DERIVATIVE SECURITIES: AN INTRODUCTION

Goal: Introduction of the basic *asset classes*, the *markets* where they are traded and the notion of *derivative*.
Discussion of elementary, but important, examples.

1. Asset classes

A. Equities:

- Stocks
- Warrants
- Corporate Debt
- Convertible bonds

B. Fixed-Income Securities:

- U.S. Treasury Bonds & Bills
- Mortgages
- Foreign sovereign bonds
- Foreign-denominated sovereign bonds ($, Yen, DEM)
- Euro-bonds , Euro-deposits ($, Yen, DEM)

C. Foreign Exchange

- Currencies (usually vs. \$, usually short-term deposits)
- Cross-Rates (e.g. Yen/DEM, FF/DEM)

D. Commodities

- Precious Metals
- Agricultural commodities (coffee, etc.)
- Oil & Gas

2. Markets

- Exchange trading
- Over the counter (electronic or telephone)

3. Participants

- Corporations
- Institutional investors (Mutual funds, pension funds)
- Governements
- Hedge funds
- Private investors
- Investment banks & financial institutions

3. Derivative Securities

Derivative Securites are financial instruments or arrangements that *depend or are contingent on, the value of other financial assets* (the underlying assets).

Basic Characteristics:

Most derivative securities involve the exchange of goods or cash-flows at a future **time** (fixed or determined by the contract and /or the notion of **optionality**: goods or money will be exchanged according to future market conditions.

Applications:

- Hedging: risk-management of cash market investments
- Leverage: enhancement of returns on   ″
- Speculation
- Market completion: more versatile investment vehicles
- Correlation of market instruments

4. Forward contracts

Forward: Agreement to buy/sell an asset at a future time at a predetermined price.

Characteristics:

- Underlying asset & delivery
- Maturity
- Forward Price
- Contract price (usually @ market - Price=0)

Features of interest:

- Over-the-counter between counterparties
- Counterparty risk

4. Futures contracts

Futures: *Exchange-traded*, standardized, proxy of forward contract with margin requirements and daily *mark-to-market.*

Characteristics:

- Underlying asset & delivery (Cash or physical)
- Expiration
- Contract notional (standardized)
- Futures price

Features:

- Exchange-traded
- Clearinghouse acts as counterparty
- Daily mark-to-market
- Margin

Most common contracts: Eurodollar deposits (LIBOR), S&P 500 Index, T-Bond

Forwards & Futures Profit-Loss Diagrams

4. Swaps

Agreement to exchange cash-flows of different types at a sequence of dates in the future

Example. Simple interest-rate swap.

- *Tenor*: (time-to-expiration) 5 yrs.
- Notional \$ 10,000,000
- A pays B interest of 7% of notional semiannually
- B pays A LIBOR interest on notional semiannually

Features:

- over the counter
- equivalent to a sequence of forward contracts (FRA s) on LIBOR rate
- Swap rate depends on credit risk

References: J. Hull: *"Options, Futures..."*, and Marshall & Kapner: *"Understanding Swaps"*, Wiley.

Example: 3-Month Eurodollar Futures Contract

Underlying asset: 3-mo Eurodollar deposits

Settlement dates: Mar, Jun, Sep, Dec

Notional amount: \$ 1,000,000

Futures price: F = 100 - 90-day LIBOR

Min. tick: 0.01 % 90-day LIBOR $\Longleftrightarrow$

$$\$\ (90/360)\times\ (1{,}000{,}000/10{,}000) = \$\ 25$$

$$\text{Futures goes up} \Longrightarrow \begin{cases} \text{Long position} & +\ \$\ 25 \\ \\ \text{Short Position} & -\ \$\ 25 \end{cases}$$

$$\text{Futures drops} \Longrightarrow \begin{cases} \text{Long position} & -\ \$\ 25 \\ \\ \text{Short position} & +\ \$\ 25 \end{cases}$$

Application:

Company expects to borrow \$1,000,000 for 90 days in Jan

- Go short 1 December contract. F=94.00

- Position fixes the borrowing rate @ 6%.

5. Options

Call option: Contract gives the *holder* the right, but not the obligation, to *buy* the underlying asset from the *writer* at a specified price on a specified date.

Put option: Contract gives the *holder* the right, but not the obligation, to *sell* the underlying asset from the *writer* at a specified price on a specified date.

European-Style: The option can be exercised only at the specified *expiration date.*

American Style: The option can be exercised *at any time before or at* the expiration date.

