MANAGING THE VOLATILITY RISK OF PORTFOLIOS OF DERIVATIVE SECURITIES: THE LAGRANGIAN UNCERTAIN VOLATILITY MODEL

by

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Abstract: We present an algorithm for hedging option portfolios and custom-tailored derivative securities which uses options to manage volatility risk. The algorithm uses a *volatility band* to model heteroskedasticity and a non-linear partial differential equation to evaluate worst-case volatility scenarios for any given forward liability structure. This equation gives sub-additive portfolio prices and hence provides a natural ordering of preferences in terms of hedging with options. The second element of the algorithm consists of a portfolio optimization taking into account the prices of options available in the market. Several examples are discussed, including possible applications to market-making in equity and foreign-exchange derivatives.

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1. Introduction

Volatility is a crucial variable in the trading and risk-management of derivative securities. The uncertain nature of forward volatility is recognized as one of the main factors that drive market-making in options and custom-tailored derivatives.

Derivative asset prices are affected by new information and changes in expectations as much as they are by changes in the value of the underlying index. Large-scale changes in implied volatilities were notorious in the crash of 1987 and in the aftermath of the Mexican Peso devaluation of December 1994. Currency markets saw significant reversals in the skewness of the implied volatility curve in the summer of 1995, as central banks moved to protect the dollar, and again in late September 1995 as the dollar fell against the Deutschemark. These events cannot be understood, forecast or modeled using "deterministic" volatility.

Incorporating heteroskedastic behavior (*i.e.* volatility of volatility) is essential for the risk-management of derivatives. Several attempts in this direction have been made, most notably with autoregressive models and with the use of stochastic differential equations to model volatility changes. However, random volatility models are more delicate and complex to implement than Black-Scholes due to the larger number of parameters that need to be estimated. More importantly, introducing multiple parameters to define a pricing measure may not necessarily be the best way to incorporate information about future market risk. Pricing models with *a priori* statistical distributions for the stochastic volatility will not capture abrupt changes in volatility expectations and the fact that, ultimately, *only one* volatility path will be realized.

Option prices provide concrete information about the market's volatility expectations. Therefore, options are crucial for hedging in an uncertain volatility environment. Typically, hedging with options is presented in two paradigms: *replication*, or synthetization of liabilities, and *dynamical hedging*, or management of portfolio sensitivities. In the first approach, a derivative product is identified through "reverse-engineering" as a series of simpler option-like payoffs. Whenever cash-flows can be matched exactly to those of market instruments, the value of a given contingent claim should be equal to the price of the synthetic portfolio, which also represents a sure hedge against market movements. In practice, however, perfect replication is seldom possible due to its high cost, and to market incompleteness and liquidity constraints. Traders are therefore faced with the problem of determining how to diversify risk by synthetization and how to price (and manage) the residual risk to be carried forward. Ideally, they would like to hedge their risk-exposure in a secondary market whenever this leads to profitable trading but it is often said that " options are too expensive ": a thorough coverage using derivatives can price a deal out of the market.

The alternative is dynamical hedging. If traders had perfect foresight on forward volatility, then Delta-hedging continuously in the cash market would be essentially riskless. In practice, continuous hedging is impossible and there is volatility risk. Such risks are usually taken into account by hedging with derivative instruments which, through their convexity,

allow for adjustments in the exposure to higher-order sensitivities of the model, such as Gamma (differential change in Delta), Vega (differential change in volatility), Rho (differential change in interest rates), etc.[1] From a theoretical point of view, the main problem with Greek matching is that (i) the method provides protection only to *differential* (small) market changes, (ii) the method is parametric, *i.e.* requires having a precise view on forward volatility and (iii) the cost of marking-to-market the dynamic hedge cannot be determined in advance.

There is an intrinsic inconsistency in the way pricing and hedging are viewed in classical frameworks of dynamic hedging. On the one hand, pricing is made with parametric models[2]. Once the "market price of risk" has been calculated, every derivative instrument is equivalent to a dynamic portfolio of basic securities. On the other hand, "matching the Greeks" — especially Gamma, Vega and other higher-order derivatives — implicitly recognizes that the probabilistic assumptions will not be valid at later times. We believe that a key factor missing in current derivative pricing models is the idea that heteroskedasticity gives rise to a preference ordering in terms of which trades should be made under particular market conditions. This ordering of preferences cannot be captured by linear present-value models in which derivative asset prices are additive functions of their future cash-flows.

From the above considerations, we contend that a reasonable model for managing the risk of derivative securities in markets with uncertain volatility should satisfy the following requirements:

- Heteroskedasticity, or uncertainty in the values of forward volatilities should be taken into account by assuming that *more than one arbitrage-free measure can realize current prices at any given time.*

- Portfolio values should be *sub-additive* for the sell-side and *super-additive* for the buy-side due to diversification of volatility risk.

- Option prices, as well as the prices of other liquid optional instruments traded in the market, should be incorporated into the model, since they provide the information necessary to "narrow down" volatility uncertainty.

This paper is an attempt to implement these ideas in a model which is both theoretically sound and easy to implement. The main elements in our approach are (i) modeling volatility uncertainty by using *volatility bands* and (ii) *optimization* over the class of admissible probabilities according to market prices of derivative instruments. The first idea is implemented using the Uncertain Volatility Model (UVM) introduced recently by Avellaneda, Levy and Parás (1994,1995), Parás (1995). This pricing method assumes that forward volatility paths vary inside a band and calculates the value of assets/liabilities

[1] It is important to note that matching the higher-order derivatives of the Black-Scholes formula (Greeks) is not a theoretical consequence of dynamical asset pricing theory, but rather a practical device used by traders, which are well-aware of the shortcomings of statistical present value models and market heteroskedasticity.

[2] In the sense of *parametric statistics*, wherein a specific probability distribution is assumed.

under the worst-case volatility scenario. Portfolio valuation is sub-additive for the sell-side and super-additive for the buy-side. The second feature, *optimization*, is fundamental: we use options as hedging instruments to minimize the worst-case scenario value of the contracted liabilites. Our approach can be summarized schematically by the equation

$$\text{Model Value } =$$

$$Min\,\{\text{ Value of Option Hedge } +\ Max\ \text{PV}(\text{ Residual Lability})\}\ ,$$

where (i) the residual liability (liability minus option payoffs) is valued under the worst-case scenario according to UVM and (ii) the minimum is take over all option portfolios, i.e. over possible hedges which use options and cash instruments. For any given liability structure, this procedure selects a position in options which, under the assumptions of the model, will cancel the portfolio risk at the least cost.

This paper describes the implementation of the algorithm for derivative securities based on a single market index. We study several examples which can lead to potential applications and analyze the sensitivity of the model to the assumptions of volatility band and on the number of input options used to hedge. Other examples indicate how the model could be used for trading equity and and foreign-exchange derivatives using a large number of input options. In particular, we consider an example based upon the Telmex options market in the first months of 1995, and an analysis of risk-management of knockout options on Dollar-Mark using recent market conditions. Finally, in an Appendix, we connect the model with the theory of arbitrage-free pricing in a market with many derivative securities based on a single underlying asset.

2. The uncertain volatility model (uvm)

Let us consider a model for contingent claim valuation based on an underlying asset with uncertain volatility. For this purpose, we assume that the spot price of the underlying follows a stochastic differential equation of the type

$$P \quad : \quad \frac{dS_t}{S_t} \ = \ \mu_t\, dt \ + \ \sigma_t\, dZ_t \ , \tag{1}$$

where μ_t and σ_t are "spot" drift and volatility parameters. These functions may depend on present and/or past market information. Since the underlying asset is a traded security, we can assume for valuation purposes that

$$\mu_t \ = \ r_t \ - \ d_t \ , \tag{2}$$

where r_t is the spot (domestic) riskless rate and d_t is the dividend rate (or foreign interest rate). Our primary concern is with volatility risk so we assume, for simplicity, that r_t and d_t are constant. For each volatility process $\{\sigma_t\}$, the symbol P in (1) represents the probability measure induced on price paths $\{S_t\}$.

Volatility uncertainty is modeled by assuming that the volatility process which drives the price in (1) will fluctuate within a band, i.e.,

$$\sigma_{min} \;\leq\; \sigma_t \;\leq\; \sigma_{max} \;, \qquad 0 \;\leqq\; t \;\leqq T \;. \tag{3}$$

Here, σ_{min} and σ_{max} are constants — or more generally, deterministic functions of the spot price and time — and T is some terminal horizon. The assumption of a forward range for the spot volatility is the only hypothesis made on heteroskedastic behavior. In particular, we shall not specify an *a priori* probability distribution for σ_t.[3] Instead, we will consider the set $\mathcal{P}$ of all possible measures P induced by any volatility processes which vary within the band.

We consider the following valuation problem: *calculate the present value of terminal assets/liabilities under the worst-case scenario for a forward volatility path, assuming that it remains inside the a-priori band.* To fix ideas, consider an agent that must deliver a stream of cash-flows

$$F_1(S_{t_1})\,, F_2(S_{t_2})\,, F_3(S_{t_3})\,, \ldots\,, F_N(S_{t_N}) \tag{4}$$

where $F_j(\cdot)$ are payoffs due at settlement dates $t_1 \;<\; t_2 \;<\; \ldots \;<\; t_N$.[4] The worst-case scenario present value estimate of his liability is given by

$$V(S_t\,, t) \;=\; \sup_{P\in\mathcal{P}} \; \mathbf{E}^P \left\{ \sum_{j=1}^{N} \, e^{-\,r(t_j\,-t)}\, F_j(S_{t_j}) \right\} \;, \tag{5}$$

where $\mathbf{E}^P$ represents the expectation operator associated with the stochastic differential equation (1)-(2).

As shown in Avellaneda, Levy and Parás (1995), $V(S\,, t)$ satisfies the nonlinear programming equation

[3] Probabilistic beliefs on the behavior of price volatility as a stochastic process, such as Hull & White (1987) can be incorporated in the model by constructing a confidence interval for the volatility path. Avellaneda, Levy and Parás (1995) applied this idea to the case of a lognormal mean-reverting volatility process. In real-world situations, a statistical analysis of the range of spot volatilities can be used to determine a suitable band for the market of interest.

[4] Under this convention, the liabilities are included with a positive sign and the assets with a negative sign. So if the agent is short a call expiring at τ for example, then $F(S_\tau) = max(S_\tau - K, 0)$, but if he was long that same call then $F(S_\tau) = -max(S_\tau - K, 0)$.

$$\frac{\partial V}{\partial t} \; + \; \frac{1}{2}\Sigma^2 \left\{ \frac{\partial^2 V}{\partial S^2} \right\} \cdot \frac{\partial^2 V}{\partial S^2} \; + \; \mu \, S \frac{\partial V}{\partial S} \; - \; r \, V \; = \; 0 \; , \tag{6}$$

where

$$\Sigma^2 \{X\} \; = \; \begin{cases} \sigma^2_{max} & \text{if } \; X \; \geq \; 0 \\ \\ \sigma^2_{min} & \text{if } \; X \; < \; 0 \; . \end{cases} \tag{7}$$

This equation is similar to the Black & Scholes (1973) PDE, with the difference that the "input volatility" is not constant: it is determined by the sign of $\frac{\partial^2 V(S,t)}{\partial S^2}$, i.e., by the convexity of V. Thus, the worst-case volatility path is $\sigma^*_\tau \; = \; \Sigma \left\{ \partial^2 V(S_\tau, \tau)/\partial S^2 \right\}$, a function of S_τ and τ which depends upon the stream of cash-flows (liabilities) under consideration.

