ARBITRAGE PRICING THEORY

Goal: Description of the theoretical principles
behind derivative security valuation

- Model a **single-period** securities market with uncertainty
- Finitely many "final states" or outcomes
- Finitely many securities traded

Key idea: Absence of arbitrage opportunities implies relations between securities prices

Methodology applies to most valuation problems in Finance by suitable generalization of the model

Sources: D. Duffie, *Dynamic asset pricing theory* and notes

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

1. The Arrow-Debreu model

Assumptions:

- N **traded securities** $s_1, s_2, \ldots, s_N$
- N **prices** $p_1, p_2, \ldots, p_N$
- M **final states** of the market, in which the securities provide cash-flows $(D_{1j}, D_{2j}, \ldots, D_{Nj})$ with $1 \le j \le M$

$\mathbf{p} = (p_1, p_2, \ldots, p_N) =$ price vector

$\mathbf{D} = (D_{ij}) =$ dividend, or cash-flow matrix

A **portfolio** is represented by an N-vector

$$\theta = (\theta_1, \theta_2, \ldots \theta_N)$$

$\theta_i =$ number of units of security s_i held in the portfolio (long or short). Assumed to be a **real number**

Investments & Arbitrage Portfolios

- Initially, investor purchases a portfolio $(\theta_1, \theta_2, \ldots \theta_N)$ @ price

$$\theta \cdot \mathbf{p} \ = \ \sum_{i=1}^{N} \theta_i \, p_i$$

- Subsequently, one of the M states ocurrs and the investor claims/pays the cashflow

$$\theta \cdot \mathbf{D}_{\cdot\, j} \ = \sum_{i=1}^{N} \theta_i \, D_{i\, j}$$

if state j ocurred

Definition: $\theta \ = \ (\theta_1, \theta_2, \ldots \theta_N)$ is an **arbitrage portfolio** iff

(i) $\quad \theta \cdot \mathbf{p} \ < \ 0 \qquad$ and

(ii) $\quad \theta \cdot \mathbf{D}_{\cdot\, j} \ \geq \ 0 \, , \qquad \forall \quad 0 \ \leq \ j \ \leq \ M$

or

(i) $\quad \theta \cdot \mathbf{p} \ < \ 0$

(ii) $\quad \theta \cdot \mathbf{D}_{\cdot\, j} \ \geq \ 0 \, , \qquad \forall \quad 0 \ \leq \ j \ \leq \ M \qquad$ and

(iii) $\quad \exists \ 0 \ \leq \ j \ \leq \ M \quad$ such that $\quad \theta \cdot \mathbf{D}_{\cdot\, j} \ > \ 0$

No-arbitrage Theorem

Theorem 1: A necessary and sufficient condition for the **non-existence** of arbitrage portfolios is that there exists a vector

$$\pi \; = \; (\,\pi_1\,,\pi_2\,,...,\pi_M\,) \quad , \qquad \pi_j \; > \; 0 \quad \forall\, j$$

such that

$$\mathbf{p} \; = \; \mathbf{D} \cdot \pi$$

or, equivalently,

$$p_i \; = \; \sum_{j=1}^{M} D_{i\,j}\, \pi_j \qquad \forall \;\; 0 \; \leq \; i \; \leq \; N \; .$$

Remarks:

- π_j s are called **state-prices**
- Set $$\hat{\pi}_j \; = \; \pi_j \; / \left(\sum_{j'=1}^{M} \pi_{j'} \right)$$
- $\hat{\pi}_j$ s are **probabilities** : $\sum_j \; \hat{\pi}_j \; = \; 1 \quad , \; \hat{\pi}_j \; > \; 0$
- Price of a riskless bond with face value $\$F \; = \; B_F \quad \Longrightarrow$

$$B_F \; = \; F \cdot \sum_{j=1}^{M} \pi_j$$

Interpretation of Theorem 1

- Assume riskless lending/borrowing exists @ (simply compounded) interest rate R. Then,

$$B_F = \frac{F}{1+R} \qquad \Longrightarrow \qquad \sum_{j=1}^{M} \pi_j = \frac{1}{1+R}$$

From Theorem 1,

$$\begin{aligned} p_i &= \sum_{j=1}^{M} D_{i\,j}\, \pi_j \\ &= \left(\sum_{j=1}^{M} \pi_j \right) \cdot \sum_{j=1}^{M} D_{i\,j}\, \hat{\pi}_j \\ &= \frac{1}{1+R} \cdot \sum_{j=1}^{M} D_{i\,j}\, \hat{\pi}_j \\ &= \frac{1}{1+R} \cdot \mathbf{E}^{\hat{\pi}} \left[D_{i\,\cdot} \right] \quad , \quad 1 < i < N \; . \end{aligned}$$

