INTEREST-RATE DERIVATIVES

Goal of the lecture

- Brief discussion of debt securities.
- Arbitrage-free pricing of interest-rate derivatives.
- Yield curve modeling.
- Binomial model for interest-rate evolution.
- Applications: valuation of optional intruments

Options on interest rates (caps & floors)

Options on bonds

Callable bonds

1. Zero-coupon bonds & Yield

Definition: A **zero-coupon bond** is a debt instrument wherein interest and principal are paid at the end of the corresponding lending period.

lend (borrow) \$ Z_0 $\longrightarrow \longrightarrow \longrightarrow \longrightarrow$ receive (pay) \$ Z_T

Yield: Implied continuously compounded interest rate

$$Z_0 \;=\; e^{-\,y_T \cdot\, T} \,.\; Z_T$$

$$y_T \;=\; \frac{1}{T} \ln \left(\frac{Z_T}{Z_0} \right)$$

Examples: 90-day to 2-year Treasury bills

Example

Time to maturity	Price (p/ $100)	Yield (%)
0.25	97.5	10.13
0.50	94.9	10.47
1.00	90.0	10.54
2.00	81.06	10.50

The right-hand column is called a **zero-coupon yield curve**.

2. Implied forward rates

Definition: The implied forward rates $f(t_1, t_2)$ are theoretical interest rates for future debt that are consistent (in the sense of no-arbitrage) with the current yield curve.

$$e^{-y_{t_2}\, t_2} \;=\; e^{-y_{t_1}\, t_1} \;\cdot\; e^{-f(t_1,\, t_2)\cdot\,(t_2\, -\, t_1)}$$

$$y_{t_2}\, t_2 \;=\; y_{t_1}\, t_1 \;+\; f(t_1,\, t_2)\cdot\,(t_2 \;-\; t_1)$$

$$f(t_1,\, t_2) \;=\; \frac{y_{t_2}\, t_2 \;-\; y_{t_1}\, t_1}{t_2 \;-\; t_1}\;.$$

Time to maturity	Yield (%)	Forward rate (%)
0.25	10.13	10.13
0.50	10.47	10.81
1.00	10.54	10.61
2.00	10.50	10.53

3. Coupon-bearing bonds

A **coupon-bearing bond** or, simply, a **bond** is a debt instrument such that the interest on the principal amount is paid in installments, or **coupons**.

Example: U.S. Govt bonds: coupons are paid semi-annually. Notional is typically $ 100,000.00

Coupon	Maturity	Price	Yield
06.875	02/97	100.04-05	6.790
07.250	02/98	100.27-29	6.904
07.125	02/00	100.07-09	7.057
07.500	05/02	101.28-00	7.137
07.500	02/05	101.27-31	7.219
07.625	02/25	102.01-05	7.444

(Prices quoted in 1/32 of a percentage point.)

Bond Yield

Definition: The **yield** of a bond is the theoretical interest rate (compounded semi-annually) that makes the bond price equal to the discounted value of its cash-flows (coupons & principal).

B = price, P = principal, C = coupon rate, Y = yield
N = number of 6 mo. periods

$$B = \sum_{j=1}^{N} \frac{P \cdot C/2}{(1 + Y/2)^j} + \frac{P}{(1 + Y/2)^N}$$

(For bonds with less than 6 months to next coupon, modify formula.)

- If bond sells **at par**, i.e. $B = P$, then $Y = C$.
- If $B > P$ ($B < P$) then $Y < C$ ($Y > C$).

4. Floating-rate notes & bonds

Cash-flow structure is like that of a bond, but coupon rate is indexed to a variable rate, e.g. LIBOR, or 90-day T-bills

Example: 10-year bond, principal $ 100,000.00, with semi-annual coupon payments @ rate LIBOR + 4.00 %

A yield can be computed from the **swap rate** for a 10-yr. LIBOR swap.

ARBITRAGE-FREE PRICING OF DEBT SECURITIES & DERIVATIVES

Assume a basic lending/borrowing rate (e.g. LIBOR or T-bill) which can vary with time. For simplicity, the rate is continuously compounded.

$$r_t \;=\; \text{interest rate for the period } (t,\, t+dt)$$

Fair price of a zero-coupon bond maturing in T years:

$$Z \;=\; \mathbf{E}\left\{ e^{-\int_0^T r_t\, dt} \right\} \cdot P \qquad (*)$$

$\mathbf{E}$ = expectation operator over a probability defined over the **forward paths** of r_t.

Proof of (*): From arbitrage pricing theorem (lect §2)

$$\text{fair value today} \;=\; e^{-r_0\, dt}\, \mathbf{E}\{\text{value at time } dt\}$$

Iteration of this formula leads to (*).