The value $V(S_t, t)$ corresponds to the *cost of dynamic hedging* with the underlying asset under the worst-case volatility path. In fact, it was shown that a dynamic strategy for risklessly hedging in this uncertain volatility environment — starting with wealth $V(S_t, t)$ — consists in maintaining a position of $\Delta \; = \; \partial V(S_\tau \, , \tau)/ \, \partial S$ contracts in the cash market and adjusting it periodically as the market moves (Avellaneda, Levy and Parás (1995)). An agent who follows this strategy will end up with a non-negative cash-flow after delivering the payoffs $F_j(S_{t_j})$. Moreover, if the volatility path actually followed the worst-case path σ^*_t, this is the only non-anticipative strategy that exactly replicates the stream of cash-flows, generating no excess returns. Therefore, this is the *least costly* dynamic hedging strategy that can be constructed using the underlying asset — no options — as hedging instrument, which never generates losses regardless of the path followed by volatility.

The partial derivatives $\partial V(S,t)/\partial S$ and $\partial^2 V(S,t)/\partial S^2$ can be viewed as "risk-adjusted" Delta and Gamma. For example, a short option position represents a convex liability $F(S)$ for the agent and is therefore priced with the maximum volatility $\sigma_\tau = \sigma_{max}$. Similarly, a long option position is valued using $\sigma_\tau \; = \; \sigma_{min}$. The quantity

$$\Gamma_t \; = \; - \, \frac{\partial^2 V(S,t)}{\partial S^2}$$

can be interpreted as a new Gamma (sensitivity of the hedge-ratio to price movements) adjusted for volatility risk. For at-the-money options, the relation between this adjusted Gamma and the classical Black-Scholes Gamma is the following: given any constant "Black-Scholes volatility" within the band (e.g., $\sigma_{BS} \; = \; \frac{1}{2} \, (\, \sigma_{min} \; + \; \sigma_{max} \,)$), the effect of the nonlinear PDE (6) is to overestimate σ_{BS} when the agent is short Gamma and thus to reduce Gamma (in absolute value) in comparison with Black-Scholes theory. When the agent is long Gamma, σ_{BS} is underestimated and thus the Gamma owned also increases.

A key feature of equation (6)-(7) is its *sub-additivity with respect to payoffs*. The sum of the worst-case-scenario values of each cash-flow will be generally greater than the value of

the stream of cash-flows considered as a "lumped" liability. This is due to the cancelation of volatility risk that occurs if the overall portfolio has mixed convexity. Hence, unlike linear models, the present model quantifies the portfolio's exposure to volatility risk.

3. Hedging with options: the Lagrangian uncertain volatility model (λ-UVM)

Consider an agent who must deliver cash-flows $F_j(S_{t_j})$, $j = 1, 2, ...N$, and wishes to hedge his exposure using a mix of options and the underlying asset. Let us assume that there are M European-style options available for hedging. Their payoffs and expiration dates are denoted respectively by $G_i(S_{\tau_i})$ and τ_i, with $\tau_1 \leq \tau_2 \leq ... \leq \tau_M$.[5] We assume that the options are available for trading at the prices $C_1 , C_2 , ... , C_M$. We omit temporarily the bid-offer spreads in these prices to simplify the discussion, but these will be incorporated later on.[6]

Suppose that, at time t, the agent purchases a portfolio of options consisting of $\lambda_1 , \lambda_2 , ... , \lambda_M$ contracts of each strike/maturity to hedge his exposure. The market value of this portfolio is

$$\sum_{i=1}^{M} \lambda_i \, C_i \ . \tag{8}$$

The combination of this position and the agent's previous liability modifies the agent's overall risk profile. After the trade, the *residual liability* in present-value terms is

$$\sum_{j=1}^{N} e^{-r(t_j - t)} \, F_j(S_{t_j}) \ - \ \sum_{i=1}^{M} \lambda_i \, e^{-r(\tau_i - t)} \, G_i(S_{\tau_i}) \ .$$

The total cost of hedging, computed by adding the cost of the options (8) and the worst-case cost of dynamically hedging the residual is

$$V(S_t , t ; \ \lambda_1 , ... , \lambda_M) \ =$$

[5] Thus, $G_i(S) = Max \ (S - K_i , 0)$ or $G_i(S) = Max \ (K_i - S , 0)$, where K_i is the strike price. Due to the put-call parity relation, it is sufficient to consider one type of option (call or put) per strike and maturity.

[6] Liquidity plays an important role in hedging, especially if the trader considers a complex hedge using a large amount of options.

$$\sup_{P\in\mathcal{P}} \mathbf{E}^P \left\{ \sum_{j=1}^{N} e^{-r(t_j-t)}\, F_j(S_{t_j}) \;-\; \sum_{i=1}^{M} \lambda_i\, e^{-r(\tau_i-t)}\, G_i(S_{\tau_i}) \right\}$$

$$+ \sum_{i=1}^{M} \lambda_i\, C_i \;. \tag{9}$$

The supremum over expectations $\mathbf{E}^P$ is calculated with UVM (equations (6)-(7)).

A portfolio of options $(\lambda_1\,,\lambda_2\,,\ldots,\lambda_M\,)$ is said to be an *optimal* hedge if it solves the optimization problem

$$\inf_{\lambda_1\,,\lambda_2\,,\ldots,\lambda_M} V\,(S_t\,,t\,;\;\lambda_1\,,\lambda_2\,,\ldots\lambda_M\,)\;. \tag{10}$$

To this optimization we give the name of *Lagrangian Uncertain Volatility Model* or λ-UVM.[7]

The vector $(\lambda_1\,,\lambda_2\,,\ldots,\lambda_M\,)$ must be restricted to vary over a suitable range of portfolio combinations. A reasonable specification for this range is

$$\Lambda_i^- \;\leq\; \lambda_i \;\leq\; \Lambda_i^+\;, \tag{11}$$

where Λ^+ and Λ^- are constants. These constraints on the option portfolio are natural from the point of view of trading limits or from liquidity considerations.[8] We shall assume that the λs can take arbitrary values inside the intervals in (11). It is convenient to assume, whenever possible, that the λs can take both negative and positive values.

The function $V\,(S_t\,,t\,;\lambda_1\,,\ldots,\lambda_M)$ is convex in $(\lambda_1\,,\ldots,\lambda_M)$. This follows from (9): the value function is a supremum of linear functions in λ (c.f. Appendix). When the λ-vector is equal to zero, this function reduces to regular UVM valuation as in Section 2 (and the agent hedges only in the cash market). In general, the minimum in (10) is attained for $\lambda_i \neq 0$: since the implied volatilities lie inside the band, they are "cheaper to buy" and "more expensive" to sell than at σ_{max} or σ_{min}. Accordingly, the cost of the efficient hedge derived from (10) will be less than the expected cost of strictly delta-hedging under the worst-case scenario.

In the examples to come, we will show that the cost of the efficient hedge can be often comparable to the Black-Scholes "fair value" using mid-market volatilities, even when the

[7] As we shall point out later, the λs can be viewed as Lagrange multipliers for a constrained optimization problem.

[8] Prices quoted by market-makers are often valid only for limited amount of contracts. "Optimal" strategies which require trading in large volumes may be meaningless since the price is likely to change with the order flow.

band is very wide (30% to 150% in the TELMEX example of §9). This is due to the fact that the algorithm constructs strike and calendar option spreads which reduce volatility risk and the need for intensive Delta-hedging.

So far, we regarded options as essentially static instruments: once the position is taken, the option hedge-ratios λ_i need not be readjusted. However, market are far from being static. "Desert island" option strategies[9] can be replaced by dynamic ones as the market permits. In fact, once the hedge is in place, the agent can take advantage of the evolution of prices and the issuance of new options to improve the position by applying λ-UVM to the liability structure inherited from the previous trading date. If a new option hedge is then selected, the new position generates a book profit without taking additional volatility risk.[10]

Calibration of the volatility band. There are important consideration to be made with respect to the choice of the volatility band. An incorrect calibration of the model may result in spurious arbitrage opportunities when the band $[\sigma_{min}\ ,\ \sigma_{max}]$ is taken to be too narrow. Taking into account expected trends in forward volatility, possibly with a band which varies with time, may be useful in order to avoid overestimating volatility risk with the UVM. This issue deserves some discussion.

We know from classical valuation theory that there is a precise mathematical relation between spot volatilities and implied volatilities obtained via the Black-Scholes formula. In fact, assuming a band for spot volatilities is essentially equivalent to assuming that the *implied volatilities* of traded options will remain within predetermined bounds over the period of interest. In other words, the band assumption implies an *upper bound on the purchase price* and a *lower bound on the sale price* of any option traded over the period of interest. For instance, if σ_{min} and σ_{max} are constants (the simplest kind of band), the bounds in (3) are consistent with the belief that

$$\sigma_{min} \ \leqq\ \sigma_{impl.}(t,\,\tau) \ \leqq\ \sigma_{max}\ ,$$

where $\sigma_{impl.}(t,\tau)$ represents the implied volatility at time t of an option maturing at time τ. If the bands are time-dependent, i.e.,

$$\sigma_{min} \ =\ \sigma_{min}(t) \qquad ,\qquad \sigma_{max} \ =\ \sigma_{max}(t)\ ,$$

the assumption (3) on the forward spot volatility is tantamount to assuming that

$$\frac{1}{\tau\ -\ t}\int_t^\tau \sigma^2_{min}(s)\,ds \ \leqq\ \sigma^2_{impl.}(t,\,\tau) \ \leqq \frac{1}{\tau\ -\ t}\int_t^\tau \sigma^2_{max}(s)\,ds\ ,$$

for any $0 \leqq t \leqq \tau \leqq T$.

[9] The hyperbole is borrowed from E. Thorp (1969).

[10] This statement is valid as long as the assumption made on the volatility band is not violated.

The calibration of the forward volatility band should be done using market data, such as the implied volatilities of liquid options, historical and seasonal information, and beliefs about abrupt changes in implied volatility levels. The implied volatilities of traded options provides an important indication about the range of σ_t. If, for instance, σ_{min} and σ_{max} are chosen to be constant, the band should contain the implied volatilities of all options considered as input instruments, to avoid "model arbitrage".[11] In addition, the band should be strictly wider than the range of the implied volatilities (taken as the total range when the band is flat or in "buckets" when the band if time-dependent) because of the additional volatility risk that exists and transaction costs (see Section 4).

In markets where implied volatilites vary significantly with the maturity of options, a more conservative approach would consists in using *implied forward-forward volatilites* to calibrate the band. Accordingly, if at time t we are given the implied volatilites of two at-the-money options with maturities T_1 and T_2, $(T_1 < T_2)$, we can derive an approximate "implied T_1- to-T_2 volatility", $\sigma_{impl.}(T_1, T_2)$, by solving the equation

$$(T_2 - t) \cdot \sigma^2_{impl.}(t, T_2) = (T_1 - t) \cdot \sigma^2_{impl.}(t, T_1) + (T_2 - T_1) \cdot \sigma^2_{impl.}(T_1, T_2) . \quad (12)$$

The band can then be chosen so as to satisfy the additional constraint that $\sigma_{min} < \sigma^2_{impl.}(T_1, T_2) < \sigma_{max}$.[12]

[11] Choosing a band that contains all implied volatilities is a generalization of the standard procedure which consists in selecting a constant spot volatility in (1), usually equal to the implied volatility of a liquid at-the-money option.

[12] Equation (12) defines an *expected*, but not necesarily realized, forward-forward volatility over the period (T_1, T_2). The bands obtained using this type of constraint are always wider than the ones obtained by using implied volatilities of options currently traded. For a study of implied forward-forward volatilities, we refer the reader to Taleb and Avellaneda (1995) (forthcoming).

4. Transaction costs and bid/offer spreads

Transaction costs increase the cost of hedging and hence affect the composition of the optimal hedging portfolio. In particular, "pure" arbitrage opportunities which might exist in the absence of market frictions may disappear once transaction costs and bid-offer spreads are properly accounted for.[13]

Bid-offer spreads for trading the underlying asset. We assume that Delta-hedging will require the agent to buy at the offer and sell at the bid. Although this may not always be the case, the assumption is consistent with the worst-case scenario approach followed so far.

