde{P}_n\,,\widetilde{Q}_n)$ is a suitable approximate solution to (3.26) in general, we now assume that (4.39) is not satisfied, and focus on Cases 3 and 4 of the backward-induction equation.[16]

Consider the finite-difference equation (3.26) with parameters π^D, π^D, γ, a, b and c as in Case 3. We have

$$\begin{cases} \gamma \;\approx (1 \;-\; \frac{A\sigma\sqrt{dt}}{2})2\,\sigma\,\sqrt{dt} \\ \\ \pi^U \;\approx\; \frac{2+A}{4} \;-\; \frac{1}{2\,\sigma}\,\big(\,\mu \;+\; (1/2)\sigma^2\,(1-\frac{A^2}{2})\,\big)\,\sqrt{dt} \\ \\ \pi^D \;\approx\; \frac{2-A}{4} \;+\; \frac{1}{2\,\sigma}\,\big(\,\mu \;+\; (1/2)\sigma^2\,(1-\frac{A^2}{2})\,\big)\,\sqrt{dt} \quad . \end{cases} \tag{4.40}$$

Substituting these formulas and the Taylor expansions of P_{n+1} and Q_{n+1} into (3.26) leads to

$$\widetilde{P}_n \,-\, \left(\pi^U\widetilde{P}^U_{n+1} + \pi^D\widetilde{P}^D_{n+1}\right) \;+\; \frac{k}{2}\,\left(\pi^U\widetilde{Q}^U_{n+1} - \pi^D\widetilde{Q}^D_{n+1}\right) \;=\; p_n \tag{4.41}$$

and

$$\widetilde{Q}_n \;-\; \frac{1}{\gamma}\left(\widetilde{P}^U_{n+1} - \widetilde{P}^D_{n+1}\right) \;+\; \frac{k}{2\gamma}\left(\widetilde{Q}^U_{n+1} - \widetilde{Q}^D_{n+1}\right) \;=\; q_n\;, \tag{4.42}$$

with error terms satisfying, respectively,

$$p_n \;=\; \frac{\sigma^2}{2}(\;A \;+\; \frac{A^2}{2}\;)\;S_n^2\,P_{SS}\big(S_n\;,t_{n+1}\big)\,dt \;+\; O\,(\;||F||\,(dt)^{3/2}\;)\;, \tag{4.43}$$

and

$$q_n \;=\; \frac{A\,S_n\,\sigma\,\sqrt{dt}}{2}\;S_n^2\,P_{SS}\big(S_n\;,t_{n+1}\big) \;+\; O\,(\;||F||\,dt\;)\;. \tag{4.44}$$

To estimate the remainders, we use the fact that $S^2\,P_{SS}(S,t)$ is small. In fact, since (4.39) does not hold, we may assume that

$$|\,S_n^2\,P_{SS}\big(S_n\;,t_{n+1}\big)\,| \;<\; \frac{4\,C\,||F||}{2-A}\;\sqrt{dt}\,. \tag{4.45}$$

[16] Roughly speaking, Cases 1 and 2 correspond to those nodes where the dollar-weighted curvature $S^2\,|P_{SS}(S,t)|$ is sufficiently large. Cases 3 and 4 correspond to the vicinity of inflection points of $P(S\,,t)$.

Substituting this inequality into (4.43) and (4.44), we conclude that p_n and q_n satisfy the error estimates (4.19) when Case 3 holds. We omit the analysis of Case 4 , which is entirely similar.

Step 6: Globally Convex Solutions. If F(S) is convex then by the maximum principle $P(S,t)$ is always convex. Therefore, according to Step 2 we conclude that the backward-induction relation is satisfied within the error bounds given by (4.19). Furthermore, the portfolios $(\widetilde{\Delta}_n, \widetilde{B}_n)$ defined by (4.8) are always k-convex according to Proposition (3.3).

Step 7: Comparison Principle. The Theorem now follow from Proposition 4.3. In fact, applying this result in conjunction with the estimates on the remainders p_n and q_n, we find that $\left[\{\widetilde{\Delta}_j\}_{j=n}^N, \widetilde{B}_n\right]$ is a δ-optimal hedging strategy with

$$\delta \;\equiv\; C\|F\|(dt)^{1/2} + \frac{A\sigma}{2} C\|F\|(dt)^{1/2} \;\sim C\|F\|(dt)^{1/2},$$

where C denotes a constant which depend on σ, μ, A and T, but not on dt or F. In particular,

$$\left|V_n(\Delta, S_n) - \left(\widetilde{P}_n + \frac{k}{2}|\widetilde{Q}_n - \Delta S_n| - \Delta S_n\right)\right| = \left|V_n(\Delta, S_n) - \left(\widetilde{B}_n + \Psi(\widetilde{\Delta}_n - \Delta)S_n\right)\right|$$

$$\leqq C\|F\|(dt)^{1/2} \;. \tag{4.46}$$

The proof of Theorem 4.4 is complete.

4.4 Application to pricing and hedging option portfolios.

We extend the result of Theorem 4.4 to option portfolios.

Theorem 4.5. *Consider a European-style contingent claim with a payoff* $(\Delta(S_N), B(S_N))$ *with* $\Delta(S_N)$ *uniformly bounded. Suppose that the function*

$$F(S_N) \;\equiv\; \Delta(S_N)\; S_N \;+\; B(S_N)$$

is piecewise linear and that $F'(S_N)$ *has finitely many discontinuities.*[17]

Let $P(S,t)$ *be the solution of the final-value problem (4.15) and set*

[17] We shall refer to this function as the *payoff function* of the contingent claim.

$$P_n = P(S_n\,,t_n) \quad , \quad Q_n \;=\; P(S_n\,,t_n) \;-\; S_n\,\frac{\partial P(S_n\,,t_n)}{\partial S}\;.$$

Then, we have

$$\left|V_n(\Delta,S_n) - \left(P_n + \frac{k}{2}|Q_n - \Delta S_n| - \Delta S_n\right)\right| \;\leqq C\,(dt)^{1/8}\;, \tag{4.47}$$

where C depends on F and S_n but not on dt.

Proof. We shall proceed by approximations, reducing the proof to the case of the bounded smooth payoffs treated in Theorem 4.4.

Step 1. Reduction to bounded payoff. Consider a derivative security with the modified final payoff

$$\begin{cases} \Delta_N(S_N) \;-\; \Delta^{(0)} \\ \\ B_N(S_N)\;, \end{cases}$$

where $\Delta^{(0)}$ represents the (constant) slope of the function F at infinity. Clearly, the payoff function $\left(\Delta_N(S_N) \;-\; \Delta^{(0)}\right)\cdot S_N + B_N(S_N) \;=\; F(S_N) \;-\; \Delta^{(0)}\cdot S_N$ is uniformly bounded. As observed in Remark 2.3, the addition or subtraction of a fixed number of shares to the final payoff of a contingent claim induces a trivial change in the optimal hedging strategy: the hedge-ratio is simply translated by the constant number of shares. Notice also that the solution of the nonlinear PDE (4.15) with the modified final payoff $F(S_N) \;-\; \Delta^{(0)}\cdot S_N$ is given by $P(S,t) \;-\; \Delta^{(0)}\cdot S$, where $P(S,t)$ solves (4.15). The modified Delta is $\frac{\partial P}{\partial S} \;-\; \Delta^{(0)}$. Thus, since the hedging problem and the PDE are both invariant under the addition of a linear function to the final payoff we may assume without loss of generality that F is bounded.

Step 2. Approximation by bounded smooth payoff. Given any bounded, continuous piecewise linear function F satisfying the assumptions of the theorem and any $\epsilon \;>\; 0$, we can construct a function $F_\epsilon(S_N)$ with the following properties:

$$|F_\epsilon(S_N) \;-\; F(S_N)| \;\leqq\; \epsilon\;, \tag{4.48}$$

$$|F'_\epsilon(S_N)\;| \;\leqq C\;, \tag{4.49}$$

where C is a constant and

$$\| F_\epsilon \| \ \leqq \ \frac{C}{\epsilon^3} \ . \tag{4.50}$$

The function $F^{(\epsilon)}$ can be obtained by a standard convolution procedure. To derive the estimate in (4.50) we used the fact that F' is uniformly bounded and that the norm $\| \cdot \|$ incorporates derivatives of F_ϵ up to third order.

Define the auxiliary payoff

$$\Delta_N^{(\epsilon)}(S_N) = \ F'_\epsilon(S_N) \qquad \text{and} \qquad B_N^{(\epsilon)}(S_N) = \ F_\epsilon(S_N) - \ F'_\epsilon(S_N)\, S_N \ ,$$

and the auxiliary sequence of portfolios $\{(\Delta_n^{(\epsilon)}, B_n^{(\epsilon)})\}$, obtained by solving the nonlinear PDE (4.15) with final condition F_ϵ. According to Theorem 4.4, the trading strategy $\left[\{\Delta_j^{(\epsilon)}\}_{j=n}^N, B_n^{(\epsilon)}\right]$ is $C\|F_\epsilon\|(dt)^{\alpha/2}$ –optimal for the auxiliary payoff $(\Delta_N^{(\epsilon)}, B_N^{(\epsilon)})$. In particular, let $V_n^{(\epsilon)}(\Delta, S)$ denote the corresponding optimal effective cost function. Then, from (4.46) and (4.50) we have

$$\left| V_n^{(\epsilon)}(\Delta, S_n) - \left(P_n^{(\epsilon)} + \frac{k}{2}|Q_n^{(\epsilon)} - \Delta S_n| - \Delta S_n \right) \right| \leqq \frac{C\,(dt)^{1/2}}{\epsilon^2} \ , \tag{4.51}$$

where $P_n^{(\epsilon)}$ and $Q_n^{(\epsilon)}$ are defined in the usual way using the solution of the PDE. From the estimates (4.48) and (4.49) we conclude that

$$|P(S,t) \ - \ P^{(\epsilon)}(S,t)| \ \leqq \epsilon$$

and

$$\left| \frac{\partial P^{(\epsilon)}(S,t)}{\partial S} \right| \ \leqq \ C \ .$$

Using these estimates and (4.51), we find that

$$\left| V_n^{(\epsilon)}(\Delta, S_n) - \left(P_n + \frac{k}{2}|Q_n - \Delta S_n| - \Delta S_n \right) \right| \leqq \frac{C\,(dt)^{1/2}}{\epsilon^3} \ + \ C\,(dt)^{1/2} \ , \tag{4.52}$$

where C is a constant that depends on F but not on dt.