- No-arbitrage $\Longrightarrow$ there exists probabiliity measure $\hat{\pi}$ on the set of **market outcomes** such that every security satisfies

$$\text{Price} = \frac{1}{1+R} \cdot \mathbf{E}^{\hat{\pi}} \left\{ \text{Cash-flows} \right\}$$

Subjective vs. risk-adjusted probabilities

No-arbitrage theorem assumes nothing about the **subjective probabilities** (frequencies) of states $1, \dots M$ (other than $\neq 0$)

Assume that investors believe that Prob.{ state j ocurrs } $= f_j$

<u>Typically, $f_j \neq \hat{\pi}_j$!!</u>

- $\frac{1}{p_i}\left[\mathrm{E}^f\,(D_{i\,\cdot}) - p_i\right] =$ <u>expected return for s_i</u>
- $\frac{1}{p_i}\left[\mathrm{E}^{\hat{\pi}}\,(D_{i\,\cdot}] - p_i\right] =$ <u>bond return</u> (R)

Historically, securities have different expected returns, which are also different from the bond return.

- In terms of **subjective probabilities** (f), we have

$$p_i = \frac{1}{1+R} \cdot \sum_{j=1}^{M} D_{i\,j}\left(\frac{\hat{\pi}_j}{f_j}\right) \cdot f_j$$

$$= \frac{1}{1+R} \cdot \mathbf{E}^f\left[D_{i\,\cdot}\left(\frac{\hat{\pi}_\cdot}{f_\cdot}\right)\right]$$

- $\frac{\hat{\pi}_j}{f_j}$ = **state-price "deflators"**

State-prices & "elementary claims"

- Add to the security space M "elementary securities" s_{N+1} ... s_{N+M} such that

$$D_{N+k\,,j} \;=\; \begin{cases} 1 & \text{if } \; j = k \\ \\ 0 & \text{if } \; j \neq k \end{cases}$$

If there are no arbitrage portfolios with the "enlarged" securities market

$$\begin{aligned} p_{N+k} \;&=\; \text{price of the}\, k^{th} \,\text{elem. security} \\ &=\; \sum_{j=1}^{M} D_{N+k\,,j}\,\pi_j \\ &=\; \pi_k \qquad , \qquad\qquad 1 \;\leq\; k \;\leq\; M \end{aligned}$$

- State price π_j $=$ price of a hypothetical derivative security that pays \$ 1 in state j and \$ 0 otherwise.

<u>Value assigned by the market to \$ 1 in state j</u>

2. Proof of Theorem 1

A. Existence of state-prices implies no-arbitrage

Suppose that θ is an arbitrage portfolio. Then

$$\theta \cdot D._j \geq 0 \qquad \forall j$$

$$\Longrightarrow \quad \theta \cdot \mathbf{p} = \theta \cdot (\mathbf{D} \cdot \pi) = (\theta \cdot \mathbf{D}) \cdot \pi \geq 0 \qquad (\pi_j > 0)$$

and

$\theta \cdot \mathbf{p}$ is stricitly positive if some $\theta \cdot D._j$ is positive. #

B. No-arbitrage implies existence of state prices

Will show: if no set of state prices exists, then there is an arbitrage portfolio

• Consider the **"distance"** function:

$$D(x_1, x_2, ...x_M) = \left| \mathbf{p} - \textstyle\sum_j x_j \mathbf{D}._j \right| \qquad 0 < x_i < +\infty$$

• **Claim**: this function satisfies

1. $Inf_{x_j > 0} \, D(x_1, x_2, ...x_M) = Min_{x_j \geq 0} \, D(x_1, x_2, ...x_M) = 0$

2. Minimum is attained at some $(x_1^*, x_2^*, ...x_M^*)$ with $x_j^* > 0 \quad \forall j$ (a state price vector)

<u>**Proof of 1.**</u> Suppose $Inf_{x_j > 0}\ D(x_1, x_2, ...x_M) \ = \ D_0 \ > \ 0$

$$\Longrightarrow \quad Inf_{x_j > 0} \left| \mathbf{p} - \sum_j x_j \mathbf{D}_{\cdot j} \right|^2 = D_0^2 > 0$$

- First-order conditions: $\partial D^2 / \partial x_j^2 \ \geq \ 0 \quad \big(> \ 0 \ \text{ iff } \ x_j^* \ = \ 0 \big)$

$$-2 \left(\mathbf{p} - \sum_j x_j^* \mathbf{D}_{\cdot j} \right) \cdot \mathbf{D}_{\cdot j} \ \geq \ 0$$

