Mathematics of Finance I: Homework # 1

Please give complete arguments and derivations for all questions. Due in three weeks.

1) The annual interest rate in country XYZ is 36%. This country has markets in which a commodity, the "widget", can be traded spot and 6 months forward. The spot price of a widget is \$ 1 and storage costs for six months are estimated at 5 cents/widget. (i) What is the fair 6 months forward price for a contract for 1000 widgets? (Neglect the "convenience value" for holding widgets, or other variables aside from interest rate and storage costs.) (ii) You have access to financing in the U.S. at a rate of 9% annually for up to \$1,000,000.00. Assuming that the 6 months forward price is given by (i), construct an arbitrage. How much could you gain from such arbitrage?

2) For what values of the strike are the prices of European puts and calls equal? For what values of the strike are calls twice as valuable as puts?

3) **Options on futures**. Written on a variety of index futures (e.g S&P500 and T-Bond futures), options on futures are exchange-traded instruments that give the holder the right to "enter in a futures contract at a specified futures price". This works as follows: the holder of a call on the S&P futures contract with strike K who exercises the option receives the difference between the futures price F and th strike price K if $F - K > 0$ and is automatically assigned a long position in the futures contract. Options on futures are typically American style. (i) Derive a theoretical put-call parity relation for options on futures, assuming that the options are European instead of American. Do you think that this relation holds for American options? (ii) "Put-call symmetry". Show that the fair values of *American* options on futures satisfy the relation:

$$P_{t,T}(F) \;=\; Ct, T(F) \;.$$

4) A security trades at \$ 100 and the 3-month calls and puts with strike price \$ 100 trade respectively at \$5 and \$4. (i) What can you conclude about the relation between the six-month forward price and the spot price? (ii) Calculate the annual dividend yield of the underlying security assuming an annualized six-month interest rate of 8%.

5) Assume that the annual interest rate is 8% and that the annualized volatility of a non-dividend paying stock over the next six months will be 15% (annualized). i) Construct binomial trees of 5, 15 and 45 periods to calculate the value of a European-style 6-month call option with strike price 10% above today's spot price. Calculate the corresponding Deltas in the three cases. Then compare your results with the Black-Scholes formula. ii) How many periods do you think are necessary to obtain an error of less than 1% with respect to the Black-Scholes price?

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6) An oil-producing company issues debt with coupons which depend on the value of crude oil. It issues a two-year note which pays coupons semi-annually and a final payment of 100,000 dollars after 2 years (there are 3 coupons). Assume that the current price of a barrel of oil is 18 dollars. The coupons are calculated according to the following rule: Let P_t = price of a barrel in dollars. Then

$$\text{Coupon payment} = \begin{cases} \$4,000 \ \text{ if } P_t \leq 18 \\ \\ \$\ \left(2 + \frac{C-18}{5}\right) \cdot 2,000 \ \text{ if } 18 < P_t < 23 \\ \\ \$6,000 \text{ if } P_t \geq 23\ . \end{cases}$$

(i) Show that the note can be decomposed into a series of discount bonds with fixed coupons and options on the price of crude oil.

(ii) Assume that the annualized volatility of oil will be 5% over this period, and that the annual interest rate is 8%. Calculate the "fair value" of this note using the Black & Scholes formula.

(iii) Recalculate the price assuming that the volatility is only 1%.

7) Using the Black-Scholes formula, derive a formula for the *strike price* of a European call option with a given Δ, assuming that the volatility, the dividend yield and the interest rate are known. Calculate the strikes and fair values, in percentage of the value of the underlying asset, of calls with $\Delta = 0.20,\ ,\ 0.25\ ,\ 0.50\ ,\ 0.75$ and 0.80, assuming that $\sigma = 0.13\ ,\ r = 5.90\%$ and $q = 4.20\%$.

8) A *call spread* is a position consisting in one call bought and another call sold, with different strikes. (i) Calculate and *graph* the value of a call spread with strikes at \$90 (long call)and \$110 (short call) with six months to expiration. Assume that $r = 0.05$ and $q = 0$ but that the volatility can take the values 10%, 20%, 30% and 40% (plot the four graphs). (ii) How does the value of the spread change when the price of the underlying is \$ 100 as the volatility changes ? Same question when the price is \$ 80. Draw some conclusions (or conjectures) about the dependency of the value of a call-spread on volatility.