Finally, let us compare $V_n(\Delta, S_n)$ and $V_n^{(\epsilon)}(\Delta, S_n)$. For this purpose, consider the situation of an agent that must deliver the payoff (Δ_N, B_N) at the expiration date but

holds instead the portfolio $(\Delta_N^{(\epsilon)}, B_N^{(\epsilon)})$. To deliver the latter payoff, he must incur an additional cost of

$$F(S_N) - F_{(\epsilon)}(S_N) + \frac{k}{2}|\Delta(S_N) - F'_{(\epsilon)}(S_N)|S_N$$

dollars. However, since from estimate (4.48), the difference in the payoff functions is $O(\epsilon)$ and that $|\Delta(S_N) - F'_\epsilon(S_N)| < C'$ for some constant C', the agent can meet the final liability by adding to the initial portfolio $(\Delta_n^{(\epsilon)}, B_n^{(\epsilon)})$ a fixed amount of shares and bonds. Namely, he can add to his initial portfolio $C'\,A\,\sigma\,\sqrt{dt}/2$ shares, and bonds with a face value of ϵ dollars. The cost of this (static) portfolio is (to leading order in $dt^{1/2}$) less than

$$\epsilon\, e^{-r(T-t_n)} + \frac{A}{2}\sigma\, C'\, S_n\, \sqrt{dt}\ .$$

Hence,

$$\left| V_n(\Delta, S_n) \ - \ V_n^{(\epsilon)}(\Delta, S_n) \ \right| \leqq \ \epsilon\, e^{-r(T-t_n)} + \frac{A}{2}\sigma\, C'\, S_n\, \sqrt{dt}\,. \tag{4.53}$$

Combining estimates (4.52) and (4.53), we obtain

$$\left| V_n(\Delta, S_n) - \left(P_n + \frac{k}{2}|Q_n - \Delta S_n| - \Delta S_n \right) \right| \ \leqq C\ \left[\epsilon \ + (dt)^{1/2} \ + \ \frac{(dt)^{1/2}}{\epsilon^3} \ \right]\ . \tag{4.54}$$

It is clear that the magnitude of the regularization parameter ϵ which minimizes the right-hand side of this last expression is $\epsilon \ \propto \ C\,(dt)^{1/8}$. The conclusion of Theorem 4.5 follows.

Remark 4.6: Portfolios of European-style calls and puts — settled in cash or in shares — satisfy the assumptions of Theorem 4.5. The theorem also applies to barrier options with continuous payoffs. We note, however, that Theorem 4.5 does not apply to contingent claims with discontinuous payoffs, such as digital options or "up-and-out" barrier options. Nevertheless, even in the case of discontinuous payoff functions, the solution of (4.15) provides satisfactory numerical approximations to the BLPS cost function (see, for instance, Figure 1.4 in the Introduction).

Remark 4.7 In the course of the proof we obtained also an $O\left(dt^{1/8}\right)$— optimal hedging strategy, in which the hedge-ratio is the derivative of a *regularized* solution of the PDE in (4.15) with payoff F_ϵ. This result is weaker than Theorem 4.4, since we have not demonstrated the dt^β — optimality of the strategy which uses the Deltas $(\frac{\partial P}{\partial S})$ of the *exact* solution of (4.15). The reason for this is that Theorem 4.4 assumed strong regularity condition of the payoff function in order to obtain error estimates for residuals. A proof of the optimality of the hedging strategy $\Delta_n \ = \ \frac{\partial P(S_n, t_n)}{\partial S}$ can be done at the expense of a more elaborate analysis, taking into account the behavior of the Deltas in a neighborhood

of the jump discontinuities of F', near expiration. We shall not elaborate on this point further, observing that the dt^β— optimality of such strategy is well-supported by Monte-Carlo simulations (Avellaneda and Parás (1994), for piecewise linear payoffs.

We note also that Theorem 4.5 does not apply directly to contingent claims with discontinuous payoff functions, such as digital options. Nevertheless, payoffs with discontinuities can be super- and sub-dominated by piecewise linear functions which approximate the payoff function everywhere except near points of discontinuity. Even in the case of contingent claims with discontinuous payoff functions, the solution of (4.15) provides an satisfactory numerical approximation to the BLPS cost function for $A < 1$ (see Figure 1.4).

5. Non-convex payoffs with $A \geqq 1$ and Path-dependent strategies

We characterize the solution of the BLPS algorithm for contingent claims with mixed convexity in the regime $A \geqq 1$. Under these circumstances, the optimal strategies are path-dependent and the BLPS algorithm does not correspond to the the backward-induction equation (3.26). The optimal strategies and their effective cost can be characterized in the limit $N \nearrow +\infty$ by means of a nonlinear PDE corresponding to an obstacle problem for a Black-Scholes PDE with an adjusted volatility.

5.1 The Obstacle Problem and its associated hedging strategy.

Definition 5.1. *We say that $P(S,t)$ is the solution to the obstacle problem with payoff $F(S)$ if*

$$P(S,t) \geqq e^{-r(T-t)} F(Se^{r(T-t)}) \ , \tag{5.1}$$

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2(1+A)S^2\frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP \leqq 0 \tag{5.2}$$

for all (S,t) and

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2(1+A)S^2\frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0$$

whenever

$$P(S,t) > e^{-r(T-t)} F(Se^{r(T-t)}) \ , \tag{5.3}$$

with

$$P(S,T) \ = \ F(S) \ . \tag{5.4}$$

Remark 5.2. This free-boundary problem has a unique solution for any bounded, continuous payoff function $F(S)$ and is mathematically well-posed. It represents a natural "continuation" of (4.15) for $A \geqq 1$, which corresponds formally to solving the final-boundary problem for a degenerate nonlinear PDE with a diffusion coefficient equal to $\sigma^2\,(1+A)$ for $P_{SS} > 0$ and 0 for $P_{SS} \leqq 0$ (in the sense of "viscosity solutions" (see Crandall, Ishii and Lions (1992)). For convex payoffs, it reduces to the well-known Black-Scholes equation with volatility $\sigma\sqrt{1+A}$ proposed by Boyle and Vorst (1992) (cf. also Section 4).

The *contact set* associated with the obstacle problem (5.1)-(5.4) is

$$\mathcal{C} \ = \ \left\{(S,t) : \ P(S,t) = \ e^{-r(T-t)} F(e^{r(T-t)}\ S) \ , \ t < T\right\} \ . \tag{5.5}$$

This is a closed set in the (S,t)-plane. We shall denote its boundary by $\partial\,\mathcal{C}$.

We will denote by $\mathcal{R}_n$ the ray in the (S,t)-plane which joins the points (S_{n-1}, t_{n-1}) and (S_n, t_n).

Definition 5.3. *We define the hedging strategy associated with the obstacle problem as the strategy with initial cash reserve given by*

$$B_0 \;=\; P(S,t) \;-\; \Delta_0\, S \tag{5.6}$$

and with a sequence of stock-holdings $\{\Delta_n\}$ obtained through the following two phase process, starting in Phase I if $(S_0\,, t_0\,) \notin \mathcal{C}$ and in Phase II otherwise.

Phase I: Delta-Hedging. The hedge-ratio is adjusted at every trading date to

$$\Delta_n \;=\; \frac{\partial P}{\partial S}(S_n,\, t_n)\;, \tag{5.7}$$

until the first trading date $t_{n'}$ at which the ray $\mathcal{R}_{n'}$ intersects $\partial\,\mathcal{C}$ or $n' = N$. If $t_{n'} = t_N$ the stock-holdings are adjusted to $\Delta_N = F'(S_N)$ and the strategy stops. Otherwise, the strategy goes into Phase II.

Phase II: Static Hedging. Denote by (S^*, t^*) the point at which $\mathcal{R}_{n'}$ intersects $\partial\,\mathcal{C}$ (see Figure 5.1).[18] Thereafter, the hedge-ratio is maintained fixed at the value

$$\Delta_n \;=\; \frac{\partial P}{\partial S}(S^*,\, t^*) \;=\; F'(S^*)\;, \tag{5.8}$$

until the first time $t_{n"}$ at which $(S_{n"}, t_{n"}) \notin \mathcal{C}$ and

$$\left|\, \frac{\partial P}{\partial S}(S_{n"},\, t_{n"}) \;-\; F'(S^*) \,\right| \;\leqq\; \left|\, \frac{\partial P}{\partial S}(S_{n"},\, t_{n"}) \;-\; \frac{\partial P}{\partial S}(S_{n"-1},\, t_{n"-1}) \,\right|\;, \tag{5.9}$$

or $n" = N$. If $t_{n"} = t_N$ the stock-holdings are adjusted to $\Delta_N = F'(S_N)$ and the strategy stops. Otherwise, the strategy goes back to Phase I.

The main result of this Section is

[18] If $n' = 0$ then $(S^*, t^*) = (S_0\,, t_0\,)$.