Characteristics of the contract:

- Underlying asset
- Exercise price (strike price)
- Maturity
- American or European
- Premium

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Option Profit-Loss Diagrams

History

Agr. forwards & equity options traded OTC until 1973

1973: CBOT starts exchage-traded stock-options

1973: F. Black & M. Scholes: *"The pricing of options and corporate liabilities"*, J. Pol. Econ., 81, p.271

late 70's - 80's: Options and futures on practically all asset classes (OTC & exchange traded). Volume $\approx$ trillions \$

Futures, Forwards, Options & Cash market $\Longrightarrow$
building blocks for constructing **structured financial products** and **investment strategies**

- Equity-linked debt
- Convertible bonds
- Callable bonds
- Mortgages (prepayment option)
- Interest rate **caps** and **floors**
- "Exotic" options

Tools of Financial Engineering

- Synthetization of **new structures** using futures, options, etc.
- **Pricing models** based on Black-Scholes and more advanced techniques
- **Dynamic strategies** for hedging portfolios
- Arbitrage pricing theory

References

J. Hull, *"Futures, Options and other derivative securities"*

D. Duffie, *"Dynamical asset pricing theory"*

S. Figlewski, W. L. Silber, M. G. Subramanyam: *"Financial Options: From theory to practice"*, NYU Solomon Center

Others

S. R. Marki, *"Derivative Financial Products"*, Harper-Collins

R. A. Klein, J. Lederman, *"Hand. of Derivatives & Synthetics"*

Selected articles, e.g., M. Rubinstein: *"Exotic Options", RISK*

Example:

Using options to construct "price-contingent claims"

BASIC ARBITRAGE RELATIONS

1. Forwards

r = interest rate (annual)

q = dividend yield "

S_t = spot price

$F_{t,T}$ = forward price

$$F_{t,T} = S_t \cdot e^{(r-q)\cdot(T-t)}$$

Proof: Equivalent portfolio

- long $e^{-q(T-t)}$ units of the underlying asset
- loan of $F_{t,T}$ dollars to be returned at time T

<u>Strategy</u>: reinvest dividends generated by holding the underlying asset

$$V_\tau = \Delta_\tau \cdot S_\tau$$

$$\begin{aligned} V\tau + d\tau &= \Delta_\tau \cdot S_{\tau+d\tau} + \Delta_\tau \cdot S_{\tau+d\tau} \cdot q\, d\tau \\ &= \Delta_{\tau+d\tau} \cdot S_{\tau+d\tau} \end{aligned}$$

$$\Longrightarrow \Delta_{\tau+d\tau} = \Delta_\tau \cdot (1 + q\, d\tau)$$

$$\Delta_\tau \;=\; e^{q(\tau\,-\,t)}\cdot\Delta_t$$

$$V_\tau \;=\; e^{q(\tau\,-\,t)}\cdot e^{-q(T\,-\,t)}\cdot S_\tau$$

<u>Final cash-flow</u> $= \; e^{+q(T-t)}\cdot e^{-q(T-t)}\cdot S_T \;-\; F_{t,T} \;=$

$$S_T \;-\; F_{t,T} \;= \text{Forward contract's c.-f.}$$

Value of initial position $= \; e^{-q(T-t)}\cdot S_t \;-\; e^{-r(T-t)}\cdot F_{t,T}$

<u>0 value for Forward</u> $\Longrightarrow \; F_{t,T} \;=\; S_t\cdot\; e^{(r-q)\cdot\,(T-t)}$ Q.E.D.

Equivalent strategy determines $F_{t,T}$.

"Cash & Carry Arbitrage": Suppose $F_{t,T} \;>\; S_t\cdot\; e^{(r-q)\cdot\,(T-t)}$

- Borrow $S_t\, e^{-q(T-t)}$ dollars @ interest r
- Buy $e^{-q(T-t)}$ units of the underlying asset
- Sell one unit of the underlying asset forward
- reinvest dividends continuously

Result:

Loan: $-S_t\, e^{(r-q)\cdot(T-t)}$

Cash market: S_T (one unit underlying)

Forward: $F_{t,T} - S_T$

<u>Net cash-flow</u> : $F_{t,T} - S_T \;+\; S_T \;-\; S_t\, e^{(r-q)\cdot(T-t)} \;>\; 0$

“Reverse Cash & Carry”: Suppose $F_{t,T} \ < \ S_t \cdot \ e^{(r-q)\cdot\,(T-t)}$

“Mirror-image” strategy

- **Short-sell** $e^{-q(T-t)}$ units of the underlying asset
- Lend proceeds $S_t\, e^{-q(T-t)}$ dollars @ interest r
- Buy one unit of the underlying asset forward

Short-sale: Borrow security from broker with promise of returning it plus dividends accrued. Sell the security @ market

Result:

Short-sale: $-S_T$ (-1 unit underlying)

Loan: $S_t\, e^{(r-q)\cdot(T-t)}$

Forward: $S_T \ - \ F_{t,T}$

<u>Net cash-flow</u> : $-S_T \ + \ S_t\, e^{(r-q)\cdot(T-t)} \ + \ S_T \ - \ F_{t,T} \ > \ 0$

Arbitrage: Trade that generates a sure profit (or a sure profit in some future state of the market and no losses) with zero cost

Arbitrages cannot subsists for long times since the market will take advantage of the situation. This tends to restore parity.