It is well-known that the impact of bid/offer spreads on Delta-hedging costs can be incorporated *a priori* into the value of derivative securities by adjusting the model volatility (see Leland (1985), Boyle and Vorst (1992), Hoggard, Whalley and Wilmott (1993), Avellaneda and Parás (1994,1995) among others). Accordingly, if the price of the underlying asset follows a geometric Brownian motion with *constant* volatility σ, the cost of Delta-hedging a derivative security, including expected future transaction costs, is obtained by solving equation (6)-(7) with

$$\Sigma\left[\frac{\partial^2 V}{\partial S^2}\right] = \begin{cases} \sigma\sqrt{1+\sqrt{\frac{2}{\pi}}\frac{k}{\sigma\sqrt{dt}}}\,, & \text{if } \frac{\partial^2 V}{\partial S^2} \geqq 0\ , \\ \\ \sigma\sqrt{1-\sqrt{\frac{2}{\pi}}\frac{k}{\sigma\sqrt{dt}}}\,, & \text{if } \frac{\partial^2 V}{\partial S^2} < 0\ . \end{cases}$$

Here k is the expected roundtrip transaction cost (expressed in percent of the value of the underlying security) and dt is the time-lag between adjustments.[14] This fundamental result can be applied to heteroskedastic pricing models. If the volatility is uncertain and varies in the band (3), then the "trasnsaction-cost-adjusted" spot volatility will vary in a wider band with upper and lower bounds given by

$$\hat{\sigma}_{max} \equiv \max_{\sigma_{min} \leqq \sigma \leqq \sigma_{max}} \left\{ \sigma\sqrt{1+\sqrt{\frac{2}{\pi}}\frac{k}{\sigma\sqrt{dt}}} \right\}$$

[13] Transaction costs arise from brokerage fees, below-market returns for margin deposits, taxes and bid-offer spreads for trading cash instruments and options. We shall be primarily concerned with two latter effects. The problem of receiving below-market interest for deposits held in margin accounts can be studied in the framework of passive/active deposit rates. This feature is actually taken into account in the numerical implementation of the algorithm, but we shall not elaborate on it here.

[14] If $\sqrt{\frac{2}{\pi}}\frac{k}{\sigma\sqrt{dt}} \geqq 1$, the lower volatility should be set to zero (Avellaneda and Parás (1994)).

and

$$\hat{\sigma}_{min} \equiv \min_{\sigma_{min} \leqq \sigma \leqq \sigma_{max}} \left\{ \sigma \sqrt{Max \left[1 - \sqrt{\frac{2}{\pi}} \frac{k}{\sigma \sqrt{dt}} , 0 \right]} \right\} \tag{13}$$

Hence, the impact of transaction costs on Delta-hedging can be incorporated by choosing the width of the band appropriately.

Bid/offer spreads in option prices. We assume that

$$C_i^{(b)} = \text{bid price for the } i^{th} \text{ option}$$

and

$$C_i^{(o)} = \text{offer price for the } i^{th} \text{ option} , \tag{14}$$

for $i = 1, 2, ..., M$. We shall operate under the assumption that options can be purchased at the offer price and sold at the bid price for the amounts specified by the $\Lambda_i^{\pm}$ constraints in (11). In this case, the cost of acquiring a portfolio $(\lambda_1 , \lambda_2 , ... , \lambda_M)$ is computed as in (8), but with C_i replaced by $C_i^{(b)}$ if $\lambda_i < 0$ or $C_i^{(o)}$ if $\lambda_i > 0$. Accordingly, this cost is given by

$$\sum_{i=1}^{M} \left[\lambda_i \left(\frac{C_i^{(o)} + C_i^{(b)}}{2} \right) + |\lambda_i| \left(\frac{C_i^{(o)} - C_i^{(b)}}{2} \right) \right] . \tag{15}$$

The total hedging cost for the derivative product with cash-flows $\{F_j(S_{t_j})\}$ is, therefore,

$$V^{(b/o)}(S_t , t ; \lambda_1 , ... , \lambda_M) =$$

$$\sup_{P \in \mathcal{P}} \mathbf{E}^P \left\{ \sum_{j=1}^{N} e^{-r(t_j - t)} F_j(S_{t_j}) - \sum_{i=1}^{M} \lambda_i e^{-r(\tau_i - t)} G_i(S_{\tau_i}) \right\}$$

$$+ \sum_{i=1}^{M} \left[\lambda_i \left(\frac{C_i^{(o)} + C_i^{(b)}}{2} \right) + |\lambda_i| \left(\frac{C_i^{(o)} - C_i^{(b)}}{2} \right) \right] . \tag{16}$$

We conclude that transaction costs for trading in the cash market and in options can be taken into account with slight modifications of the algorithm presented in Section 3.

5. Barrier options

We discuss how to apply the model to barrier options. *Knockout* options become automatically worthless when the price of the underlying asset reaches a pre-established level and, similarly, *knock-in* options are "activated" when a particular price level is attained. The sensitivities of the option premium near the barriers are of particular interest from the point of view of hedging. In fact, the payoff of "reverse knockouts" and "reverse knock-ins" is a discontinuous function of the index. This results in large values of Delta and inversed Gamma exposures that make delta-hedging near expiration difficult if not outright impossible.[15] The possibility of constructing cost-effective hedges using vanilla options seems particularly appealing in the world of barrier options.

We consider the problem of hedging a knockout option by diversification into the vanilla options market. Suppose that the agent decides to use an option portfolio with hedge-ratios $(\lambda_1 , ... , \lambda_M)$. The principal difference with the case of European-style cash-flows without barriers is that pricing a portfolio which combines barrier options and vanilla options requires studying the position after the option knocks out and the agent is left with a vanilla option position that must be managed or unwound.

For simplicity, we consider the case of a single knockout option with a constant barrier. The (worst-case) liability along the barrier corresponds to the λ-UVM value of the outstanding vanilla options position after the knockout. It is given by

$$V^{barr}(S,t;\lambda_1 , ... , \lambda_M) \;=\; \sup_{P \,\in \mathcal{P}} \mathbf{E}^P \left\{ - \sum_{\tau_i \geq t}^{M} \lambda_i \, e^{-r(\tau_i - t)} \, G_i(S_{\tau_i}) \right\} , \qquad (17)$$

for (S,t) along the knockout barrier. This function is calculated using the nonlinear PDE (6)-(7). To calculate the value of the residual liability for the agent prior to hitting the barrier — this includes the exposure to the still-active knockout option — we must solve a boundary-value problem for equation (6)-(7) with boundary data (17). The model value, denoted again by $V(S,t;\lambda_1 , ... , \lambda_M)$, is obtained by adding the value of the residual liability to the cost of the option portfolio.

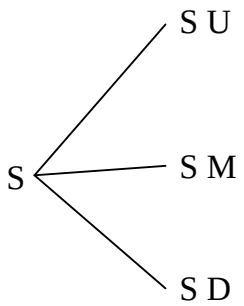
According to the theory of §2, the residual liability (solution of the boundary-value problem) corresponds to the portion of the portfolio which requires active Delta-hedging in the cash market. The Delta is equal to $\partial V(S,t;\lambda_1 , ... , \lambda_M) \,/\, \partial S$.

The algorithm can be used to hedge, in an aggregate mode, a portfolio of barrier options with different barriers, strikes and expirations.

[15] Substantial losses have resulted by the impossibility of unwinding knockout positions requiring a disproportionate offset in the cash market. An interesting account of techniques for trading barrier options is given in Taleb (1995).

6. Numerical implementation

In general, the nonlinear equation (5)-(6) does not have closed-form solutions and must be solved numerically. A simple approach, described in Avellaneda, Levy and Parás (1994,1995) and Parás (1995) relies on an explicit finite-difference solver for the PDE. It is useful to regard a finite-difference scheme as an exact valuation algorithm for a discrete model approximating the stochastic differential equation (1).[16] Accordingly, let us consider a trinomial model in which the underlying asset can change after each trading period to one of three different levels.



After we impose a risk-adjusted drift (eq. (2)), the trinomial tree has one degree of freedom at each node, since the choice of risk-adjusted probabilities $\{P_U, P_M, P_D\}$ is not unique. This degree of freedom is used to model heteroskedasticity: probabilities that assign more weight to the extreme nodes will yield a larger spot volatility than those assigning more weight to the center. Thus, by fixing U, M, and D and allowing the risk-adjusted probabilities to vary over a one-dimensional set, we can accommodate a range of variances according to the band (3) that we wish to model. A simple choice of parameters is

$$U = e^{\sigma_{max}\sqrt{dt}+\mu\, dt}, \quad M = e^{\mu\, dt} \quad \text{and} \quad D = e^{-\sigma_{max}\sqrt{dt}+\mu\, dt},$$

dt being the time-mesh.[17] The one-parameter family of risk-adjusted pricing probabilities is given by

$$P_U = p\left(1 - \frac{1}{2}\sigma_{max}\sqrt{dt}\right),$$

$$P_M = 1 - 2p$$

[16] This interpretation is not essential to derive the finite-difference approximation to (5)-(6). It provides a "concrete" interpretation of the uncertain volatility model and the nonlinear PDE in the language of probability trees.

[17] For stability reasons the volatility assigned to the tree has to be greater or equal to σ_{max}. This restriction is equivalent to imposing on the pricing probabilities to be positive.

and

$$P_D = \; p \left(1 \; + \; \frac{1}{2} \sigma_{max} \sqrt{dt} \right) \; .$$

Here, p is a variable parameter that satisfies

$$\frac{\sigma^2_{min}}{2\,\sigma^2_{max}} \; \leqq \; p \; \leqq \; \frac{1}{2}$$

to reflect different choices for risk-adjusted probabilities or, equivalently, for spot volatilities at each node.

The numerical implementation of equation (4) along these lines takes the form

$$V^j_n \;\; = \;\; F^j_n \;\; + \;\; e^{-r\,dt} \cdot \;\; \left[\; V^j_{n+1} \;\; + \;\; p\, L^j_{n+1} \; \right] \;\; , \tag{18}$$

where

$$L^j_{n+1} \;\; \equiv \;\; \left(1 - \frac{1}{2} \sigma_{max} \sqrt{dt} \right) \; V^{j+1}_{n+1} \;\; + \;\; \left(1 + \frac{1}{2} \sigma_{max} \sqrt{dt} \right) \; V^{j-1}_{n+1} \;\; - \;\; 2\, V^j_n \;\; . \tag{19}$$

The term F^j_n appearing in (18) represents the cash-flow due at the n^{th} trading date. The parameter p is chosen according to the rule

$$p \;\; = \;\; \begin{cases} 1/2 \qquad \text{if} \;\; L^j_{n+1} \;\; \geq \;\; 0 \\ \\ \sigma^2_{min} / \, 2\sigma^2_{max} \qquad \text{if} \;\; L^j_{n+1} \;\; < \;\; 0 \;\; . \end{cases} \tag{20}$$

For $p = 1/2$, the extreme branches (U and D) carry 100% of the probability and the local variance achieves the maximum value, $\sigma^2_{max}\, dt$. For $p = \; \sigma^2_{min}/2\,\sigma^2_{max}$, the local variance achieves the minimum value, $\sigma^2_{min}\, dt$. The expression L^j_{n+1} can be interpreted as a discrete approximation to the second-derivative of V.

Equations (18) and (20) are the discrete analogues of (6) and (7). The reader is referred to Parás (1995) for the proof of convergence of the algorithm as $dt \;\; \rightarrow \;\; 0$.