Theorem 5.4. *Assume that $A \geqq 1$. Let $F(S)$ be a function satisfying the regularity condition (4.14) and let $P(S,t)$ be the solution to the obstacle problem (5.1)-(5.4). Consider the European-style contingent claim with payoff*

$$\begin{cases} \Delta_N(S_N) = F'(S_N) \\ \\ B_N(S_N) = F(S_N) - S_N F'(S_N) \ , \end{cases} \tag{5.10}$$

and let $[\{\Delta_n\}_{n=0}^N, B_0]$ be the hedging strategy associated with the obstacle problem (cf. (5.6)-(5.9)). Then, $[\{\Delta_n\}_{n=0}^N, B_0]$ is a $C \| F \| (dt)^{1/2}$-optimal hedging strategy for (Δ_N, B_N), where C is a constant which depends on μ, σ, A and T.

Proof. The proof of Theorem 5.4 is done by deriving upper and lower bounds on the effective cost function $V_n(\Delta,\, S)$. Proposition 5.10 hereafter shows that $P(S,t) - \Delta\, S - C \| F \| (dt)^{1/2}$ is a lower bound of the minimum effective cost. Proposition 5.14 shows that $[\{\Delta_n\}_{n=0}^N, B_0 + C \| F \| (dt)^{1/2}]$ dominates the final payoff and hence $V_n(\Delta,\, S) \leqq P(S,t) - \Delta\, S \, + \, C \| F \| (dt)^{1/2}$. The Theorem follows immediately from these two propositions.

Remark 5.5 The assumptions on the payoff and on the regularity of the function F are made for mathematical convenience. The generalization of Theorem 5.4 to payoffs corresponding to option portfolios is done as in the case $A < 1$ (See Paragraph 4.4).

We assume hereafter that $r = 0$, which amounts to measuring values in dollars-at-expiration rather than in present value. The letter C will denote a generic constant which depends on μ, σ, A and T and which may vary from one inequality to another.

5.2 The lower bound.

For the purposes of comparing various cost functions, it is convenient to consider models in which the transaction cost parameter may vary from one node of the binomial tree to another, i.e.

$$k \ = \ k(n\,,j) \ , \qquad 0 \leqq j \leqq n \leqq N \ . \tag{5.11}$$

The following Lemma is self-evident:

Lemma 5.6. *Consider the BLPS model with two transaction cost functions such that*

$$k^{(1)}(n\,,j) \ \leqq \ k^{(2)}(n\,,j) \ .$$

Let $V_n^{(1)}(\Delta, S_n)$ and $V_n^{(2)}(\Delta, S_n)$ represent the minimal effective cost functions of a derivative security under the two transaction cost regimes. Then

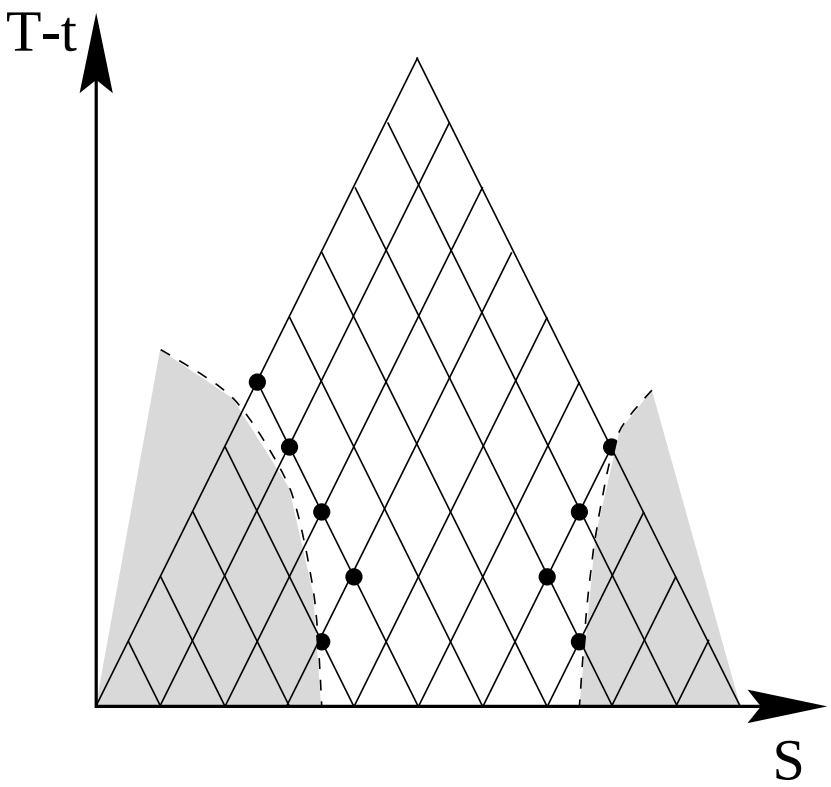


FIGURE 5.1. Schematic rendering of the binomial lattice near the contact set $\mathcal{C}$. The heavy dots represent the points in $D\,\mathcal{C}$.

$$V_n^{(1)}(\Delta, S_n) \;\leqq\; V_n^{(2)}(\Delta, S_n)\;.$$

Lemma 5.7. *Let $D\mathcal{C}$ be the set of nodes (n,j) in the binomial tree such that*

$$(i) \qquad (S_n^j, t_n) \;\notin\; \mathcal{C}$$

and

$$(ii) \quad \left[(S_n^j, t_n)\,,\,(S_n^j U, t_{n+1})\right] \cap\; \mathcal{C} \;\neq\; \emptyset \quad \textit{or} \quad \left[(S_n^j, t_n)\,,\,(S_n^j D\,, t_{n+1})\right] \;\cap\; \mathcal{C} \;\neq\; \emptyset.^{19}$$

For any $k \;=\; A\,\sigma\,\sqrt{dt}$ with $A \;\geqq\; 1$, define the transaction cost function

$$k^{(1)}\,(n\,,j) \;=\; \begin{cases} k & \textit{if } \;(n,j) \;\notin\; D\mathcal{C} \\ \\ \sigma\,\sqrt{dt} & \textit{if } \;(n,j) \;\in\; D\mathcal{C}\;. \end{cases} \tag{5.12}$$

[19] We use the notation $[A, B]$ to describe the segment connecting the points A and B in the (S,t) -plane. Thus, $D\,\mathcal{C}$ can be viewed as the "discrete outer boundary" of $\mathcal{C}$, formed of the nodes which are not in $\mathcal{C}$ but which "connect" immediately to $\partial\mathcal{C}$.

Let $V_n^{(1)}(\Delta, S_n)$ denote the minimum effective cost for the derivative security with payoff (5.10) assuming the transaction cost function (5.12). Then, there exists a constant $C = C(\sigma, \mu, A, T)$ such that

$$V_n^{(1)}(\Delta_{n-1}, S_n) \;\geqq\; F(S_n\;) \;-\; \Delta_{n-1}S_n \;+\frac{\sigma\sqrt{dt}}{2}\;|\Delta_{n-1}\;|\,S_n \;-\;\; C\|F\|\;(dt)^{1/2} \qquad (5.13)$$

for all n, S_n and Δ_{n-1}.

Proof. Given any $A' \;<\; 1$, define $k' \;=\; A'\,\sigma\,\sqrt{dt}$ and let $V_n'(\Delta\,, S_n)$ denote the optimal effective cost function corresponding to the derivative security with payoff (5.10) with k' as the transaction cost parameter. Then, by Lemma 5.6,

$$V_n^{(1)}(\Delta\,, S_n) \;\geqq\; V_n'(\Delta\,, S_n)\;.$$

Let $P^{(A')}(S,t)$ denote the solution of the final-value problem (4.15) with $r = 0$ and Leland number A'. From this last inequality and Theorem 4.4, we have

$$\begin{aligned} V_n^{(1)}(\Delta, S_n) \;&\geqq\; V_n'(\Delta, S_n) \\ &\geqq\; P^{(A')}(S_n, t_n) \;-\; \Delta\, S_n \\ &\quad +\;\frac{k'}{2}\left|\frac{\partial P^{(A')}}{\partial S}(S_n, t_n) \;-\; \Delta\right| S_n \;-\; C\,\|F\|\,(dt)^{1/2} \\ &\geqq\; P^{(A')}(S_n, t_n) \;-\; \Delta\, S_n \;+\;\frac{k'}{2}\;|\Delta|\;S_n - \;C\,\|F\|\,(dt)^{1/2}\;, \end{aligned}$$

where we used the fact that $S\,P_S^{(A')}$ is uniformly bounded. The desired estimate, (5.13), will follow by letting $A' \nearrow 1$, after we establish that

$$\lim_{A'\uparrow 1} P^{(A')}(S,t) \;\geqq\; F(S)\;. \qquad (5.14)$$

To prove (5.14), we introduce the auxiliary function

$$W(S,t) \;\equiv\; F(S) \;-\; \nu(T-t)\;,$$

where ν is a positive constant. We claim that $W(S,t)$ is a sub-solution of the equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 \left[1 + A' \operatorname{sign}\left(\frac{\partial^2 P}{\partial S^2}\right)\right] S^2 \frac{\partial^2 P}{\partial S^2} = 0 \tag{5.15}$$

for A' sufficiently close to 1. In fact, since $\left|S^2 \frac{d^2 F}{dS^2}\right| \leqq \|F\|$,

$$\begin{aligned}
&\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 \left[1 + A' \operatorname{sign}\left(\frac{\partial^2 W}{\partial S^2}\right)\right] S^2 \frac{\partial^2 W}{\partial S^2} \\
&\qquad = \nu + \frac{1}{2}\sigma^2 \left[1 + A' \operatorname{sign}\left(\frac{d^2 F}{dS^2}\right)\right] S^2 \frac{d^2 F}{dS^2} \\
&\qquad \geqq \nu - \frac{1}{2}\sigma^2 [1 - A'] \|F\| \\
&\qquad \geqq 0
\end{aligned}$$

whenever $A' > 1 - \frac{2\nu}{\|F\|\sigma^2}$. Assume henceforth that A' is in this range. Since $W(S,t)$ is a sub-solution of (5.15) and $P^{(A')}(S,t)$ is a solution and, moreover, $W(S,T) = P^{(A')}(S,T) = F(S)$, we conclude from the Maximum Principle that $P^{(A')}(S,t) \geqq W(S,t)$ for $t < T$. Therefore,

$$\lim_{A' \uparrow 1} P^{(A')}(S,t) \geqq W(S,t) = F(S) - \nu(T-t) \qquad \nu > 0 \ .$$

Letting $\nu \downarrow 0$ in this last inequality, we conclude that (5.14) holds, and the proof is complete.