2. Futures

Forward-Futures Parity: If the interest rate is **constant** between t and T and $\Phi_{t,T}$ is the futures price, then

$$\Phi_{t,T} \;=\; F_{t,T}.$$

Proof: Consider the strategy

- Hold $\Delta_\tau \;=\; e^{r(\tau-t)}$ futures at time τ (dynamic)

Value of this strategy:

$V_t \;=\; 0$

$$V_{\tau+d\tau} \;=\; V_\tau \;+\; V_\tau \cdot r\,d\tau \;+\; e^{r(\tau-t)}\,(\,\Phi_{\tau+d\tau,T} \;-\; \Phi_{\tau,T}\,)$$

$$dV_\tau \;=\; V_\tau\, r\, d\tau \;+\; e^{r(\tau-t)}\, d\,\Phi_{\tau,T}$$

$$\Longrightarrow \qquad V_T \;=\; e^{r(T-t)}\,(S_T \;-\; \Phi_{t,T})$$

- Sell -$e^{r(T-t)}$ forward contracts @ time t (static)

Final value: $\;-e^{r(T-t)}\,(S_T \;-\; F_{t,T})$

Combined strategy long futures – short forwards has

- zero cost
- Riskless profit-loss = $\qquad e^{r(T-t)}\,(F_{t,T} \;-\; \Phi_{t,T})$

$$\text{No-Arbitrage} \quad\Longrightarrow\quad F_{t,T} \;=\Phi_{t,T}$$

3. Futures - cash arbitrage

Reverse Cash & Carry

If $\Phi_{t,T} \; < \; S_t \cdot e^{(r-q)\cdot(T-t)}$,

- Short-sell $e^{-q(T-t)}$ units of the underlying asset
- Lend $S_t \cdot e^{-q(T-t)}$ dollars @ interest rate r
- Dynamic futures hedge with $\Delta_\tau \; = \; e^{-r(T-\tau)}$ contracts held **long**

Direct Cash & Carry

If $\Phi_{t,T} \; > \; S_t \cdot e^{(r-q)\cdot(T-t)}$,

- Borrow $S_t \cdot e^{-q(T-t)}$ dollars @ interest rate r
- Buy $e^{-q(T-t)}$ units of the underlying asset
- Dynamic futures hedge with $\Delta_\tau \; = \; e^{-r(T-\tau)}$ contracts held **short**

"Tailing the hedge": the futures hedge-ratio is less than 1, and converges to 1 at the settlement date

Reference: D.R. Siegel & D.F. Siegel, *"The futures markets"*, Probus,1994

OPTION PARITY RELATIONS

1. Put-Call parity

$C_{t,T}$ = price of a European call with expiration date T

$P_{t,T}$ = price of a European call with expiration date T

K = common strike price

$$P_{t,T} \; - \; C_{t,T} \; = \; e^{-r(T-t)} \, (\, K \; - \; F_{t,T} \,)$$

Proof:

$$P_{T,T} \; = \; Max \, [\, 0 \, , \, K \, - S_T \,] \qquad C_{T,T} \; = \; Max \, [\, 0 \, , \, S_T \; - \; K \,]$$

$$\begin{aligned} P_{T,T} \; - \; C_{T,T} \; &= \; K \; - \; S_T \\ &= \; \underset{\text{(cash)}}{K \; - \; F_{t,T}} \; + \; \underset{\text{(forward)}}{F_{t,T} \; - \; S_T} \end{aligned}$$

Long Put, Short Call =

$(K - F_{t,T})$ \$, Short Forward

Equivalent form:

$$P_{t,T} \; - \; C_{t,T} \; = \; e^{-r(T-t)} \, K - e^{-q(T-t)} \, S_t$$

Synthetic positions using put-call parity

Synthetic Put = Long Call + Long bond + Short Underlying

Synthetic Call = Long Put + Short bond + Long Underlying

Synthetic Bond = Long Put + Short Call + Long Underlying

Synth. Underlying = Long Call + Short Put + Long bond

Applications:

- Lend/borrow money via options position
- Simulate short-sale using options
- Arbitrage (interest rate or dividend rates)