The minimization of the function $V(S_t\,, t\,; \lambda_1\,, ...\,, \lambda_M)$ in λ-UVM is done with the finite-difference solver for equation (6), in the formulation (18)-(19)-(20), coupled to a minimization routine. For our computations we used a quasi-Newton routine provided in the NAG library.

We implemented λ-UVM on a SUN workstation. Our algorithm incorporates a *term-structure of volatility bounds*, thus allowing the band to change with time. It also allows for a term-structure of deposit rates, taking into account differences between borrowing

and lending rates (and thus, for instance, interest rates on margin accounts). Using the formulation of §4 we are also able to take into account transaction costs in the cash and options markets. Finally, the algorithm also offers the possibility of calculating hedging strategies that impose limits on Delta.[18]

The running time of the computations grows linearly with M, the number of input options, and is essentially independent of the number of target cash-flows. A (non-optimized) version of the algorithm used in the preparation of this paper solved the optimization problem in approximately 37 user-seconds for $M = 3$ and 160 user-seconds for $M = 10$. Running times for hedging barrier options are longer due to mesh-refinement near the barrier but still scale linearly with M. The longest calculations for barrier options, involving 25 input instruments, took 15 user-minutes.

7. Example: Hedging OTC options with different strikes with at-the-money options

This example illustrates the theory in the simplest case: option pricing in a market with uncertain volatility in which there is a single traded option considered as input. We assume that the latter option is at-the-money and that it trades at some known implied volatility. Using this information in conjunction with a volatility band, we price call options with different strikes having the same maturity.

Experiment 1. We assume that the riskless interest rate is 7%. The underlying asset is assumed to pay no dividends over the time-period under consideration. The maturity of all options considered is 183 days. The at-the-money six-month option (with strike equal to the price of the underlying asset) is assumed to trade at an implied volatility $\sigma_{impl.} = 0.16$.

We assumed a volatility band with $\sigma_{min} = 0.08$ and $\sigma_{max} = 0.32$ in this first experiment. We applied the optimality algorithm to compute the prices and hedge-ratios for options with strikes at 85%, 90%, 95%, 105%, 110% and 115% of the spot. The results are illustrated in the following Table.

Table 1: Option pricing with band (.08-.32) and at-the-money options with volatility .16 (M=1)

K/S	0.85	0.90	0.95	1.00	1.05	1.10	1.15

[18] All these are non-linear constraints on the dynamic programming equation (6), which is modified accordingly.

BS value	.1770	.1327	.0932	.0609	.0368	.0206	.0105
Δ_{BS}	.9600	.8944	.7804	.6261	.4569	.3021	.1817
V	.1870	.1440	.1019	.0609	.0495	.0399	.0317
Δ	.5920	.4103	.2091	.0000	$-$ .0758	$-$ .0909	$-$ .0817
λ	.4166	.6070	.8061	1.0000	.9428	.8188	.6783
$\sigma_{impl.}$	.2538	.2304	.1995	.1600	.2052	.2345	.2549

The second and third rows indicate the Black-Scholes values and Deltas of the different options, computed with 16% volatility. All option values are expressed as a percentage of the spot price. The fourth row gives the value V which results from solving the optimality algorithm assuming a band $0.08 - 0.32$ and a single hedging instrument: at-the-money options with implied volatility 0.16. The next two rows show the optimal hedge-ratios with respect to the underlying security (Δ) and the at-the-money option (λ). The last row expresses the optimal value for each strike in terms of implied volatility.

Notice that the mix of options and shares is different according to the strike — a larger proportion of options is used when the strike is close to the money. Consistently with the theory, the model value of the at-the-money option is identical to the market price, .0609, and the hedge is purely synthetic with $\Delta = 0.000$ and $\lambda = 1.000$. As a rule, the amount of at-the-money options held long (λ) diminishes as the strike goes away from the money.

Experiment 2. We use the same model parameters as before, except for the volatility band, which is taken to be $\sigma_{min} = 0.04$ and $\sigma_{max} = 0.64$. The results, given in Table 2, give an approximate idea of how sensitive the pricing is to the width of the volatility band (compare the last rows of Table 1 and Table 2).

Table 2: Option pricing with band (.04-.64) and at-the-money options with volatility .16 (M=1)

K/S	0.85	0.90	0.95	1.00	1.05	1.10	1.15
V	.1954	.1504	.1056	.0609	.0567	.0526	.0486
Δ	.3500	.2365	.1195	.0000	$-$ .0067	$-$ .0084	$-$ .0052
λ	.6513	.7651	.8818	1.0000	.9409	.8770	.8105

$\sigma_{impl.}$	.3067	.2633	.2154	.1600	.2307	.2811	.3209

The option hedge-ratios obtained with the $0.04 - 0.64$ band are generally larger than the ones with the narrower band. Note also that the implied volatilities are larger in Table 2 and increasingly so as the strikes move away from the money.

Experiment 3. A further assessment of the effect of changing the width of the band can be made by considering the limiting case $\sigma_{min} = 0$, $\sigma_{max} = \infty$. In this case, the optimality algorithm for hedging one option with $M = 1$ admits an explicit solution, namely

$$\tilde{V}_K \;=\; \begin{cases} \left(1 - \frac{K}{S}\right) \;+\; \left(\frac{K}{S}\right) \cdot \tilde{V}_S & K \;<\; S \;, \\ \\ \tilde{V}_S & K \;\geqq\; S \;. \end{cases}$$

Here, $\tilde{V}_K$ represents the value of a call with strike K expressed in percentage of the price of the underlying. In particular, the $(0, \infty)$ hedge-ratios are

$$\Delta \;=\; \begin{cases} \left(1 - \frac{K}{S}\right) & K \;<\; S \;, \\ \\ 0 & K \;\geqq\; S \;, \end{cases}$$

and

$$\lambda \;=\; \begin{cases} \frac{K}{S} & K \;<\; S \;, \\ \\ 1.00 & K \;\geqq\; S \;. \end{cases}$$

These hedge-ratios correspond to trivial (cheapest) static strategies using stocks and at-the-money options. In the example considered above, we have, $\tilde{V}_S \;=\; 0.0609$. Using the above formulas, we generate the corresponding table for $\sigma_{min} = 0 \;,\; \sigma_{max} = \infty$.

Table 3: Option pricing with band (0-∞) and at-the-money options with volatility .16 (M=1)

K/S	0.85	0.90	0.95	1.00	1.05	1.10	1.15
V	.2018	.1548	.1078	.0609	.0609	.0609	.0609

Δ	.1500	.1000	.0500	.0000	.0000	.0000	.0000
λ	.8500	.9000	.9500	1.0000	1.0000	1.0000	1.0000
$\sigma_{impl.}$	.3424	.2846	.2248	.1600	.2456	.3112	.3668

The interest of the table resides in the last row, which provides absolute upper bounds on the implied volatilities, conditionally on the fact that the implied volatility of at-the-money options is 0.16.

8. Example: Constructing volatility term structures from market data for at-the-money options

The values of options on a particular asset with different expiration dates are often quoted using a *term structure of implied volatilities* for liquidly traded contracts, which are usually the at-the-money options. When trading over-the-counter (OTC) options with odd expiration dates, agents must estimate the option values using market data. In this example, we use the optimality algorithm to estimate the prices of options with odd-expiration dates. We derive in this way a worst-case volatility term structure that interpolates between the implied volatilities corresponding to maturities which are traded. Both the "sell-side" and the "buy-side" are considered.

We shall assume in the example that there exist four liquidly traded at-the-money option contracts maturing in 7, 35, 70 and 183 days. The input data is as follows

Maturity	**Implied volatility**
7 days	19.6%
35 days	14.9%
70 days	12.0%
183 days	16.0%

As before, we assume an interest rate of 7% and neglect transaction costs. The limits for trading options were taken to be ± 10 contracts for each maturity (as we shall see, these limits were not reached in the calculations made below). The volatility band was chosen to be $\sigma_{min} = .08$, $\sigma_{max} = 0.32$. We considered the problem of pricing at-the-money options (calls) with maturities of 10, 20, 50, 60, 100, 120 and 150 days.

Table 4. At-the-money call options held short

Mat.(days)	10	20	50	60	100	120	150
V	.0183	.0199	.0248	.0260	.0527	.0558	.0586
Δ	− .0443	− .0751	− .1060	− .0489	− .0435	− .0823	− .0508
λ_7	.1005	− .0047	− .0151	− .0062	− .0013	−	−
λ_{35}	.9216	1.0862	− .0634	− .0334	− .1642	− .0261	− .0069
λ_{70}	−	−	1.1976	1.0923	.4189	.0721	.0149
λ_{183}	−	−	−	−	.7545	1.0212	1.0367
$\sigma_{impl.}$	.2647	.1954	.1391	.1288	.2129	.2004	.1798

In this table, λ_x represents the hedge-ratio for the option maturing in x days. The last row gives the implied volatilities for each options calculated using V. An examination of the option hedge-ratios shows that the next closest maturity usually carries the largest part of the hedge.

For example, the implied volatility for the 10-day option is 26.5%, and the amount of 7-day options used to hedge volatility risk is very small ($\lambda_7 = .1005$). The point is that holding the 7-day option does not offer much protection against an increase in the 10-day volatility. Given the wide expected range of volatilities given by the band, and the comparative low cost of 35-day options — the 35-day volatility is only 14.5% — the algorithm yields $\lambda_{35} = .9246$, i.e. an option hedge using predominantly the 35-day option.

Consider instead the option maturing in 60 days, which is ten days before the expiration of the 70-day trading at 12% volatility. The algorithm values this option on the offer side at 12.88% volatility, which is reasonably close to the 12%, especially given the width of the UVM band. In this case, the option hedge-ratios are essentially negligible with the exception of the one corresponding to the nearest-next maturity option, $\lambda_{70} = 1.1976$. Similar analyses can be made for the other maturities.

We now study the pricing and hedging of *long* at-the-money options with intermediate maturities. The "buy-side" problem is equivalent to asking what is the *least possible* arbitrage-free value of a derivative security conditionally on the prices of input instruments.

Table 5. At-the-money call options held long

Mat.(days)	10	20	50	60	100	120	150

V	.0117	.0133	.0230	.0243	.0307	.0338	.0384
Δ	− .0154	− .0869	.0727	− .1300	− .1599	− .2354	− .3171
λ_7	− .9847	− .8895	− .0188	− .0296	− .0217	− .0298	− .0342
λ_{35}	−	−	− .8984	− .8275	− .1777	− .2029	− .2033
λ_{70}	−	−	−	−	− .6106	− .4817	− .3776
$\sigma_{impl.}$	.1646	.1243	.1266	.1180	.1042	.1003	.0876

As expected, the implied volatilities are lower in comparison with the values for short positions. The buy-side implied volatilities approach the market values when there is an option expiring near, and *before*, the option under consideration. For instance, the 10-day option has a buy-side volatility of 16.5% — which is essentially in the middle of the band and 2.5 percentage points below the 7-day volatility. The corresponding hedge ratio is $\lambda_7 = -.9874$, which represents essentially a "one-for-one hedge" consisting in shorting one 7-day option. In contrast, the 150-day option has a volatility which is practically near the lower bound implied by the band (of .08). The corresponding option position consists in shorting the 35- and 70-day options and implies substantial Delta-hedging. Notice that the 183 option, which expires after all maturities considered here, is not selected by the algorithm on the buy-side.

9. Telmex options

This examples involves a larger set of input options. We consider the Telmex (TMX) Advanced Depository Receipts and options on this security traded in the NYSE in the months following the December 1994 Peso devaluation. During this period, the market exhibited very large implied volatilities in comparison with 1994 levels. Variations in implied volatilities on a given trading date were also large, according to both strike levels and maturities. Overall, the volatility term-structure was inverted, decreasing as the maturity increased. A complex volatility structure such as this one is well-suited for applying the optimality algorithm.