Lemma 5.8. *Let $t_{n'} \leqq t_N$ be the first hitting time of the set $D\mathcal{C}$ (with $t'_n = t_N$ if the path does not hit $D\mathcal{C}$). Assume that the transaction cost is given by (5.12), and let $\widehat{V'_n}(\Delta, S_n)$ represent the minimum effective cost of the contingent claim with payoff*

$$\begin{cases} \Delta_{n'}(S_{n'}) = F'(S_{n'}) \\ \\ B_{n'}(S_{n'}) = F(S_{n'}) - S_{n'} F'(S_{n'}) \quad . \end{cases} \tag{5.16}$$

Then, for any $n < n'$ and any initial endowment Δ_{n-1},

$$V_n^{(1)}(\Delta_{n-1}, S_n) \;\geqq\; \widehat{V_n'}(\Delta_{n-1}, S_n) \;-\; C\,\|F\|(dt)^{1/2} \;. \tag{5.17}$$

Proof. Suppose that an agent has an endowment of $\Delta_{n'-1}$ shares at time $t_{n'}$ (before rebalancing). The effective cost of delivering the payoff at time $t_{n'}$ for $n' < N$ will be, according to (5.16),

$$\begin{aligned}
\widehat{V_{n'}'}(\Delta_{n'-1}, S_{n'}) \;&= \Psi\,(F'(S_{n'}) \;-\; \Delta_{n'-1})\; S_{n'} \;+\; F(S_{n'}) \;-\; S_{n'}\,F'(S_{n'}) \\
&= F(S_{n'}) \;-\Delta_{n'-1}\,S_{n'} \;+\frac{k^{(1)}}{2}|F'(S_{n'}) \;-\Delta_{n'-1}|S_{n'} \\
&= F(S_{n'}) \;-\Delta_{n'-1}\,S_{n'} \;+\frac{\sigma\sqrt{dt}}{2}|F'(S_{n'}) \;-\Delta_{n'-1}|S_{n'} \;.
\end{aligned}$$

Thus, by Lemma 5.7 and the boundedness of $S\,F'(S)$, we conclude that

$$V_{n'}^{(1)}(\Delta_{n'-1}, S_{n'}) \;\geqq\; \widehat{V_{n'}'}(\Delta_{n'-1}, S_{n'}) \;-\; C\,\|F\|\,(dt)^{1/2} \;. \tag{5.18}$$

Inequality (5.17) then follows from Proposition 2.4. Q.E.D.

In the next Lemma, we estimate the function $\widehat{V_N'}(\Delta, S)$.

Lemma 5.9. *Let $P(S,t)$ be the solution of the obstacle problem (5.1)-(5.4) with $r = 0$ and set*

$$P_n \;=\; P(S_n, t_n)\;, \qquad Q_n \;=\; S_n\,\frac{\partial P(S_n, t_n)}{\partial S}\;, \tag{5.19}$$

Let $t_{n'}$ be stopping time defined in Lemma 5.8. Consider a contingent claim which delivers the payoff

$$\begin{cases} \widetilde{\Delta}_{n'}(S_{n'}) \;=\; Q_{n'}/S_{n'} \\ \\ \widetilde{B}_{n'}(S_{n'}) \;=\; P_{n'} \;-\; Q_{n'}\;, \end{cases} \tag{5.20}$$

when $t = t_{n'}$ and denote its minimum effective cost function under the transaction cost function (5.12) by $\widehat{W}_n(\Delta, S_n)$. Then

$$(i) \qquad \widehat{V_n'}(\Delta, S_n) \;\geqq\; \widehat{W}_n(\Delta, S_n) \;-\; C\,\|F\|\,(dt)^{1/2} \tag{5.21}$$

for all $n \leqq n'$ *and all* Δ*; and*

$$(ii) \qquad \left|\widehat{W}_n(\Delta\,,\,S_n) \;-\left(\;P_n \;+\; \frac{k}{2}\,|Q_n \;-\; \Delta\,S_n\,| \;-\; \Delta\,S_n\right)\right| \;\leqq\; C\,||F||\,(dt)^{1/2}\;. \tag{5.22}$$

Proof. (i) The functions P and $\frac{\partial P}{\partial S}$ are uniformly continuous along the boundary of the contact set and

$$\left|\;P(S_{n'}, t_{n'}) - F(S_{n'})\;\right| \leqq C\|F\|(dt)^{1/2},$$

$$\left|\frac{\partial P}{\partial S}(S_{n'}, t_{n'}) - F'(S_{n'})\right| \leqq C\|F\|(dt)^{1/2}\;.$$

This ensures that the payoff (5.16) dominates (both in the number of shares and in bonds) the payoff (5.10), up to and error of order $C\,\|F\|\,(dt)^{1/2}$. Hence, (5.21) follows.

(ii) Since $P(S,t)$ is convex in the complement of $\mathcal{C}$, the portfolios $(\widetilde{\Delta}_n, \widetilde{B}_n) = (Q_n/S_n, P_n - Q_n)$, with P_n and Q_n given by (5.22) are k-convex as well as $k^{(1)}$-convex. We know that $\binom{P_n}{Q_n}$ satisfies the recursion relation

$$\begin{pmatrix}P_n\\Q_n\end{pmatrix} \;=\; \Phi\begin{pmatrix}P_{n+1}\\Q_{n+1}\end{pmatrix} \;+\; \begin{pmatrix}p_n\\q_n\end{pmatrix} \qquad n \leqq n'-1,$$

with remainders satisfying

$$|p_n| \leqq C\|F\|\,(dt)^{3/2} \qquad \text{and} \qquad |q_n| \leqq C\|F\|\,dt\;, \tag{5.23}$$

The estimate (5.23) follows from Taylor expansions, as in the proof of Step 2, Theorem 4.4, using the fact that $P(S,t)$ satisfies the PDE (5.3). Now, since $k^{(1)} = k$ in the complement of $D\mathcal{C}$, we have also

$$\begin{pmatrix}P_n\\Q_n\end{pmatrix} \;=\; \Phi^{(1)}\begin{pmatrix}P_{n+1}\\Q_{n+1}\end{pmatrix} \;+\; \begin{pmatrix}\hat{p}_n\\\hat{q}_n\end{pmatrix} \qquad n \leqq n'-1\;, \tag{5.24}$$

where $\Phi^{(1)}$ denotes the backward-induction operator corresponding to the transaction cost function $k^{(1)}$. Notice the the remainders $\binom{\hat{p}_n}{\hat{q}_n}$ are identical to $\binom{p_n}{q_n}$ for $n \leqq n'-2$.

To derive an estimate for the remainders $\binom{\hat{p}_{n'-1}}{\hat{q}_{n'-1}}$, observe that k and $k^{(1)}$ differ only by by $(A-1)\,\sigma\,\sqrt{dt}$ along $D\,\mathcal{C}$. Hence, the backward-induction operators Φ and $\Phi^{(1)}$ satisfy

$$\| \ \Phi \ - \ \Phi^{(1)} \ \| \ \propto \ \sqrt{dt}$$

at the nodes of $D\,\mathcal{C}$. Therefore, since we have

$$\binom{\hat{p}_{n'-1}}{\hat{q}_{n'-1}} \ = \ \binom{p_{n'-1}}{q_{n'-1}} \ + \ \left(\Phi \ - \ \Phi^{(1)}\right) \binom{P_{n'}}{Q_{n'}},$$

and $P_{n'}$ and $Q_{n'}$ are uniformly bounded, the remainders in (5.24) for $n = n' - 1$ can be at most of order $O(\sqrt{dt})$.

The estimate (5.22) for $\widehat{W}_n(\Delta, S_n)$ now follows from Proposition 4.3.[20]

Proposition 5.10. *Let $V_n(\Delta, S_n)$ denote the minimum effective cost function for the contingent claim with payoff (5.10), with constant $k \ \geqq \ 1$. Let $P(S,t)$ represent the solution of the obstacle problem with Leland number $A \ = \ k/(\sigma\,\sqrt{dt})$. Then,*

$$V_n(\Delta\,, S_n) \ \geqq \ P(S_n, t_n) \ \ - \Delta\, S_n \ + \ \frac{k}{2} |S_n\, P'_S(S_n, t_n) \ - \Delta S_n| \ - \ C\,\|F\|\,(dt)^{1/2} \ . \quad (5.25)$$

Proof. By Lemmas 5.6 and 5.8, we have

$$V_n(\Delta\,, S_n) \ \geqq \ V_n^{(1)}(\Delta, S_n) \ \geqq \ \widehat{V'_n}(\Delta\,, S_n) \ - \ C\,\|F\|\,(dt)^{1/2} \ . \quad (5.26)$$

Using Lemma 5.9, eq. (5.21), we conclude that $V_n(\Delta, S_n)$ satisfies the lower bound (5.25). Q.E.D.