- Multiplying by x_j^* and adding,

$$(\mathbf{p} - \mathbf{p}^*) \cdot \mathbf{p}^* = 0 \quad , \qquad \mathbf{p}^* = \sum_j x_j^* \mathbf{D}_{\cdot j}$$

- Choose $\theta \ = \ -(\mathbf{p} - \mathbf{p}^*) \quad \Longrightarrow \quad \theta \cdot \mathbf{D}_{\cdot j} \ \geq \ 0 \quad \forall j$

& $\quad \theta \cdot \mathbf{p} \ = \ -(\mathbf{p} - \mathbf{p}^*) \cdot \mathbf{p} \ = \ -|\mathbf{p} - \mathbf{p}^*|^2 \ = \ D_0^2 \ < \ 0$

Hence θ is an arbitrage portfolio. By contradiction, there exist **non-negative** x_j^* s such that $\mathbf{p} \ = \ \sum_j x_j^* \mathbf{D}_{\cdot j}$.

<u>Proof of 2.</u> Suppose that $\mathbf{p} \; = \sum_j \, x_j^* \, \mathbf{D}_{\cdot\, j}$, but

no state price vector exists.

$$\Longrightarrow \qquad \exists k \;\; \text{such that} \;\; \forall \epsilon \; > \; 0$$

$$\underset{x_1,\ldots,x_M \,\text{pos.}}{Inf} \; \left| \mathbf{p} - \epsilon \, \mathbf{D}_{\cdot\, k} \, - \sum_j \, x_j \, \mathbf{D}_{\cdot\, j} \right| \; > \; 0 \; .$$

This is due to the **convexity** of state price vectors: a convex combination of SPV s is an SPV

• Define $\quad \mathbf{p}_\epsilon \;=\; \mathbf{p} \, - \epsilon \, \mathbf{D}_{\cdot\, k}$

Repeating the previous argument, we find that

$$\exists \qquad \mathbf{p}_\epsilon^* \; = \; \sum_j \, \tilde{x}_j^* \, \mathbf{D}_{\cdot\, j} \qquad \text{such that}$$

• $\quad - \, (\, \mathbf{p}_\epsilon \, - \, \mathbf{p}_\epsilon^* \,) \cdot \mathbf{D}_{\cdot\, j} \;\geq\; 0$

• $\quad - \, (\, \mathbf{p}_\epsilon \, - \, \mathbf{p}_\epsilon^* \,) \cdot \mathbf{p}_\epsilon \;\; = \;\; - \, |\mathbf{p}_\epsilon \, - \, \mathbf{p}_\epsilon^* \, |^2 \; < \; 0$

Consider the portfolio $\theta \;\; = \;\; - \, (\, \mathbf{p}_\epsilon \, - \, \mathbf{p}_\epsilon^* \,)$,

$$\begin{aligned} \theta \cdot \mathbf{p} \;\; &= \;\; - \, (\, \mathbf{p}_\epsilon \, - \, \mathbf{p}_\epsilon^* \,) \cdot \mathbf{p}, \\ &= \;\; - \, (\, \mathbf{p}_\epsilon \, - \, \mathbf{p}_\epsilon^* \,) \cdot (\mathbf{p}_\epsilon \; + \; \epsilon \, \mathbf{D}_{\cdot\, k} \,) \\ &= \;\; - \, (\, \mathbf{p}_\epsilon \, - \, \mathbf{p}_\epsilon^* \,) \cdot \mathbf{p}_\epsilon \; - \; (\, \mathbf{p}_\epsilon \, - \, \mathbf{p}_\epsilon^* \,) \cdot \epsilon \, \mathbf{D}_{\cdot k} \end{aligned}$$

Continued

$$\theta \cdot \mathbf{p} \;=\; -\left|\mathbf{p}_\epsilon \;-\; \mathbf{p}^*_\epsilon\right|^2 \;-\; (\,\mathbf{p}_\epsilon \;-\; \mathbf{p}^*_\epsilon\,) \cdot \epsilon\,\mathbf{D}_{\cdot k}$$