The following table describes the Telmex market on March 10, 1995.[19]

[19] Prices are per ADR and the contract size is 100 ADRs. Source: *The New York Times*, March 11, 1995. Closing prices do not represent market data at a particular instant in time. We do not take into consideration bid/offer spreads, volume traded, open interest, etc. Our sole aim is to illustrate how the algorithm could be applied to a market with many options and to analyze the hedges that result.

Table 6: Telmex, March 10, 1995

March 10 closing: TMX= 28.625 Assumed interest rate= 7%

Mat.	Strike	Call	Ivol	Put
Mar	25	3.6250	0.0000	0.25
Mar	30	0.4375	0.5938	2.0000
Mar	35	0.0625	0.8345	6.250
Mar	40	0.0625	1.3221	12.6250
Apr	22.5	6.6250	0.7028	0.4375
Apr	25	4.5000	0.6306	0.875
Apr	30	1.5000	0.5642	2.750
Apr	35	0.3750	0.5675	6.50
May	22.5	6.8750	0.5888	0.6875
May	25	5.1250	0.6193	1.25
May	30	2.3125	0.5794	3.50
May	35	0.8125	0.5478	7.250
May	40	0.3750	0.5959	12.50
Aug	22.5	7.3750	0.4251	1.25
Aug	25	5.8750	0.4764	2.125
Aug	30	3.2500	0.4689	4.250
Aug	35	1.7500	0.4777	7.625
Aug	40	0.8750	0.4761	14.00

The options are American-style. The implied volatilities of the calls — computed using the Black-Scholes formula with an interest rate of 7% — appear in the column "Ivol". Puts appear only for reference purposes and are not used as inputs. Call options can be treated as being essentially European-style, due to a a well-known property of options on stocks that pay no dividends. In contrast, American puts can be exercised early and hence cannot be used in the optimality algorithm.[20]

The maximum range for implied volatilities is in the front month, from 0% (Mar 25) to 132% (Mar 40). These volatilities reflect (most likely) the substantial transaction costs and risk-premia associated with far-from-the-money options expiring in a few days. We

[20] For instance, an agent that sells a put to hedge an OTC derivative position runs the risk that the put will be exercised early and hence that the hedge will be "lifted".

discard these two options from the set of input instruments, reagarding them as illiquid. We see that the next two extreme volatilities are 43% (Aug 22.5) and 83% (Mar 35).

To implement the uncertain volatility model we choose a flat band (from March to October) with

$$\sigma_{min} = 0.3 \quad , \quad \sigma_{max} = 1.5$$

Notice that the range of implied volatilities of the remaining input instruments is $0.4251 - 0.8345$, which is well-contained within the chosen band. We have even assumed a "comfort zone" on each side to account for transaction costs in the cash market and possible volatility fluctuations outside the range of the implied volatilities of the input options. The limits on the λs of the hedge portfolio are taken to be $\Lambda_i^+ = 10$ and $\Lambda_i^- = -10$. Transaction costs for trading options are not taken into account.

Experiment: *Writing a European-style digital option.* Consider a contingent claim with the following terms:

Expiration date: third Friday of May 1995

Payoff: \$10,000.00 if $TMX < 20$ or \$0 if $TMX > 20$ at the expiration date.

We used only the 11 calls which expire in March, April and May (excluding the Mar 25 and Mar 40 calls) as input instruments.

The optimality algorithm yields the following output:

$$\textbf{Value } (V) = \$1426$$

$$\Delta = -0.2344 \approx 2344 \text{ ADRs sold short per \$ 10,000 notional}$$

Call hedge-ratios:[21]

Mat.	Strike	Quantity (λ)
Apr	22.5	-0.15
Apr	25	0.05
May	22.5	+0.34

[21] We use the standard mathematical convention that λ represents the number of options to buy *one share*. Therefore, the number of contracts is actually $\lambda/100$. Given that the notional amount of the digital is \$ 10,000, only options with λ s greater than 0.01 are reported.

Figure 1.

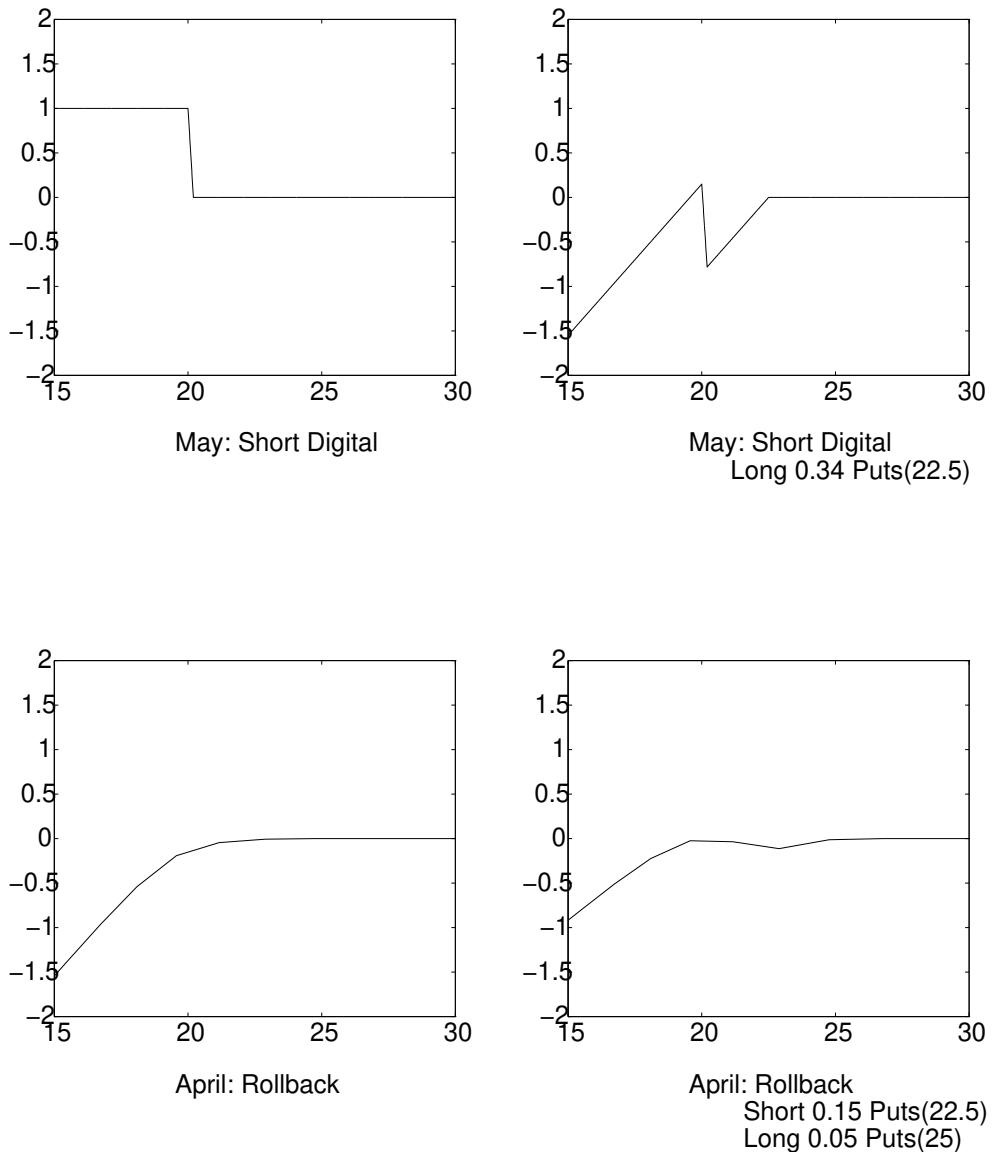


FIGURE 1. Residual liability values corresponding to the short digital Telmex option and the portfolio of synthetic puts shown at different dates, per dollar notional. The upper-left box represents the liability corresponding to the digital option with May expiration. The upper-right box shows the liability including the option hedge consisting of 0.34 long May 22.5 synthetic puts. The lower-left box represents the value of the liability after the April expiration date. The lower-right box represents the liability in April, including the position in the two options expiring in April. Notice that the hedge selected by the algorithm gives rise to flat residual liabilities. Thus, the algorithm reduces market risk significantly. Present values are calculated with the 0.3-1.5 volatility band.

We analyze this output from the point of view of pricing first. The value $V = \$1426$ represents the total hedging costs (options plus residual). The worst-case scenario value with a band $0.3 - 1.5$ *without* using options turns out to be \$5729, or about three times higher. This gives a measure of the efficiency of using options to reduce hedging costs. An assessment of the "competitiveness" of $V = \$1426$ as an offer price for the digital option can be made by calculating the nominal (implied) volatility which would make V equal to the Black-Scholes "fair value " of the digital. This volatility turns out ot be $\sigma_{impl.} = 0.7685$. Given that the implied volatilities of the May options nearest-to-the-money are 57.9% and 61.9% to the nearest decimal, we conclude that an implied volatility of 76.85% is reasonable for this "exotic" product, and even more so when the full range of implied volatilities is taken into account.

The result is also interesting from the point of view of hedging. What kind of protection was achieved using options? Using the put-call parity relation, we see that the option component of the hedge portfolio is equivalent to

Long: 34 May 22.5 synthetic European put contracts

5 Apr 25 synthetic European put contracts

Short: 15 Apr 22.5 synthetic European put contracts

The hedge consists essentially in buying synthetic May puts to dominate the liability of the digital payoff, selling April synthetic puts with higher volatility and covering the exposure by short-selling TMX ADRs. Figure 1 represents the liability profiles for this hedge at different expiration dates. The portfolio of synthetic puts eliminates almost completely the volatility risk of the digital option.

10. Example: Dollar/Mark foreign-exchange barrier options

We consider the problem of pricing and hedging knockout options with in-the-money barrier ("reverse knockouts") using vanilla options and the spot market. The data used in this example corresponds to the actual \$/DEM market on August 23, 1995 and incorporates bid-offer spreads. It consisted of

- spot DEM/USD exchange rate on that date (\$1 = DEM 1.4885 - 1.4890)
- USD deposit rates for 1, 2, 3, 6 and 9 months (bid-offer)
- DEM deposit rates for the same period (bid-offer)
- volatility curve for 50 Δ options with these maturities (bid-offer)
- 25 Δ risk-reversals curve for these maturities (bid-offer)

Maturity	Type	Strike	bid	offer
30 days	Call	1.5421	0.0064	0.0076
	Call	1.5310	0.0086	0.0100
	Call	1.4872	0.0230	0.0238
	Put	1.4479	0.0085	0.0098
	Put	1.4371	0.0063	0.0074
60 days	Call	1.5621	0.0086	0.0102
	Call	1.5469	0.0116	0.0135
	Call	1.4866	0.0313	0.0325
	Put	1.4312	0.0118	0.0137
	Put	1.4178	0.0087	0.0113
90 days	Call	1.5764	0.0101	0.0122
	Call	1.5580	0.0137	0.0160
	Call	1.4856	0.0370	0.0385
	Put	1.4197	0.0141	0.0164
	Put	1.4038	0.0104	0.0124
180 days	Call	1.6025	0.0129	0.0152
	Call	1.5779	0.0175	0.0207
	Call	1.4823	0.0494	0.0515
	Put	1.3902	0.0200	0.0232
	Put	1.3682	0.0147	0.0176
270 days	Call	1.6297	0.0156	0.0190
	Call	1.5988	0.0211	0.0250
	Call	1.4793	0.0586	0.0609
	Put	1.3710	0.0234	0.0273
	Put	1.3455	0.0173	0.0206

Table 2.. Options prices used as input in the λ-UVM algorithm for hedging barrier options. The prices are in DEM per dollar notional because we performed the calculations using DEM as the "domestic currency" and USD as the "underlying asset". Subsequently all prices and Deltas were converted back to dollars per dollar notional.