[20] In order to obtain the right estimate from Proposition 4.3, δ must be obtained by first adding the residuals $|p_n| + \frac{k}{2}|q_n|$ which are of order $O\big((dt)^{3/2}\big)$, and then adding the $O\big((dt)^{1/2}\big)$ term from $|p_{n'-1} - \hat{p}_{n'-1}| + \frac{k}{2}|q_{n'-1} - \hat{q}_{n'-1}|$.

5.3 Analysis of the associated strategy and upper bound.

In this paragraph, we prove that the strategy associated with the obstacle problem dominates the final payoff. We shall denote by Π_n the mid-value of the hedge portfolio, i.e.,

$$\Pi_n \;\equiv\; \Delta_n\, S_n \,+\, B_n \;. \tag{5.27}$$

We shall also make use of the sequence

$$\kappa_n \;\equiv\; \Pi_n \,-\, P_n \qquad n = 0, 1, ..., N \;, \tag{5.28}$$

where P_n is defined in (5.19). In particular, κ_N represents the overall profit/loss generated by the strategy associated with the obstacle problem.

Proposition 5.11. *Assume that the hedging strategy goes through Phase I starting at t_n until $t_{n'}$. Then, for every $j \in [n, \ldots, n'-2]$,*

$$\Delta_j\,(S_{j+1} - S_j) \,-\, \frac{k}{2}\,|\Delta_{j+1} - \Delta_j|\,S_{j+1} \;=\; P_{j+1} \,-\, P_j \,+\, \rho_{j+1} \;,$$

$$\textit{with} \qquad\qquad |\rho_{j+1}| \;\leqq\; C\,\|F\|\,(dt)^{3/2} \;. \tag{5.29}$$

Proof. According to (5.7), $\Delta_j \equiv P_S(S_j, t_j)$ for all $j \in [n, \ldots, n'-1]$. Therefore, it follows by Taylor expansion that for all $j \in [n, \ldots, n'-2]$

$$\begin{aligned} \frac{k}{2}\,|\Delta_{j+1} \,-\, \Delta_j|\,S_{j+1} \;&=\; \frac{k}{2}\,|P_{SS}(S_j, t_j)\,(S_{j+1} - S_j)|\,S_{j+1} \,+ \hat{\rho}'_{j+1} \\ &=\; \frac{A}{2}\,\sigma^2\,S_j^2\,|P_{SS}(S_j, t_j)|\,dt \;+\; \hat{\rho}_{j+1} \;, \end{aligned} \tag{5.30}$$

$$\text{with} \qquad\qquad |\hat{\rho}_{j+1}| \;\leqq\; C\,\|F\|\,(dt)^{3/2} \;.$$

Similarly, by Taylor expansion

$$P_{j+1} - P_j - \Delta_j (S_{j+1} - S_j) = P_t(S_j, t_j)\, dt + \frac{1}{2} P_{SS}(S_j, t_j)\, (S_{j+1} - S_j)^2 + \bar{\rho}'_{j+1}$$

$$= P_t(S_j, t_j)\, dt + \frac{1}{2} P_{SS}(S_j, t_j)\, S_j^2\, \sigma^2\, dt + \bar{\rho}_{j+1} , \tag{5.31}$$

$$\text{with} \qquad |\bar{\rho}_{j+1}| \leqq C \|F\| (dt)^{3/2} .$$

Combining (5.30) and (5.31), we get

$$P_{j+1} - P_j - \Delta_j (S_{j+1} - S_j) + \frac{k}{2} |\Delta_{j+1} - \Delta_j| S_{j+1}$$

$$= \left[P_t(S_j, t_j) + \frac{1}{2} \sigma^2 (1 + A) S_j^2 \; P_{SS}(S_j, t_j) \right] dt + \rho_{j+1} , \tag{5.32}$$

$$\text{with} \qquad |\rho_{j+1}| \leqq C \|F\| (dt)^{3/2} .$$

Since $(S_j, t_j) \notin \mathcal{C}$, it follows from (5.3) (with r=0) that

$$P_t(S_j, t_j) + \frac{1}{2} \sigma^2 (1 + A) S_j^2 \; P_{SS}(S_j, t_j) = 0 . \tag{5.33}$$

Substituting (5.33) in (5.33) the result of Proposition 5.11 follows.

Proposition 5.12. *Under the same assumption as in Proposition 5.11,*

$$|\kappa_{n'-1} - \kappa_n| \leqq C \|F\| \frac{t_{n'-1} - t_n}{t_N - t_0} (dt)^{1/2} . \tag{5.34}$$

Proof. Expanding $\Pi_{n'-1}$ and then applying (5.29) of Proposition 5.11 we obtain

$$\Pi_{n'-1} = \Pi_n + \sum_{j=n}^{n'-2} \left[\Delta_j (S_{j+1} - S_j) - \frac{k}{2} |\Delta_{j+1} - \Delta_j| S_{j+1} \right]$$

$$= \Pi_n + \sum_{j=n}^{n'-2} [P_{j+1} - P_j] + \sum_{j=n+1}^{n'-1} \rho_{j+1}$$

$$= \Pi_n + P_{n'-1} - P_n + \sum_{j=n+1}^{n'-1} \rho_{j+1} .$$

Therefore

$$|\kappa_{n'-1} - \kappa_n| = |(\Pi_{n'-1} - P_{n'-1}) - (\Pi_n - P_n)| \leqq \sum_{j=n+1}^{n'-1} |\rho_{j+1}| . \tag{5.35}$$

Applying inequality (5.30) to (5.35), the conclusion follows.

Proposition 5.13. *Assume that the hedging strategy goes through Phase II starting at time $t_{n'}$ until time $t_{n''}$. Then,*

$$\kappa_{n''} - \kappa_{n'} \geqq - C \|F\| (dt)^{3/2} . \tag{5.36}$$

Proof. Denote by (S^*, t^*) the point where $\partial\mathcal{C}$ intersects $\mathcal{R}_{n'-1}$. The proof of the proposition is divided into several steps.

For any given pair $(\bar{S}, \bar{t}) \in \partial\mathcal{C}$ we use the following convention

$$P_t(\bar{S}_-, \bar{t}_-) \equiv \lim_{\substack{(S,t)\to(\bar{S},\bar{t}) \\ (S,t)\notin\mathcal{C}}} P_t(S,t) ,$$

$$P_{SS}(\bar{S}_-, \bar{t}_-) \equiv \lim_{\substack{(S,t)\to(\bar{S},\bar{t}) \\ (S,t)\notin\mathcal{C}}} P_{SS}(S,t) .$$

As shown in the Appendix, for any $(S,t) \in \partial\mathcal{C}$, we have

$$P_t(S_-, t_-) = P_{SS}(S_-, t_-) = 0 . \tag{5.37}$$

We shall give a proof of the Proposition under the assumption that $(S_{n''-1}, t_{n''-1}) \notin \mathcal{C}$. The case $(S_{n''-1}, t_{n''-1}) \in \mathcal{C}$ is analogous. Under this assumption, denote by S^o the value of S at which the segment joining $(S_{n'}, t_{n''-1})$ and $(S_{n''-1}, t_{n''-1})$ intersects $\partial\mathcal{C}$ (see Figure 5.2.A).

Step 1: Change in value of the hedging portfolio. The mid-value of the portfolio at time $t_{n''}$ is

$$\Pi_{n''} = \Pi_{n'-1} + P_S(S_{n'-1}, t_{n'-1}) (S_{n'} - S_{n'-1}) - \frac{k}{2} |P_S(S_{n'-1}, t_{n'-1}) - F'(S^*)| S_{n'}$$

$$+ F'(S^*) (S_{n''} - S_{n'}) - \frac{k}{2} |P_S(S_{n''}, t_{n''}) - F'(S^*)| S_{n''} . \tag{5.38}$$

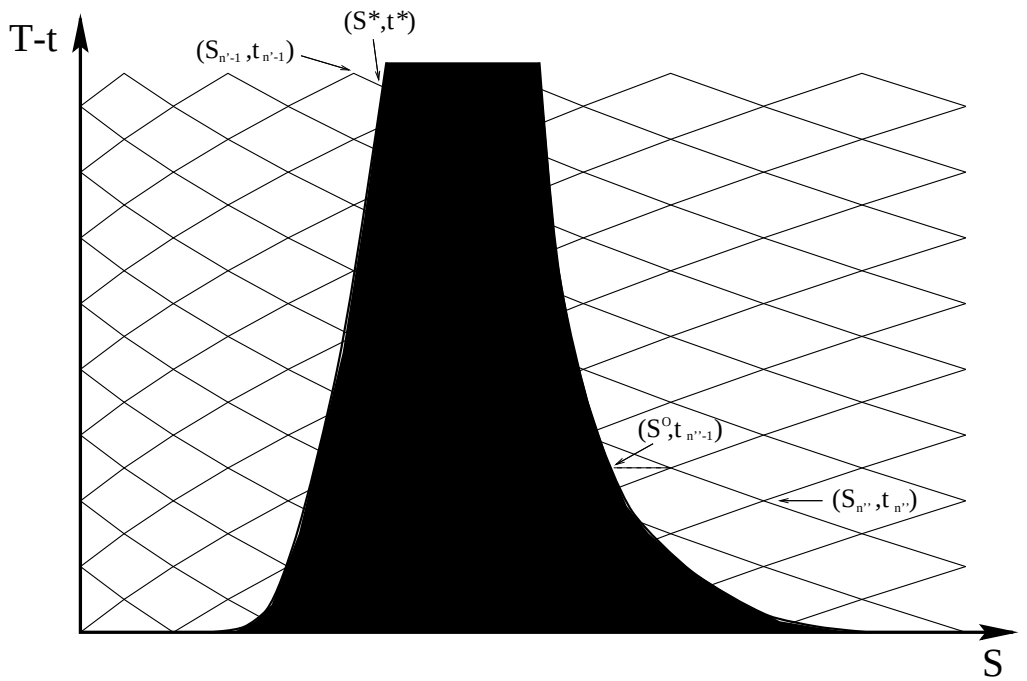


FIGURE 5.2.A. Rendering of the boundary of the contact set and of the points $(S_{n'}, t_{n'})$ $(S_{n''}, t_{n''})$ and S_0, used in the proof of Proposition 5.13.