2. No-arbitrage inequalities for European options

1. Positivity: $C_{t,T}(K) \;\geq\; 0 \qquad P_{t,T}(K) \;\geq\; 0$

2. Dominate forward: $C_{t,T}(K) \;\geq e^{-r(T-t)}\;(F_{t,T} \;-\; K\,)$

$$P_{t,T}(K) \;\geq e^{-r(T-t)}\;(K \;-\; F_{t,T}\,)$$

3. Monotonicity:

$$K \;\leq\; L \quad\Longrightarrow\quad C_{t,T}(K) \;\geq\; C_{t,T}(L)$$

$$\Longrightarrow\quad C_{t,T}(K) \;\leq\; C_{t,T}(L) \;+ e^{-r(T-t)}\;(L \;-\; K)$$

4. Time-decay: $T \;\leq\; T' \quad\Longrightarrow\quad C_{t,T}(K) \;\leq\; C_{t,T'}(K)$

5. Convexity: $L \;=\; \alpha\, K \;+\; (1-\alpha)\, M \quad (0 \;\leq\; \alpha \;\leq\; 1)$

$$C_{t,T}(L) \;\leq\; \alpha \cdot C_{t,T}(K) \;+\; (1-\alpha) \cdot C_{t,T}(M)$$

$$P_{t,T}(L) \;\leq\; \alpha \cdot P_{t,T}(K) \;+\; (1-\alpha) \cdot P_{t,T}(M)$$

Convexity & Butterfly Spreads

Example: Options on TMX (Teléfonos de Mexico)

March 10 closing: TMX= 28.625

Mat.	Strike	Call	Ivol	Put
Mar	25	3.6250	0.0000(*)	0.25
Mar	30	0.4375	0.5938	2.0000
Mar	35	0.0625	0.8345	6.250
Mar	40	0.0625	<u>1.3221</u>	12.6250
Apr	22.5	6.6250	0.7028	0.4375
Apr	25	4.5000	0.6306	0.875
Apr	30	1.5000	0.5642	2.750
Apr	35	0.3750	0.5675	6.50
May	22.5	6.8750	0.5888	0.6875
May	25	5.1250	0.6193	1.25
May	30	2.3125	0.5794	3.50
May	35	0.8125	0.5478	7.250
May	40	0.3750	0.5959	12.50
Aug	22.5	7.3750	<u>0.4251</u>	1.25
Aug	25	5.8750	0.4764	2.125
Aug	30	3.2500	0.4689	4.250
Aug	35	1.7500	0.4777	7.625
Aug	40	0.8750	0.4761	14.00

3. State-prices & Contingent-Claim Pricing

Suppose:

- For all $0 \; < \; K \; < \; +\infty$, there is a call w/ maturity T and strike K.
- $C_{t,T}(K)$ is twice differentiable in K.
- Set $\quad \Pi_{t,T}(S_t\,;\,K) \;\equiv\; e^{r(T-t)} \cdot \frac{d^2 C_{t,T}(K)}{dK^2}$ ("limit butterfly")

Then:

1. $\Pi_{t,T}(S_t\,;\,\bullet)$ is a **probability density**

$$\int_0^\infty \Pi_{t,T}(S_t\,;\,Y)\,dY \;=\; 1$$

2. A derivative security with terminal value $F(S_T)$ and no intermediate cash-flows has value

$$V(S_t,t,T) \;=\; e^{-r(T-t)} \int_0^\infty F(Y)\,\Pi_{t,T}(S_t\,;\,Y)\,dY$$

$\Pi_{t,T}(S_t\,;\,\bullet) \;=\;$ **Risk-neutral** probability measure

$e^{-r(T-t)}\;\Pi_{t,T}(S_t\,;\,Y) \;=\;$ **price of state** Y

Option Pricing

$$
\begin{aligned}
C_{t,T}(K) &= C(S_t, t, T, K) \\
&= e^{-r(T-t)} \int_0^\infty \mathrm{Max}\,[0\,,\, Y - K]\; \Pi_{t,T}(S_t\,;\,Y)\, dY
\end{aligned}
$$

Assumption: The "value" of state Y depends only on the **return** $R = (Y - S_t)/S_t \equiv Z - 1$

$$
\Longrightarrow \qquad \Pi_{t,T}(S_t\,;\,Y)\, dY = Q_{t,T}(Z)\, dZ \;, \qquad \left(Z = \tfrac{Y}{S_t}\right)
$$

$$
C(S, t, T, K) = e^{-r(T-t)} \int_0^\infty \mathrm{Max}\,[0\,,\, S\,Z - K]\; Q_{t,T}(Z)\, dZ
$$

Under this assumption, $C(S, t, T, K) =$ convex in S

- $$
\begin{aligned}
C(S_t, t, T, K) &\approx e^{-r(T-t)} \left\{ S_t \left(\int Z\, Q_{t,T}(Z)\, dZ \right) - K \right\} \\
&= e^{-r(T-t)} \left(F_{t,T} - K \right) , \qquad S_t \gg 1
\end{aligned}
$$

- $lim_{S \to 0}\, C(S_t, t, T, K) = 0$