Option prices and bid/offer spreads in Table 2 were calculated using the data and standard conventions for Dollar-Mark options. The range of implied volatilities of the options considered was approximately 12.5% - 15%, taking into account bid-offer spreads and the risk-reversal curve. To "calibrate" the λ-UVM, we used

- a volatility band $\sigma_{min} = 12\%$, $\sigma_{max} = 16\%$
- a risk-adjusted $\mu = -0.0164$ (average DM-USD spread)
- a DEM carry $r = 0.0412/0.0441$ (lend/borrow)

The risk-adjusted drift and discounting rates were computed using average deposit rates for μ and bid-offer spreads (for r).

Experiment 1. *Barrier option with 6 months to expiration.* Consider a reverse-knockout dollar put with strike $K = 1.4800$ DEM/\$ and knockout at $H = 1.3800$ DEM/\$ with six months (180 days) to expiration. Assume that this option is written on a notional amount of 20,000,000.00 U.S. dollars (one bp= \$ 2,000.00).

In the optimization procedure, we considered 25 "input" vanilla options with maturities of 1, 2, 3 , 6 and 9 months, and with 5 strikes at each maturity: the 20 Δ dollar puts and calls, the 25 Δ dollar puts and dollar calls and the 50 Δ calls. The results were as follows:[22]

Price (offer) = 0.0045 USD per 1 USD notional

Hedge in Options =

type	strike	maturity	quantity/\$ notional
put	1.6025	180 days	0.1571
put	1.5580	90 days	-0.0538
put	1.4856	90 days	-0.0948
put	1.5469	60 days	-0.0182
put	1.4866	60 days	-0.0612
put	1.4312	60 days	-0.0110
put	1.5310	30 days	-0.0025
put	1.4872	30 days	-0.0076

New Delta = 0.0216 (spot USD to be held in the hedge portfolio per dollar notional)

Figures 2 and 3 give a graphical analysis of the *residual liability*, i.e, the present value of the combined position "short knockout option, long 180 option, short 1,2 and 3 month options", The graphs represent the residual liability at several dates. Essentially, the strategy consists in buying six-month Gamma by buying 0.1544 deep-in-of-the-money dollar puts and, simultaneously, selling three-month Gamma by selling 90-days at-the-money and in-the-money dollar puts.

The effect of buying the 180-day puts is to reduce the size of the liability at the knockout barrier (USD = 1.38 DEM) by about 33% from the onset: if the option doesn't knock out but ends up near the barrier, the agent will own the put and hence offset part of the risk. Notice that the discontinuity at the knockout barrier is 10 pfennig or \$ 0.0671 per dollar

[22] Out-of-the money calls were converted into in-the-money puts to clarify the analysis of the hedge.

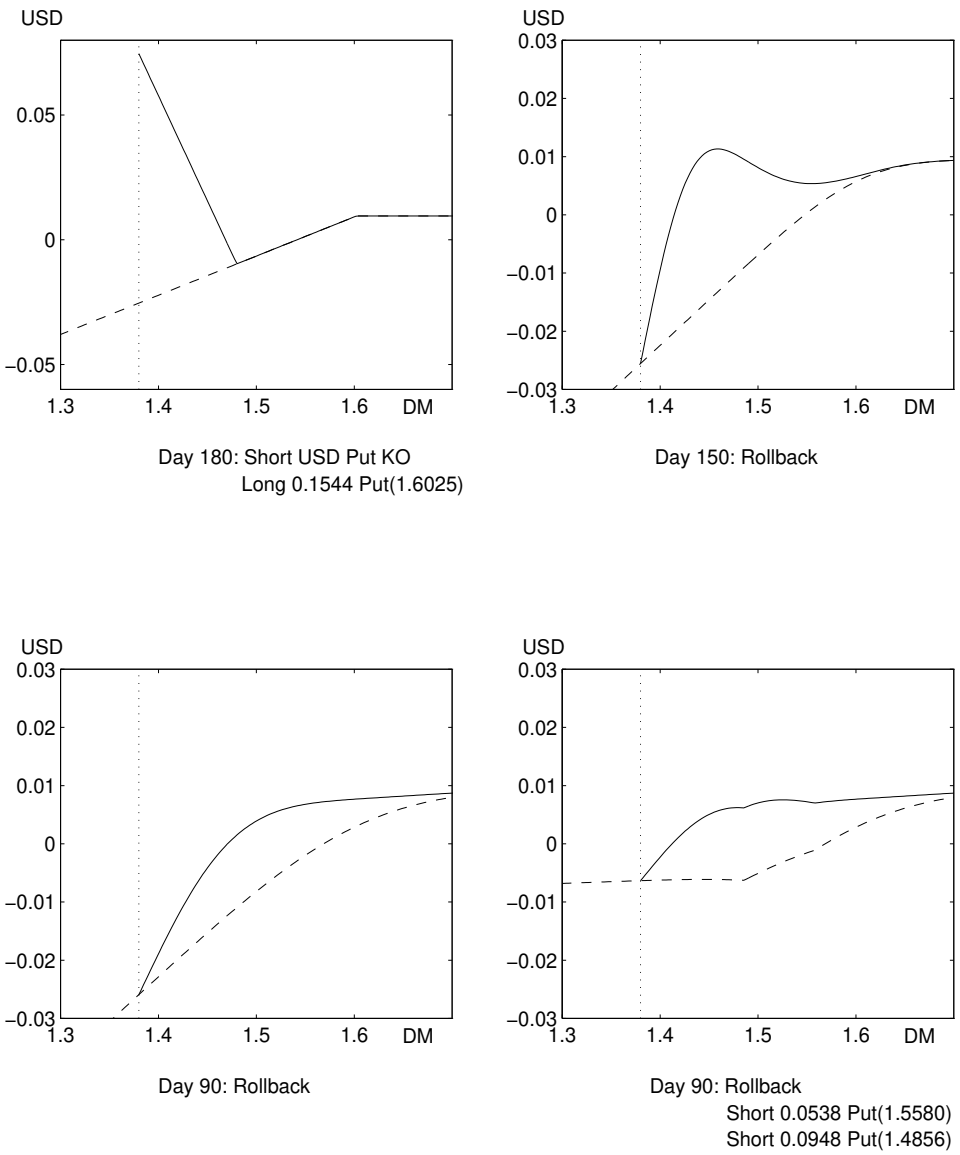


Figure 2. (a) Liability diagram for 180 day USD 1.48 Put with Knock-Out barrier at 1.38, 0.1544 six month Put 1.6025. (b) Rollback (with UVM) of the liability viewed at day 150. (c) Rollback at day 90 after exercise of the 90 day put in the hedge. (d) Liability at day 90 before put exercise. The diagrams take into account the premia received for the options sold (KO premium not included). Dotted lines represent the liability if the option knocks out.

notional, which represents a discontinuity $ 1,342,000.00. The reduction of the "jump" due to the long put option (computed conservatively at intrinsic value) is of approximately $ 442,860.00.[23]

The effect of selling 3-month Gamma is to gain some value and at the same time to

[23] An advantage in case the spot exchange rate is slightly higher than 1.38 DEM/USD near expiration, in which case dynamical hedging is very costly.

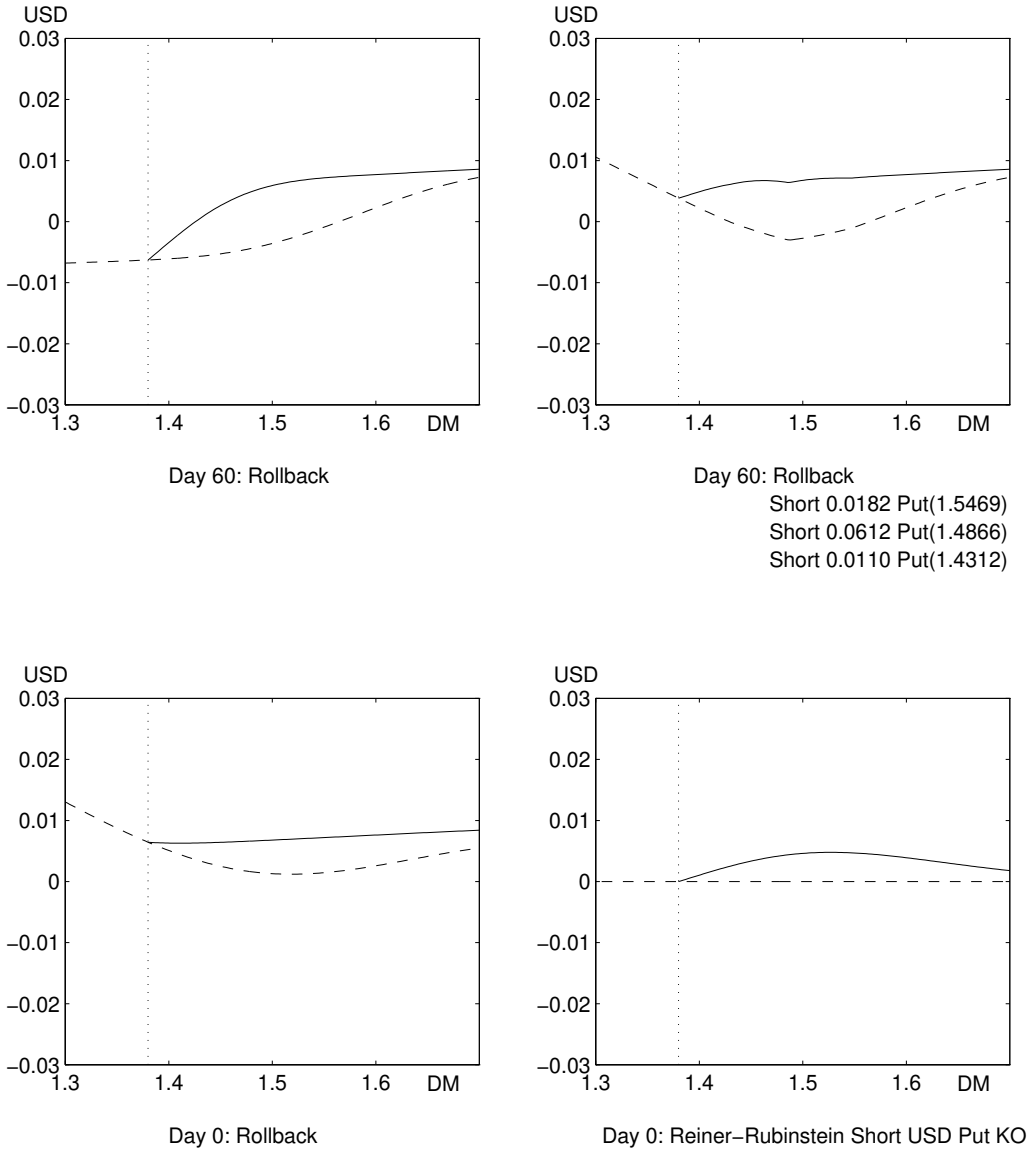


Figure 3. (a) Liability diagram, viewed at day 60 before exercise of short 60 day puts in the hedge. (b) Rollback to day 60 before exercise. (c) Rollback to day 0. (d) Reiner-Rubinstein premium for the 180 day KO; $\sigma = 0.13$. Notice, comparing (c) and (d), the difference in the Delta/Gamma profiles between the λ-UVM solution and the Reiner-Rubinstein solution.

"flatten" out the residual risk. The graphs clearly show that the 3-months liability is quite flat. Moreover, if the exotic option knocks out early, the short position in front-month options is compensated by the long-deep-in-the money put.

We observe finally that the "fair" value of the knockout option using a 13% six-month volatility (computed assuming a lognormal model; cf., Rainier & Rubinstein (198?)) is \$ 0.0030 per dollar notional, or about 30% cheaper than the offer price calculated from the algorithm. Nevertheless, the model price is comparable to offer prices quoted by market-

makers in exotic options, due to the additional risk premium charged for these instruments.