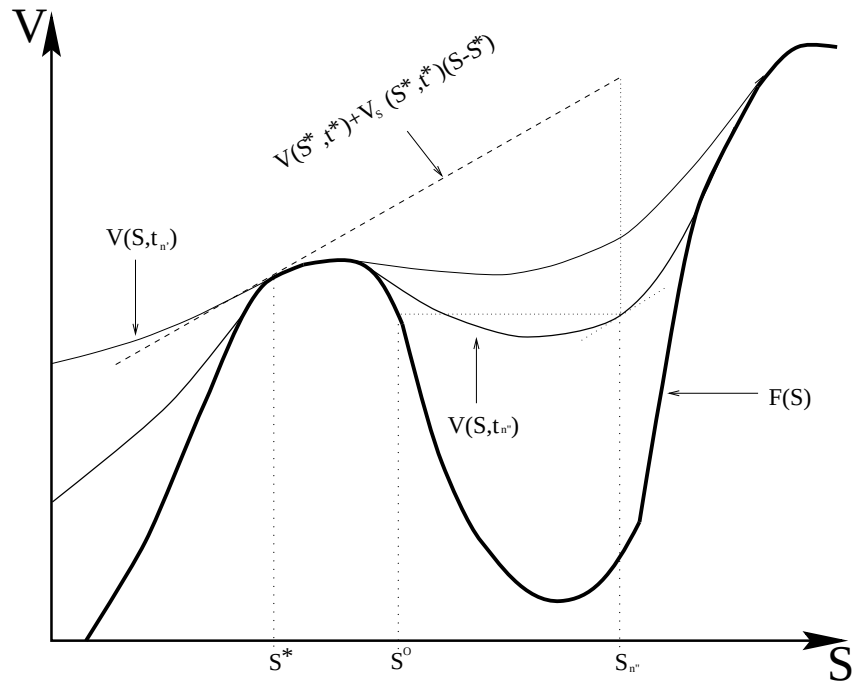


FIGURE 5.2.B. Representation of the solution of the obstacle problem and the associated hedging strategy after the point (S, t) enters $\mathcal{C}$. A temporary holding strategy is implemented after the first hitting time, t_*. The value of the portfolio will then vary along the tangent line to the graph of $F(S)$. Delta-hedging resumes after the portfolio Delta (the slope of the line) and the theoretical Delta coincide at the price level $S_{n''}$. Resuming Delta-hedging at that time allows the agent to remain above the intrinsic value $F(S)$ if the market continues to rally.

By Taylor expansion of the seond and third terms, we find that

$$P_S(S_{n'-1}, t_{n'-1}) \;=\; F'(S^*) \,+\, \frac{1}{2}\, P_{SS}(S^*_-, t^*_-)\, (S_{n'-1} \,-\, S^*) \,+\, \rho_1 \;, \tag{5.39}$$

$$|P_S(S_{n'-1}, t_{n'-1}) \,-\, F'(S^*)| \;=\; P_{SS}(S^*_-, t^*_-)\, (S_{n'-1} \,-\, S^*) \,+ \rho_2 \;, \tag{5.40}$$

and, according to (5.9)

$$-\,\frac{k}{2}\, |P_S(S_{n"}, t_{n"}) \,-\, F'(S^*)|\; S_{n"} \;\geqq\; -\,\frac{k}{2}\, |P_S(S_{n"}, t_{n"}) \,-\, P_S(S_{n"-1}, t_{n"-1})|\; S_{n"}$$

$$=\; -\,\frac{A}{2}\,\sigma^2\; S^2_{n"}\, P_{SS}(S_{n"}, t_{n"})\, dt \,+\, \rho_3 \;, \tag{5.41}$$

where

$$|\rho_1\; (S_{n'} \,-\, S_{n'-1}) \,+\, \frac{k}{2}\,\rho_2\, S_{n'} \,+ \rho_3| \;\leqq\; C\, \|F\|\, (dt)^{3/2} \;.$$

Substituting (5.39)-(5.41) into (5.38), and using (5.37) we get

$$\Pi_{n"} \,-\, \Pi_{n'-1} \;\geqq\; F'(S^*)\, (S_{n"} \,-\, S_{n'-1}) \,-\, \frac{A}{2}\,\sigma^2\; S^2_{n"}\, P_{SS}(S_{n"}, t_{n"})\, dt \,+\, \rho_4 \;, \tag{5.42}$$

where

$$\rho_4 \;=\; \rho_1\; (S_{n'} \,-\, S_{n'-1}) \,+\, \frac{k}{2}\,\rho_2\, S_{n'} \,+\, \rho_3 \;.$$

Step 2: Change in the value function. A Taylor expansion of $P(S_{n"}, t_{n"})$ gives

$$P(S_{n"}, t_{n"}) \;= P(S_{n'-1}, t_{n'-1}) \,+\, F'(S^*)\; (S_{n'-1} \,-\, S^*) \,+\, P_t(S^*_-, t^*_-)\; (t_{n'-1} - t^*)$$

$$+\; \frac{1}{2}\, P_{SS}(S^*_-, t^*_-)\; (S_{n'-1} \,-\, S^*)^2 \,+\, \int_{S^*}^{S^o} F'(S)\, dS$$

$$+\; \int_{S^o}^{S_{n"-1}} P_S(S, t_{n"-1})\, dS \,+\, P_S(S_{n"-1}, t_{n"-1})\; (S_{n"} \,-\, S_{n"-1})$$

$$+\; P_t(S_{n"-1}, t_{n"-1})\; dt \,+\, \frac{1}{2}\,\sigma^2\; S^2_{n"-1}\, P_{SS}(S_{n"-1}, t_{n"-1})\, dt \,+\, \rho_5 \;,$$

with

$$|\rho_5| \;\leqq\; C\, \|F\|\, (dt)^{3/2} \;.$$

Therefore, using (5.37)

$$P(S_{n^{\prime\prime}}, t_{n^{\prime\prime}}) - P(S_{n^\prime - 1}, t_{n^\prime - 1}) \;=\; F'(S^*)\,(S_{n^\prime - 1} \,-\, S^*) \,+\, \int_{S^*}^{S^o} F'(S)\,dS$$

$$+ \int_{S^o}^{S_{n^{\prime\prime}-1}} P_S(S, t_{n^{\prime\prime}-1})\,dS \,+\, P_S(S_{n^{\prime\prime}-1}, t_{n^{\prime\prime}-1})\,(S_{n^{\prime\prime}} \,-\, S_{n^{\prime\prime}-1})$$

$$+\, P_t(S_{n^{\prime\prime}-1}, t_{n^{\prime\prime}-1})\,dt \,+\, \frac{1}{2}\,\sigma^2\, S^2_{n^{\prime\prime}-1}\, P_{SS}(S_{n^{\prime\prime}-1}, t_{n^{\prime\prime}-1})\,dt \,+\, \rho_5 \;. \tag{5.43}$$

Step 3. Change in value of κ. Subtracting (5.43) from (5.42), and taking into account that (5.33) holds at $(S_{n^{\prime\prime}-1}, t_{n^{\prime\prime}-1})$, we have

$$\kappa_{n^{\prime\prime}} \,-\kappa_{n^\prime - 1} \;\geqq\; \int_{S^*}^{S^o} [F'(S^*) \,-\, F'(S)]\;dS \,+\, \int_{S^o}^{S_{n^{\prime\prime}-1}} [F'(S^*) \,-\, P_S(S, t_{n^{\prime\prime}-1})]\;dS$$

$$+\,[F'(S^*) \,-\, P_S(S_{n^{\prime\prime}-1}, t_{n^{\prime\prime}-1})]\,(S_{n^{\prime\prime}} \,-\, S_{n^{\prime\prime}-1})$$

$$-\left[P_t(S_{n^{\prime\prime}-1}, t_{n^{\prime\prime}-1}) \,+\, \frac{1}{2}(1 \,+\, A)\,\sigma^2\, S^2_{n^{\prime\prime}-1}\, P_{SS}(S_{n^{\prime\prime}-1}, t_{n^{\prime\prime}-1})\right] dt \,+\, \rho_6 \;,$$

$$= \int_{S^*}^{S^o} [F'(S^*) \,-\, F'(S)]\;dS \,+\, \int_{S^o}^{S_{n^{\prime\prime}-1}} [F'(S^*) \,-\, P_S(S, t_{n^{\prime\prime}-1})]\;dS$$

$$+\,[F'(S^*) \,-\, P_S(S_{n^{\prime\prime}-1}, t_{n^{\prime\prime}-1})]\,(S_{n^{\prime\prime}} \,-\, S_{n^{\prime\prime}-1}) \,+\, \rho_6 \;, \tag{5.44}$$

where

$$|\rho_6| \;=\; |\rho_4 \,+\, \rho_5| \;\leqq\; C\,\|f\|\,(dt)^{3/2} \;. \tag{5.45}$$

Step 4. Estimating the lower bound in (5.44). Without loss of generality, assume that $S^* < S_{n^{\prime\prime}}$, which implies that $S^* \,<\, S^o \,<\, S_{n^{\prime\prime}-1} \,<\, S_{n^{\prime\prime}}$. Since $F(S)$ is concave in $[S^*, S^o]$ then

$$\int_{S^*}^{S^o} [F'(S^*) \,-\, F'(S)]\;dS \;\geqq\; 0 \;. \tag{5.46}$$

Also, $P(S, t_{n"-1})$ is convex in the interval $[S^o, S_{n"-1}]$, therefore

$$F'(S^*) \ > \ P_S(S_{n"-1}, t_{n"-1}) \tag{5.47}$$

or else the first time condition in (5.9) would be violated. The convexity in the interval plus condition (5.47) imply

$$\int_{S^o}^{S_{n"-1}} [F'(S^*) \ - \ P_S(S, t_{n"-1})] \, dS \ + \ [F'(S^*) \ - \ P_S(S_{n"-1}, t_{n"-1})] \, (S_{n"} \ - \ S_{n"-1}) \ \geqq \ 0. \tag{5.48}$$

A similar argument shows that (5.46) and (5.48) also hold when $S^* \ > \ S^o \ > \ S_{n"-1} \ > \ S_{n"}$, while inequality (5.47) is reversed.