Case 1: $\tilde{x}^*_k \;>\; 0$. Then,

$$(\,\mathbf{p}_\epsilon \;-\; \mathbf{p}^*_\epsilon\,) \cdot \mathbf{D}_{\cdot\, k} \;=\; 0$$

$$\Longrightarrow \qquad \theta \cdot \mathbf{p} \;<\; 0$$

Case 2: $\tilde{x}^*_k \;=\; 0$. Then,

$$\mathbf{p}_\epsilon \;-\; \mathbf{p}^*_\epsilon \;=\; -\,\epsilon\,\mathbf{D}_{\cdot k}$$

$$\Longrightarrow \qquad \theta \cdot \mathbf{p} \;=\; 0$$

but also

$$\theta \cdot \mathbf{D}_{\cdot\, k} \;=\; -\,(\,\mathbf{p}_\epsilon \;-\; \mathbf{p}^*_\epsilon\,) \cdot \mathbf{D}_{\cdot\, k} \;>\; 0$$

Thus, unless $x^*_j \;>\; 0 \quad \forall j$ there exists an arbitrage portfolio.

3. Example: One-period binomial model

Two securities, a stock and a bond ($N = 2$), two states ($M = 2$).

Stock value satisfies

- **Risk-neutral probabilities**: π_D , π_U .

$$S = \frac{1}{1+R} [\, \pi_U \cdot S\,U + \pi_D \cdot S\,D \,]$$

$$\begin{cases} \pi_U + \pi_D = 1 \\ \pi_U \cdot U + \pi_D \cdot D = 1 + R \end{cases}$$

Solving this 2×2 linear system of equations,

$$\pi_U = \frac{(1+R) - D}{U - D} \qquad \pi_D = \frac{U - (1+R)}{U - D}$$

No-arbitrage condition: $D < 1 + R < U$

If additional securities are introduced, their market values are completely determined by the risk-neutral probabilities

Example: "Contingent claim" has cash-flows V_U and V_D in states U , D. The no-arbitrage value is

$$V \;=\; \frac{1}{1+R}\left[\,\pi_U \cdot V_U + \;\pi_D \cdot V_D\,\right]$$

Replicating portfolio

Problem: Determine a portfolio of stocks & bonds that has cash-flows V_U , V_D in each state.

Solution: $\Delta \;=\; \#$ of shares, $B \;=\;$ dollars in money-market account

$$\begin{cases} \Delta \cdot S\,U + \; B\,(1+R) \;=\; V_U \\ \Delta \cdot S\,D + \; B\,(1+R) \;=\; V_D \end{cases}$$

$$\Delta \;=\; \frac{V_U - V_D}{S\,(U-D)} \qquad\qquad B \;=\; \frac{1}{1+R}\,\frac{U\,V_D \;-\; D\,V_U}{U-D}$$

Arbitrage strategies

- If $$V \;<\; \frac{1}{1+R}\left[\,\pi_U \cdot V_U + \;\pi_D \cdot V_D\,\right]$$

Buy contingent claim, sell Δ shares $\implies$ riskless profit.

- If $$V \;>\; \frac{1}{1+R}\left[\,\pi_U \cdot V_U + \;\pi_D \cdot V_D\,\right]$$

Sell contingent claim, buy Δ shares for a riskless profit.

Market completeness

Definition: An Arrow-Debreu model (market) is said to be **complete** iff there exists a **unique** state-price vector π.

Proposition 1: Market completeness holds iff $M = N$ and the cash-flow matrix $\mathbf{D}$ is **invertible** and there is no arbitrage.

Proposition 2: A market is complete iff any additional contingent claim can be replicated using the existing N securities.

Proof: If there are two (different) state price vectors π_1 , π_2,

- $\mathbf{D} \cdot (\pi_1 - \pi_2) = 0 \implies \mathbf{D}$ not invertible
- $\mathbf{D}$ not invertible $\implies \exists \mathbf{y} \quad \mathbf{D} \cdot y = 0$

$\pi_\epsilon = \pi + \epsilon \, \mathbf{y} =$ state price vector if ϵ is small

<u>As a rule, financial markets are incomplete.</u>

MULTIPERIOD ARBITRAGE PRICING THEORY

Assumptions:

- Trading dates: t_0 (today), t_1 , t_2 , ... , t_n , ... t_N

- Market can be in one of several **states** at any particular date.

- A state represents information about prices, forecasts, market "mood" , etc., as well as past information.