Experiment 2. *3-months reverse-knockout dollar put.* We consider the case of a reverse-knockout option which is in-the-money and has 3 months before expiration. The characteristics of the option are:

Strike = 1.5300 DEM/$

Barrier = 1.43 DEM/$

Maturity = 90 days

Black-Scholes Fair Value (vol .13) = 0.0043 USD per 1 USD notional

The result of applying the λ-UVM to this option is

Price (offer) = 0.0058 USD per 1 USD notional

Hedge in Options =

type	strike	maturity	quantity (/$ notional)
put	1.5469	60 days	-0.0283
put	1.5310	30 days	-0.1629
put	1.4872	30 days	-0.1501

New Delta = -0.0263 (spot USD to be held in the hedge portfolio per dollar notional)

Figure 3 presents a graphical analysis of the residual liability for this hedge at different times to maturity.

The λ-UVM solution consists of *selling options* with one and two months to maturity. This solution, which seems surprising at first, can be analyzed as follows. At a level of spot of approximately 1.49 DEM/USD, the option is well in the money. Consequently, the agent which is short the knockout is long Gamma and will be subjected to large Deltas as the dollar falls towards the barrier. By selling 30 and 90 day volatility in the way specified by the output, the agent becomes instead practically Gamma-neutral and the Delta is no longer moving against the market but instead practically static, positioned short dollar/long Mark. Thus the hedge of the residual resembles more that of a "capped dollar put" than that of a reverse knockout. The solution minimizes volatility risk by flattening the residual profile. An early knockout will leave the agent short vanilla options, but this has already been priced into the value of the barrier option.

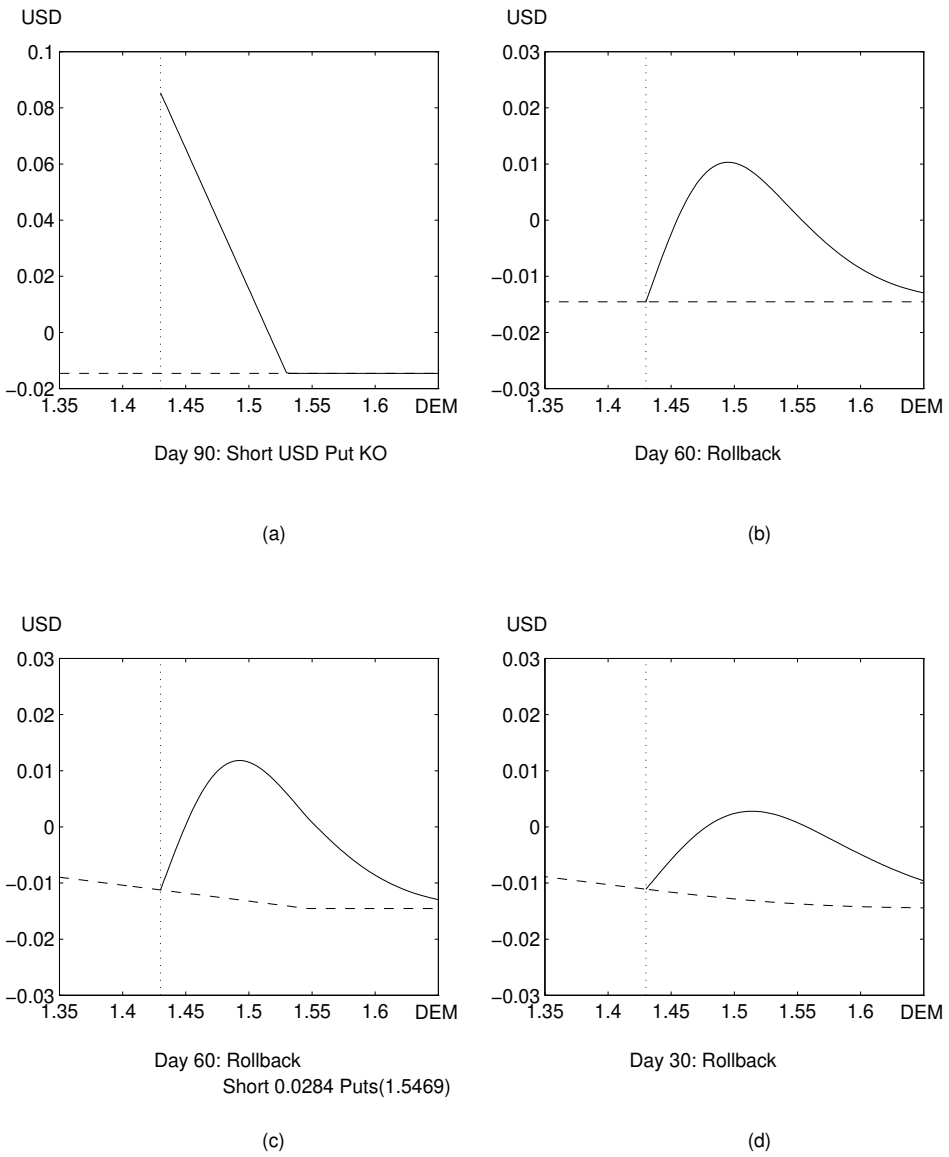


Figure 4. (a) Liability diagram for 90 day USD 1.53 Put with Knock-Out barrier at 1.43. (b) Rollback (with UVM) of the liability viewed at day 60 after exercise of the 60 day put in the hedge. (c) Liability at day 60 before put exercise. (d) Rollback to day 30 of short KO, short 0.0284 puts position. The diagrams take into account the premia received for the options sold (KO premium not included). Dotted lines represent the liability if the option knocks out.

11. Conclusion

We proposed an algorithm for calculating optimal hedging strategies for managing volatility risk of option portfolios and OTC derivatives using vanilla options and cash instruments. The algorithm is based on a theoretical model, the UVM, which takes into account uncertainty of volatility, or heterorskedasticity, by assuming that volatility can

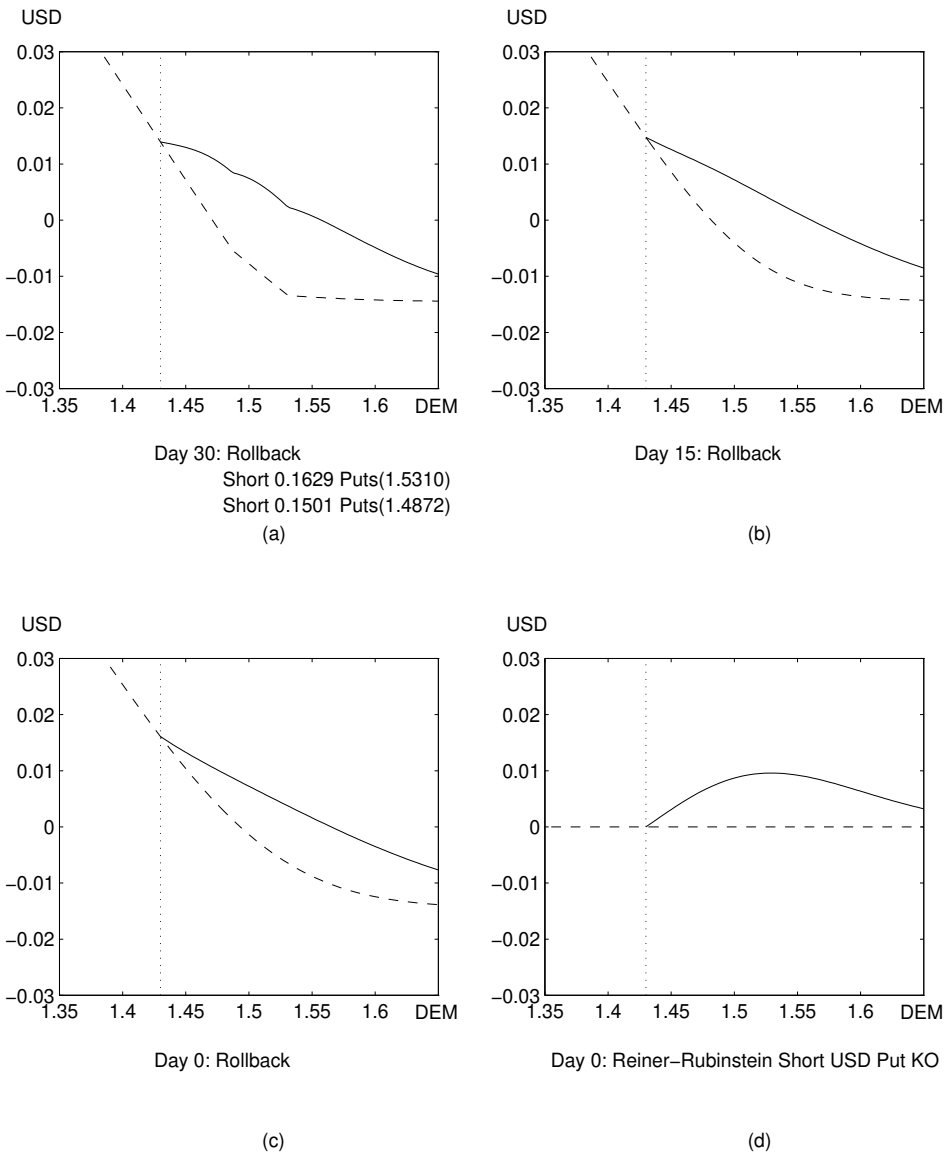


Figure 5. (a) Liability diagram, viewed at day 30 before exercise of short 30 day puts in the hedge. (b) Rollback to day 15 of short KO, short 60 and 30 day puts. (c) Rollback to day 0. (d) Reiner-Rubinstein premium for the 90 day KO; $\sigma = 0.13$. Notice, comparing (c) and (d), the difference in the Delta/Gamma profiles between the λ-UVM solution and the Reiner-Rubinstein solution.

vary inside a "band" ($\sigma_{min} \leq \sigma_t \leq \sigma_{max}$). This range is easily determined from the implied volatilities of input options and the trader's expectations and risk-aversion to extreme volatility moves. The UVM, combined with an optimization algorithm that consists of a regression using option prices, gives rise to the λ-UVM (Lagrangian Uncertain Volatility Model). The algorithm selects cost-efficient option hedges which take into account the worst-case scenarios for the unhedged cash-flows. Through the examples studied, we learned that the λ-UVM can produce competitive bid/offer prices for derivative securi-

ties in heteroskedastic markets and simultaneously reduce volatility risk and the need for intensive mark-to-market of volatility.