Substituting (5.46) and (5.48) in (5.44), and using estimate (5.45) the result of the Proposition follows. (5.36) follows, and the proof of the Proposition is complete.

Proposition 5.14. *Let $[\{\Delta_n\}_{n=0}^N, B_0]$ be the strategy associated with the obstacle problem. Then, there exists a constant $C = C(\sigma, \mu, A, T)$, independent of F and dt, such that $[\{\Delta_n\}_{n=0}^N, B_0 + C\|F\|(dt)^{1/2}]$ is dominating for (Δ_N, B_N).*

Proof. The proof is equivalent to showing that $\kappa_N \geqq \ - C\|F\|(dt)^{1/2}$. In order to do this, define the sequence of stopping times $\{t_{\eta_j}\}_{j=0}^J$ as the sequence of stopping times at which the strategy moves from one Phase to the other, with $\eta_0 \ = \ 0$ and $\eta_J \ = \ N$. Then by Proposition 5.12

$$\sum_{j=0}^{[\frac{J-1}{2}]} \kappa_{\eta_{2j+1}} \ - \ \kappa_{\eta_{2j}} \ \geqq \ - \ C \, \|F\| \, (dt)^{1/2} \ \sum_{j=0}^{[\frac{J-1}{2}]} \frac{t_{\eta_{2j+1}} \ - \ t_{\eta_{2j}}}{t_N \ - \ t_0} \ \geqq \ - \ C \, \|F\| \, (dt)^{1/2} \ . \tag{5.49}$$

We also have by Proposition 5.13 that

$$\sum_{j=1}^{[\frac{J}{2}]} \kappa_{\eta_{2j}} \ - \ \kappa_{\eta_{2j-1}} \geqq \ - \ N \, C \, \|F\| \, (dt)^{3/2} \geqq \ - \ C \, \|F\| \, (dt)^{1/2} \ . \tag{5.50}$$

Since $\kappa_0 = 0$, it follows from (5.49)-(5.50) that

$$\kappa_N \ = \ \sum_{j=1}^{J} \kappa_{\eta_j} \ - \ \kappa_{\eta_{j-1}} \ \geqq \ - \ C \, \|F\| \, (dt)^{1/2} \ ,$$

and so the proof is complete.

Remark 5.15 If $A \geqq 1$ and the payoff has mixed convexity, the optimal hedging strategy is not unique, as we now show. The strategy associated with the obstacle problem is such that the agent ceases to transact as soon as the path (S_n, t_n) reaches the contact set and resumes transactions only when (i) the path exits the contact set and (ii) the number of shares in the portfolio is near the nominal Delta (cf. equation (5.9)). A moment of thought reveals that, since the portfolio value exceeds $F(S)$ as soon as the path passes through $\mathcal{C}$ (assuming $r = 0$), the agent could choose to continue making stock transactions, as long as the net portfolio value is constrained to remain above the obstacle. An extreme case along these lines would be to transact the *maximum amount of shares* (long or short) that would still leave the agent with a portfolio value above the obstacle at the next period. We believe that the latter prescription corresponds to the strategies originally proposed in Bensaid *et al.*(1992). Different dynamic strategies can be implemented according to the agent's preference for allocating the excess returns arising from super-replicating the payoff.

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Appendix: Regularity of Solutions of the Nonlinear Diffusion Equations and Obstacle Problems

The proofs of Theorems 4.4 and 5.4 required the regularity of the solutions of the nonlinear PDE problems (4.15) and (5.1)-(5.4). The main result that was used is

Proposition A.1. *(i) Let F satisfy (4.14) and let $P(S,t)$ be the solution of the nonlinear problem (4.15). Then*

$$\sup_{S,t} \left| S^j \frac{\partial^j P}{\partial S^j} \right| \leqq C \|F\|$$

for $0 \leqq j \leqq 3$ and

$$\|P_t\|_{t,1/2} \leqq C \|F\| ,$$

where

$$\|P_t\|_{t,1/2} \equiv \sup_{S,t} |P_t(S,t)| + \sup_{S,t,t'} \frac{|P_t(S,t) - P_t(S,t')|}{|t-t'|^{1/2}}$$

is the Holder norm of order $1/2$ in t.

(ii) The same estimates hold for the solution of the obstacle problem for (S,t) in the complement of the contact set $\mathcal{C}$.

We give only an outline of the proof, and refer the reader to Friedman (1964, 1982) and Kinderlehrer-Stampacchia (1980) for for the general regularity theory of parabolic equations.

A.1 The case $A < 1$.

We shall assume without loss of generality that $r = 0$. This changes the PDE in (4.15) to the simpler form

$$P_t + \frac{1}{2} \tilde{\sigma}^2 S^2 [P_{SS}] P_{SS} = 0 , \tag{A.1}$$

with

$$\tilde{\sigma}^2 [P_{SS}] = \sigma^2 [1 + A\, sign(P_{SS})] .$$

We define a new independent variable, $y = \ln S$. It is straightforward to verify that

$$S\,P_S \;=\; P_y \tag{A.2.a}$$

$$S^2\,P_{SS} \;=\; P_{yy} \;-\; P_y \tag{A.2.b}$$

and

$$S^3\,P_{SSS} \;=\; P_{yyy} \;-3P_{yy} \;+2P_y \;. \tag{A.2.c}$$

In particular, equation (A.1) can be rewritten in the form

$$P_t \;+\; \frac{1}{2}\,\tilde{\sigma}^2\,[\,P_{yy} \;-\; P_y]\cdot(P_{yy} \;-\; P_y) = \;0 \tag{A.3}$$

with $y \;\in\; (-\infty,+\infty)$ and $t \;\in (0,T)$. The desired estimates for the solution of (4.15) follow immediately from

Proposition A 2. *Let $F(y)$ be a smooth function satisfying*

$$\|F\| \;\equiv \sum_{j=0}^{4} \;\underset{y,t}{sup}\; \left|\,\frac{d^j\,F}{dy^j}(y,t)\;\right| \;<\; \infty \;. \tag{A.4}$$

Then,

$$\sum_{j=0}^{3} \;\underset{y,t}{sup}\; \left|\,\frac{\partial^j P}{\partial y^j}(y,t)\;\right| \;<\; C\;\|F\|\;, \tag{A.5}$$

and

$$\|P_t\|_{t,1/2} \;<\; C\;\|F\|\;, \tag{A.6}$$

where C is a constant that depends only on σ, A and T.

Proof: (i) The uniform boundedness of P follows immediately from the Maximum Principle applied to equation (A.3).

(ii) The estimate for P_y follows by differentiating formally (A.3) with respect to y. Accordingly, setting $Q \;\equiv\; P_y$, we obtain

$$Q_t \;+\; \frac{1}{2}\;\partial_y\;\left[\;\tilde{\sigma}^2\cdot(Q_y \;-\; Q\,)\,\right] \;=\; 0\;. \tag{A.7}$$

The Nash estimates for the fundamental solutions of parabolic equations with bounded, measurable coefficients (see Nash (1958), Aronson (1967)) imply that there exists a constant C, which depends only on $\sigma\, A$ and T, such that

$$\sup_{y,t \leqq T} |Q(y,t)| \;\leqq\; C \;\sup_{y} |Q(y,T)| \;\; .$$

(iii) To estimate the second derivative with respect to y, we rewrite equation (A.7) in the form

$$Q_t \;+\; \frac{1}{2}\, \partial_y\, \mathcal{G}\,(Q_y \;-\; Q\,) \;=\; 0\;, \tag{A.8}$$

where $\mathcal{G}(X) \;= \sigma^2\, X \;+\; \sigma^2\, A\, |X|$. Notice that $\mathcal{G}$ is a uniformly Lipschitz continuous function. Set $Q^{(h)}(y,t) \;\equiv\; Q(y+h,t)$. Then, from (A.8) we have

$$\begin{aligned} Q^{(h)}_t \;-\; Q_t \;&=\; -\;\frac{1}{2}\, \partial_y\, \mathcal{G}\left(Q^{(h)}{}_y \;-\; Q^{(h)}\right) \\ &\quad +\; \frac{1}{2}\, \partial_y\, \mathcal{G}\,(Q_y \;-\; Q\,) \\ &=\; -\;\frac{1}{2}\, \partial_y \left\{\; \mathcal{H}\cdot\left(Q^{(h)}{}_y \;-\; Q_y - \;Q^{(h)} \;+\; Q\right)\right\} \;\; . \end{aligned} \tag{A.9}$$

where $\mathcal{H}$ is a a function that satisfies $\sigma^2\,(1-A) \leqq\; \mathcal{H} \;\leqq\; \sigma^2\,(1+A)$ ($\mathcal{H}$ is the derivative of $\mathcal{G}$ evaluated an intermediate point). Next, define the difference quotient

$$D^{(h)} \;=\; \frac{1}{h}\,\left(Q^{(h)} \;-\; Q\right)\;.$$

Equation (A.9) implies that $D^{(h)}$ satisfies the equation

$$D^{(h)}_t \;+\; \frac{1}{2}\, \partial_y\, \left[\,\mathcal{H}\cdot\left(D^{(h)}_y \;-\; D^{(h)}\right)\right] \;=\; 0\;, \tag{A.10}$$

which is structurally analogous to (A.7). In particular, we conclude that

$$D^{(h)}(y,t)| \;\leqq\; \left\| \frac{F^{(h)}_S \;-\; F_S}{h} \right\|_0 \;\; ,$$

for all h. Letting $h \to 0$ in this last estimate, we conclude that

$$P_{yy}(y,t) \;=\; \lim_{h\to 0}\, D^{(h)}(y,t)$$

is uniformly bounded.