- Collection of states at time t_n : Σ_n (finite set)

- State of the market at time n : X_n

- Prices of securities and dividends paid to holders at time t_n are functions of X_n

$S(t_n) \ = \ S_n \ = \ S_n(\,X_n)$ represents a security price

$D(t_n) \ = \ D_n \ = \ D_n(\,X_n)$ " dividend
or cash-flow payable to the holder at time t_n

Short-term lending

There exists **short-term lending** from one date to the next:

- Interest rate at time $t_0 \; = \; R_0$ (known today)
- Interest rate at time $t_n \; = \; R_n \; = \; R_n(X_n)$ (known at date t_n)

Risk-neutral probability measures

Suppose that at date t_n the market is in state $X_n \in \Sigma_n$

Holding security with price S_n for the single period (t_n, t_{n+1}) and collecting the dividends + selling it at time t_{n+1} produces the cashflow (not counting cost of purchase)

$$S_{n+1} + D_{n+1} = S_{n+1}(X_{n+1}) + D_{n+1}(X_{n+1})$$

One period APT $\implies \exists \; \pi(X_{n+1}, X_n) \quad X_{n+1} \in \Sigma_{n+1}$,

$$\pi(X_{n+1}, X_n) > 0 \quad \& \quad \sum_{X_{n+1}} \pi(X_{n+1}, X_n) = 1$$

such that

$$S_n(X_n) = \frac{1}{1 + R_n(X_n)} \times$$

$$\sum_{X_{n+1}} [S_{n+1}(X_{n+1}) + D_{n+1}(X_{n+1})] \cdot \pi(X_{n+1}, X_n)$$

$$S_n = \frac{1}{1+R_n} \mathbf{E}^{\pi(\bullet, X_n)} \{ S_{n+1} + D_{n+1} \}$$

Probabilities on the set of state paths

- $\pi(X_{n+1}, X_n)$ can be interpreted as a **risk-adjusted** probability for the ocurrence of state X_{n+1} conditional on X_n

- For any $n > 0$, define

$$P_0(X_0\,; X_1, \ldots X_N) \;=$$

$$\pi(X_1, X_0)\cdot\pi(X_2, X_1)\;\cdots\;\pi(X_N, X_{N-1})$$

Proposition : $P_0(X_0\,;\cdot, \ldots \cdot) \;=$ is a probability on the **set of state paths** $(X_1\,,\, X_2\,,\, \ldots,\, X_N)$, for $X_i \;\in\; \Sigma_i \quad, 1 \;\leq\; i \;\leq N$

In fact,

$$\sum_{(X_1\,,\,X_2\,,\,\ldots\,,\,X_N)} P_0(X_0\,; X_1, \ldots X_N) \;=\; 1$$

$$P_0(X_0\,; X_1, \ldots X_N) \;>\; 0$$

Multiperiod risk-neutral valuation

Set $\quad B_{0,n} \; = \; \frac{1}{1+R_0} \cdot \frac{1}{1+R_1} \cdot \; \cdot \; \cdot \frac{1}{1+R_{n-1}}.$

<u>Discount factors</u> for lending/borrowing from time t_0 to time t_n.

Fundamental Theorem of Dynamic Asset Pricing:

(i) If there are no arbitrage opportunities, there exists a probability measure P_0 on the set of state-paths $\{ \, (X_1 \, , \, X_2 \, , \, ... \, , \, X_N \,) \, \}$ such that every traded security satisfies

$$S_0 \quad = \quad \mathbf{E}^{P_0} \left\{ \sum_{i=1}^{n} B_{0,i} \, D_i \; + \; B_{0,n} \, S_n \right\} \qquad (1 \; \leq \; n \; \leq \; N)$$

$$\text{Today's price} \; = \begin{cases} \textbf{expected discounted dividends} \text{ up to time } t_n \\ \qquad\qquad\qquad + \\ \textbf{expected discounted market price} \text{ at time } t_n \end{cases}$$

Martingale Property of prices under P_0

(ii) Set

$$B_{m,n} \;=\; B_{0,n} \,/\, B_{0,m} \;=\; \tfrac{1}{1+R_m} \cdot \tfrac{1}{1+R_{m+1}} \cdot\cdot\cdot \tfrac{1}{1+R_{n-1}}.$$

The price of any security at time t_m satisfies

$$S_m \;=\; \mathbf{E}^{P_0}\{ \sum_{i=m+1}^{n} B_{m,i}\, D_i \;+\; B_{m,n}\, S_n \;\big|\; X_m\}$$

$$(1 \;\le\; m \;\le\; n \;\le\; N)$$

• $\mathbf{E}^{P_0}\{ \bullet \;\big|\; X_m\}$ = conditional probability, given info. up to time t_m. In particular, if a security pays no dividends, then

$$B_{0,m}\, S_m \;=\; \mathbf{E}^{P_0}\{ B_{0,n}\, S_n \;\big|\; X_m\}$$

• Prices of securities that do not pay dividends, deflated by the cost of money, are **martingales** under P_0

$B_{0,m}\, S_m$ = price deflated by return on money-market account