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Appendix : Arbitrage-free pricing and λ-UVM

A pricing measure P is said to be *arbitrage-free* if security prices are equal to the expected values of their discounted cash-flows under P (Harrison and Kreps (1979), Duffie (1992)). In the present framework, a measure of type (1)-(2) is arbitrage-free if an only if

$$C_i \;=\; \mathbf{E}^P\left\{e^{-\,r(\tau_i-t)}\,G_i(S_{\tau_i})\right\} \qquad i=1\,,2\,,3\,,....M\;.$$

The λ-UVM bears a strong relation with the problem of constructing arbitrage-free measures. In fact, we will show that, from the point of view of pricing, λ-UVM is equivalent to determining the *worst-case arbitrage-free value for the contracted liability, conditionally on the prices of input options.*

More precisely, we shall establish the following statements:

Proposition 1. *Suppose that the minimization problem (15) admits a solution* $(\lambda_1^*\,,\ldots\lambda_M^*)$ *with* $\Lambda_i^- \;<\; \lambda_i \;<\; \Lambda_i^+$ *for all* $i\in\{1,2,..,M\}$. *Let* P^* *represent the probability measure that realizes the worst-case scenario for the residual liability*

$$\sum_{j=1}^{N} e^{-\,r\,(t_j-t)}\,F_j(S_{t_j}) \;-\; \sum_{i=1}^{M}\lambda_i^*\,e^{-r\,(\tau_i-t)}\,G_i(S_{\tau_i})\;.$$

Then, P^* *solves the program:*

$$\underset{P\,\in\,\mathcal{P}}{Maximize} \qquad \mathbf{E}^P\left\{\sum_{j=1}^{M} e^{-\,r\,(t_j-t)}\,F_j(S_{t_j})\right\}$$

$$\textit{subject to} \quad \mathbf{E}^P\left\{e^{-r\,(\tau_i-t)}\,G_i(S_{\tau_i})\right\} \;=\; C_i\;,\quad i=1,\,2,\,\ldots M\;. \tag{A.1}$$

Proposition 2. *If*

(i) there exists at least one arbitrage-free probability measure of type (1)-(2) and

(ii) there are no option portfolios $(\lambda_1\,,\ldots,\lambda_M)$ *such that*

$$\sum_{i=1}^{M} \lambda_i C_i \;-\; \underset{P \in \mathcal{P}}{Inf}\, \mathbf{E}^P \left\{ \sum_{i=1}^{M} e^{-r(\tau_i - t)}\, G_i(S_{\tau_i}) \right\} \;=\; 0 \tag{A.2}$$

then the cost of the efficient hedging portfolio, $V(S_t, t\,;\, \lambda_1^*, \ldots \lambda_M^*)$, obtained via the optimization problem (10) with $\Lambda_i^- = -\infty$ and $\Lambda_i^+ = +\infty$ is the value of the program (A.1).

To prove these statements, we shall make use of the following facts:

(i) the function $V(S_t\,, t\,; \lambda_1\,, \ldots \lambda_M)$ is convex in $(\lambda_1\,, \ldots \lambda_M)$, and

(ii) for each $(\lambda_1\,, \ldots \lambda_M)$, the probability $P \in \mathcal{P}$ which achieves the supremum in (9) is unique.

The first property follows from the fact that $V(S_t\,, t\,; \lambda_1\,, \ldots \lambda_M)$ is a supremum of linear functions in the variables $(\lambda_1\,, \ldots \lambda_M)$. More precisely, we have

$$V(S_t\,, t\,; \lambda_1\,, \ldots \lambda_M) \;=\; \underset{P \in \mathcal{P}}{sup} \left\{ \sum_{i=1}^{M} a_i(P)\, \lambda_i \;+\; b(P) \right\}, \tag{A.3}$$

where

$$a_i(P) \;=\; C_i \;-\; \mathbf{E}^P \left\{ e^{-r(\tau_i - t)}\, G_i(S_{\tau_i}) \right\} \tag{A.4}$$

and

$$b(P) \;=\; \mathbf{E}^P \left\{ \sum_{j=1}^{N} e^{-r(t_j - t)}\, F(S_{t_j}) \right\} \;. \tag{A.5}$$

The uniqueness of the probability P^* realizing the worst-case scenario follows from the formula

$$\sigma_t^* \;=\; \sigma^*(S_t, t) \;=\; \begin{cases} \sigma_{max} & \text{if} \quad \partial V/\partial S \;\geq\; 0 \\ \\ \sigma_{min} & \text{if} \quad \partial V/\partial S \;<\; 0 \end{cases} \tag{A.6}$$

which expresses the extremal volatility in terms of the second derivative of the solution of the UVM (see Section 2). The latter is determined by the cash-flows $\{F_j(S_{t_j})\}$ and by the

portfolio $(\lambda_1\,,\,\ldots\,,\lambda_M)$. The probability P^* is determined by (A.6).[24] In particular, the graph of the function

$$(\lambda_1\,,\ldots\lambda_M)\;\longrightarrow\;V(S_t\,,t\,;\lambda_1\,,\ldots\lambda_M)$$

has a unique supporting hyper-plane passing through each point. Therefore, $V(S_t\,,t\,;\lambda_1\,,\ldots\lambda_M)$ is everywhere differentiable and its gradient at the point $(\lambda_1\,,\ldots\lambda_M)$ is given by

$$\frac{\partial V(S,t;\lambda_1\,,\ldots\lambda_M)}{\partial\lambda_i}\;=\;a_i(P^*)\;=\;C_i\;-\;\mathbf{E}^{P^*}\left\{\,e^{-r(\tau_i\,-\,t)}\,G_i(S_{\tau_i})\,\right\}. \qquad \text{(A.7)}$$

Proof of Proposition 1. Consider first the case $\left|\Lambda_i^{\pm}\right|\;=\;\infty$ (i.e., when there are no constraints on the λs). Since $V(S,t;\lambda_1\,,\ldots\lambda_M)$ is convex and differentiable, the first-order condition $\partial V/\partial\lambda_i\;=\;0,\;0\leqq i\leqq M$, is both necessary and sufficient for a minimum to occur. In view of (A.7), this first-order condition is equivalent to

$$\mathbf{E}^{P^*}\left\{\,e^{-r(\tau_i\,-\,t)}\,G_i(S_{\tau_i})\,\right\}\;=\;C_i\;,\qquad 1\;\leqq\;i\;\leqq\;M\;. \qquad \text{(A.8)}$$

We conclude that

If the λ-UVM algorithm has a solution, then the probability P^* associated with the optimal portfolio $(\lambda_1^*\,,\ldots\lambda_M^*)$ is arbitrage-free.

Clearly, the argument extends to the case of finite $\Lambda_i^{\pm}$ provided that the optimal portfolio lies in the *interior* of the set of constraints, *i.e.*,

$$\Lambda_i^-\;<\;\lambda_i^*\;<\;\Lambda_i^+\;,\qquad 1\;\leqq\;i\;\leqq\;M\;.$$

Next, we establish that the value obtained with λ-UVM coincides with the supremum of the discounted expected cash-flows $\{F_j(S_{t_j})\}$ as P ranges over all *arbitrage-free* probabilities (1) satisfying the band constraint (2).

In fact, using (A.3), (A.4) and (A.5), we obtain

[24] We neglect degenerate cases in which the second-derivative of the value-function vanishes on a set of positive measure. In the latter case, σ_i^* , is not uniquely defined. These cases are uninteresting, since they basically correspond to forward transactions which have no optionality.

$$\begin{aligned}
\inf_{\lambda} V(S,t;\lambda_1,\ldots\lambda_M) &= \inf_{\lambda} \sup_{P\in\mathcal{P}} \left\{ \sum_{i=1}^{M} a_i(P)\,\lambda_i \;+\; b(P) \right\}, \\
&\geqq \inf_{\lambda} \sup_{P\,,\,a_i(P)=0} \left\{ \sum_{i=1}^{M} a_i(P)\,\lambda_i \;+\; b(P) \right\}, \\
&= \sup_{P\,,\,a_i(P)=0} b(P) \\
&= \sup_{P\,\text{arb. free}} \mathbf{E}^P \left\{ \sum_{j=1}^{N} e^{-r(t_j-t)}\, F(S_{t_j}) \right\} . \qquad \text{(A.9)}
\end{aligned}$$

This gives a lower bound on the best worst-case scenario price in terms of the supremum of expectations over arbitrage-free probabilities. But this bound is in fact an equality, since we have shown that the extremal measure P^* corresponding to the optimal portfolio is arbitrage-free.

Proof of Proposition 2. Suppose that the class of no-arbitrage probabilities $P \in \mathcal{P}$ is non-empty and that the supremum of the discounted cash-flows over all such probabilities is finite. According to (A.9), the λ-UVM value is Moreover, if an optimal portfolio of the λ-UVM, $(\lambda_1^*,..,\lambda_M^*)$ exists (i.e. the minimum is attained at finite values of λ_i) we know from Proposition 1 that the value of the λ-UVM is equal to the the supremum of the discounted cashflows

$$\mathbf{E}^P \left\{ \sum_{j=1}^{N} e^{-r(t_j-t)}\, F(S_{t_j}) \right\}$$

as P ranges over all arbitrage-free probabilities in the class $\mathcal{P}$. Therefore, to prove Proposition 2, we need to show that the optimal portfolio has finite Lambdas. To see this, notice that for $|\lambda| = \sqrt{\sum_i \lambda_i^2} \gg 1$, we have

$$V(S,t;\lambda_1\ldots\lambda_M) = \sum_{i=1}^{M} \lambda_i\, C_i \;-\; \operatorname*{Inf}_{P\in\mathcal{P}} \mathbf{E}^P \left\{ \sum_{i=1}^{M} e^{-r(\tau_i-t)}\, G_i(S_{\tau_i}) \right\} \;+\; O(1) \qquad \text{(A.10)}$$

which is a statement that the liability $\{F_j\}$ is irrelevant as $|\lambda| \to \infty$. In fact, the portfolio consists predominantly of options.

There are altogether three possibilities, according to the behavior of

$$\Phi(\lambda_1\,,\ldots\Lambda_M)\;=\;\sum_{i=1}^{M}\lambda_i\,C_i\;-\;\underset{P\,\in\mathcal{P}}{Inf}\;\mathbf{E}^P\left\{\sum_{i=1}^{M}e^{-r\,(\tau_i-t)}\,G_i(S_{\tau_i})\right\}\tag{A.11}$$

for large $|\lambda\,|$.

First, if Φ converges to $+\infty$ as $|\lambda\,|\;\rightarrow\;\infty$, then the function $V(S,t;\lambda_1\,,\ldots\lambda_M)$ must achieve its minimum at some λ-vector $(\lambda_1^*\,,..\,,\lambda_M^*\,)$ with all Lambdas finite.

Second, if Φ converges to $-\infty$ along some direction as $|\lambda\,|\;\rightarrow\;\infty$, the model cannot be arbitrage-free, in the sense that equation (A.7) cannot hold of any P. In fact, in fact, if (A.7) were true for some P_0 then, clearly,

$$\sum_{i=1}^{M}\lambda_i\,C_i\;-\;\underset{P\,\in\mathcal{P}}{Inf}\;\mathbf{E}^P\left\{\sum_{i=1}^{M}e^{-r\,(\tau_i-t)}\,G_i(S_{\tau_i})\right\}\;\geqq$$

$$\sum_{i=1}^{M}\lambda_i\,C_i\;-\;\mathbf{E}^{P_0}\left\{\sum_{i=1}^{M}e^{-r\,(\tau_i-t)}\,G_i(S_{\tau_i})\right\}\;=0\;,$$

for all $\{\lambda_i\}_{i=1}^{M}$.

It remains to analyze the case in which Φ remains bounded as $|\lambda\,|\;\rightarrow\;\infty$. Notice that $\Phi(\lambda_1\,,\ldots\Lambda_M)$ is a *homogeneous function of degree one*, i.e.,

$$\Phi(\theta\,\lambda_1\,,\ldots\theta\,\lambda_M)\;=\;\theta\cdot\Phi(\lambda_1\,,\ldots\lambda_M)\;.$$

Therefore, the boundedness of Φ as $|\lambda\,|\;\rightarrow\;\infty$ (along any sequence) implies that there must exist at least one portfolio $(\lambda_1\,,..\,,\lambda_M\,)\;\neq\;0$ such that (A.2) holds. This contradicts the second assumption in the statement of Proposition 2. Hence, only the first of the three cases can actually ocurr and the proof is complete.

Remark. The existence of a portfolio satisfying (A.2) would imply that there exists a non-trivial combination of options has *model value zero* under the worst-case volatility scenario. This situation corresponds to a "quasi-arbitrage" in the sense that

(i) the sum of the cost of taking the position and of Delta-hedging the residual is zero and hence the agent is "riskless"

(ii) the agent stands to make a profit by Delta-hedging using the UVM solution if the volatility path does not follow the "worst-case" scenario.

Of course, the marginal case (A.2) can always be avoided by making the volatility band slightly wider. In the latter case, this "quasi arbitrage" will dissapear.