(iv) Next, we show that P_{yyy} is uniformly bounded. Setting $D = P_{yy}$ and differentiating both sides of (A.8) with respect to y, we obtain

$$D_t \;+\; \frac{1}{2}\,\partial^2_{yy}\;\{\mathcal{G}\,(D\;-\;Q)\;\}\;=\;0\;. \tag{A.11}$$

Subtracting equation (A.8) from (A.11), we conclude that, if $\delta \;=\; D - Q$, then

$$\partial_t\delta\;+\frac{1}{2}\,\partial^2_{yy}\;\mathcal{G}\,(\delta)\;-\frac{1}{2}\,\partial_y\;\mathcal{G}\,(\delta)\;\;=\;0. \tag{A.12}$$

We analyze the difference-quotients for δ, namely

$$\mu^{(h)}(y,t)\;\equiv\;\frac{1}{h}\,[\,\delta(y+h,t)\;-\;\delta(y,t)]\;\;.$$

It is readily seen, using the intermediate-value theorem, and equation (A.12), that these difference-quotients satisfy the PDEs

$$\mu^{(h)}_t\;+\;\frac{1}{2}\,\partial^2_{yy}\,\left[\mathcal{H}\cdot\,\mu^{(h)}\,\right]\;-\;\frac{1}{2}\,\partial_y\,\left[\mathcal{H}\cdot\mu^{(h)}\right]\;=\;0. \tag{A.13}$$

where $\mathcal{H}$ is defined as before. We can regard this last equation as the adjoint of

$$\partial_t\,\mu\;=\;\frac{1}{2}\,\mathcal{H}\cdot\partial^2_{yy}\,\mu\;+\;\frac{1}{2}\,\mathcal{H}\cdot\partial_y\,\mu\;, \tag{A.14}$$

which is the Fokker-Plank equation of an $\mathcal{L}_1$-bounded semigroup (see Krylov (1985)). This implies that the adjoint semigroup is bounded on $\mathcal{L}_\infty$.[21] Hence, the difference-quotients $\mu^{(h)}$ are bounded uniformly in h in terms of the initial data. Passing to the limit as $h \to 0$, we conclude that P_{yyy} is uniformly bounded in terms of $\|F\|$.[22]

(v) *The Holder-1/2 estimate for P_t.* Since $|S^2\,P_{SS}\,|$ is uniformly bounded, we conclude from (A.1) that P_t is bounded in terms of $\|F\|$.

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[21] $\mathcal{L}_1$ and $\mathcal{L}_\infty$ denote, respectively, the spaces of Lebesgue integrable and uniformly bounded functions.

[22] The third derivative of P is not continuous. This can be checked by writing equation (A.11) in divergence form (as in (A.10). This equation implies that the "fluxes" $\mathcal{H}\cdot(D_y\;-\;D)$ must be continuous across the curves where P_{SS} vanishes. Expressing this in terms of the variable S, we find that the ratio of the third derivatives along such curves is given by $P^+_{SSS}/P^-_{SSS}\;=\;(1-A)/(1+A)$, where the superscripts $\pm$ correspond to the sign of the second derivative on each side of of the interface.

Let us analyze the modulus of continuity of P_t in the variable t. Differentiating formally equation (A.3) with respect to t and setting $\Theta \equiv P_t$, we find that

$$\Theta_t + \frac{1}{2}\tilde{\sigma}^2 [P_{yy} - P_y] \cdot (\Theta_{yy} - \Theta_y) = 0 . \tag{A.15}$$

The boundary condition for Θ is

$$\Theta(y,T) = -\frac{1}{2}\tilde{\sigma}^2 [F_{yy} - F_y] (F_{yy} - F_y) = -\frac{1}{2} \mathcal{G} (F_{yy} - F_y) ,$$

which is a Lipschitz function in view of the regularity of F. The Holder-1/2 continuity of $U(y,t)$ in t follows from standard theory.

A.2 The obstacle problem for $A \geqq 1$: Analogy with the Stefan problem of heat conduction.

The obstacle problem (5.1)-(5.4) has many features in common with classical free-boundary problems discussed in the PDE literature (Friedman (1964, 1982), Kinderlehrer (1980)). To derive the regularity properties of solutions, we shall first analyze the compatibility conditions satisfied by P along the free boundary. As usual, we assume that $r = 0$.

Let us postulate, heuristically, that the free boundary can be represented locally as a graph, i.e. that

$$(S,t) \in \partial\mathcal{C} \iff S = S(t) ,$$

where $S(t)$ is a smooth function. With this notation, we have

$$P(S(t),t) = F(S(t)) \tag{A.16}$$

and

$$P_S(S(t),t) = F_S(S(t),t) . \tag{A.17}$$

Here, the left-hand side is understood as the limit of the partial derivative of P as S approaches the obstacle with $(S,t) \notin \mathcal{C}$. The same convention will be used for higher-order derivatives evaluated along $\partial\mathcal{C}$.

Differentiating formally (A.16) with respect to t and using (A.17), we obtain

$$P_S(S(t),t)\cdot\dot{S}(t) \;+\; P_t(S(t),t) \;=\; F_S(S(t))\cdot\dot{S}(t)\ ,$$

where $\cdot$ denotes the time derivative, and thus

$$P_t(S(t),t) \;=\; 0\ . \tag{A.18}$$

Using the equation satisfied by P in the complement of $\mathcal{C}$,

$$P_t \;+\; \frac{1}{2}\,\sigma^2\,(1+A)\,S^2\,P_{SS} \;=\; 0, \tag{A.19}$$

we conclude also that

$$P_{SS}(S(t),t) \;=\; 0\ . \tag{A.20}$$

Another interesting relation is obtained by differentiating (A.17) with respect to t. Accordingly, we obtain

$$P_{SS}(S(t),t)\cdot\dot{S}(t) + \;P_{St}(S(t),t) \;=\; F_{SS}(S(t),t)\cdot\dot{S}(t)\ ,$$

or, from (A.19) and (A.20),

$$P_{St}(S(t),t) \;=\; F_{SS}(S(t),t)\cdot\dot{S}(t)\ . \tag{A.21}$$

Introduce as before $\Theta \;\equiv\; \partial_t P$, we find that

$$\Theta_t \;+\; \frac{1}{2}\sigma^2\,(1+A)\,S^2\,\Theta_{SS} \;=\; 0, \qquad \text{for}\ \ \Theta \;<\; 0\ , \tag{A.22}$$

and

$$\dot{S}(t) \;=\; -\;\frac{1}{F_{SS}(S(t))}\;\;\Theta_S(S(t),t)\ , \tag{A.23}$$

where $S = S(t)$ denotes the boundary of the set where Θ vanishes. This problem is known as the *Stefan problem.*[23]

Notice that the speed of the free-boundary is determined by the "Stefan constant"

[23] Classically, the Stefan problem describes the evolution of the temperature profile of a two-phase medium, such as ice and water. The interface condition (A.23) expresses that the velocity of the phase-boundary is proportional to the temperature flux across it.

$$K(S(t)) \;=\; -\,\frac{1}{F_{SS}(S(t))}\;, \tag{A.24}$$

which depends on the curvature of the payoff function F. Herein lies an important difference with regards to the classical Stefan problem. If $F_{SS}(S(t)) \;=\; -\infty$, then the free-boundary will move with zero speed. Also, if $F_{SS}(S(t)) \;=\; 0$, it moves with infinite speed, which means that the function $P(S,t)$ at such times can be regarded as the new data for a Stefan problem with a different location of the free-surface.

The two extreme cases, far from being pathological, correspond in fact the case of option portfolios, when the payoff F is piecewise-linear. The solution of the obstacle problem for these cases was completely described in Avellaneda and Parás (1994). Due to the infinite curvature, the free boundaries are "pinned" exactly at those points S^* where the payoff function has a discontinuous second derivative and is *locally concave*, *i.e.* $F_{SS} \;\propto\; -\,\delta(S - S^*)$. The solution of the obstacle problem reduces then to "pasting together" solutions of two-point boundary-value problems for equation (A.19). The regularity properties of solutions follow immediately. Aside from a final time-interval near the expiration date T, where the payoff function F may have concave kinks outside $\mathcal{C}$, the solution of the obstacle problem for piecewise-linear payoffs satisfies the estimates

$$\sum_{j=0}^{3} \sup_{(S,t)\notin \mathcal{C}\,,t \;\geqq\; \epsilon} \left| S^j \,\frac{\partial^j P}{\partial S^j} \right| \;<\; \infty \qquad \epsilon \;>\; 0$$

as well as a Holder-1/2 estimate for P_t in time for $t \;\geqq\; \epsilon$.

If F_{SS} is smooth and locally $F_{SS} \;\neq\; 0$, the obstacle problem has a similar structure. It can be mapped to a sequence of classical Stefan-type problems, with finite "heat constant" K. The regularity properties for $P(S,t)$ stated in Proposition A.1 can then be deduced from classical results for the Stefan problem.