5. Valuation of fixed-income securities using "path-integration"

A. Bond

$$B \;=\; \sum_{j=1}^{N} \frac{P\,C}{2}\,\mathbf{E}\left\{e^{-\int_0^{j\times 0.5}\, r_t\, dt}\right\} \;+\; P\,\mathbf{E}\left\{e^{-\int_0^{N\times 0.5}\, r_t\, dt}\right\}$$

More generally, if coupon payments made every Δt years

$$B \;=\; \sum_{j=1}^{N} P\,C\,\Delta t\,\mathbf{E}\left\{e^{-\int_0^{j\cdot\Delta t}\, r_t\, dt}\right\} \;+\; P\,\mathbf{E}\left\{e^{-\int_0^{N\cdot\Delta t}\, r_t\, dt}\right\}$$

Note: Cash flows can be decomposed as a "strip" of zeros

B. Floating rate bonds

Δt = duration between coupons

N = number of periods

S = **spread** over LIBOR

$\overline{r}_{j\,\Delta t}$ = simply compounded rate

for period $(j\,\Delta t\ ,\ (j+1)\,\Delta t)$

$$\overline{r}_{j\,\Delta t}\cdot\,\Delta t\ =\ e^{\int_{j\cdot\,\Delta t}^{(j+1)\cdot\,\Delta t}\ r_t\ dt}\ -1$$

$$B\ =\ \sum_{j=1}^{N} P\ \mathbf{E}\left\{(S\ +\ \overline{r}_{(j-1)\cdot\,\Delta t})\,\Delta t\ e^{-\int_0^{j\cdot\,\Delta t}\ r_t\ dt}\right\}\ +$$

$$P\,\mathbf{E}\left\{e^{-\int_0^{N\cdot\,\Delta t}\ r_t\ dt}\right\}$$

B = straight bond with coupon rate S +

$$\sum_{j=1}^{N} P\ \mathbf{E}\left\{\overline{r}_{(j-1)\cdot\,\Delta t}\,\Delta t\cdot e^{-\int_0^{j\cdot\,\Delta t}\ r_t\ dt}\right\}$$

Interest payments are made **in arrears**: $\overline{r}_{(j-1)\Delta t}$

and not $\overline{r}_{j\Delta t}$

C. Forward Rate Agreements

Contract to borrow money at a fixed rate for a specified period in the future.

R = fixed rate

t_1 = starting date

t_2 = maturity

Cashflows : Pay premium today , receive PV of notional at time t_1, (at fixed rate), return notional at time t_2.

$$FRA = P\, e^{-R\,(t_2-t_1)}\, \mathbf{E}\{e^{-\int_0^{t_1} r_t\, dt}\} - P\, \mathbf{E}\{e^{-\int_0^{t_2} r_t\, dt}\}$$

$$= P\, e^{-R\,(t_2-t_1)}\, Z(0,t_1) - P\, Z(0,t_2)$$

$$FRA = 0 \text{ (\textbf{at market})} \iff R = f(t_1, t_2)$$

FRAs and forwards with simply compounded rates

$$FRA \;=\; P \;\cdot\; \mathbf{E}\left\{ (\overline{R} - \overline{r}_{t_1})\Delta t \cdot\; e^{-\int_0^{t_1} r_t\, dt} \right\}$$

$\Delta t \;=\; t_2 \;-\; t_1$

$\overline{R} \;=\;$ simply compounded fixed rate

Simply compounded forward rate satisfies:

$$\overline{f}(t_1,\, t_2) \;=\; \frac{\mathbf{E}\left\{ \overline{r}_{t_1} \cdot\; e^{-\int_0^{t_1} r_t\, dt} \right\}}{\mathbf{E}\left\{ e^{-\int_0^{t_1} r_t\, dt} \right\}}$$

Forward rate = **weighted average** of future spot rate

Swap = Strip of FRAs

Floating rate bond= Straight bond + strip of FRAs

D. Interest-rate futures

T = settlement date of the contract

Futures price = $100 \cdot (1 - \overline{r}_{f,t,T}) = \Phi_{t,T}$

$\overline{r}_{f,t,T}$ = implied interest rate (time t)

$\overline{r}_T$ = settlement interest rate (eg. 6-month LIBOR)

$$\overline{r}_{f,0,T} = \mathbf{E}\{\overline{r}_T\} \qquad (*)$$

Proof: We have seen that the futures price is a **martingale**, i.e.

$$\Phi_{t,T} = \mathbf{E}\{\Phi_{t+dt,\,T} \,|\, r_s,\ s \leq t\}$$

Since future settles at $100 \cdot (1 - \overline{r}_T)$,

$$\Phi_{0,T} = \mathbf{E}\{100 \cdot (1 - \overline{r}_T)\}$$

This implies (*).

• Implied futures rate $\neq$ forward rate, in general.

E. Caps and floors

"Caplet": Call option on future spot rate

"Floorlet": Put option on future spot rate

Payoffs: $P \cdot Max\left[(\overline{r}_T - \overline{R}) \cdot \Delta t\ ,\ 0\right]$

$P \cdot Max\left[(\overline{R} - \overline{r}_T) \cdot \Delta t\ ,\ 0\right]$

P = notional amount

T = maturity date

$\overline{R}$ = strike rate

Δt = period of validity of the rate

$\overline{r}_T$ = underlying simply compounded rate

$$CAPLET = P \cdot \mathbf{E}\left\{ Max\left[(\overline{r}_T - \overline{R}) \cdot \Delta t\ ,\ 0\right] \cdot e^{-\int_0^{T+\Delta t} r_t\, dt} \right\}$$

$$FLOORLET = P \cdot \mathbf{E}\left\{ Max\left[(\overline{R} - \overline{r}_T) \cdot \Delta t\ ,\ 0\right] \cdot e^{-\int_0^{T+\Delta t} r_t\, dt} \right\}$$

Cap: A series of caplets

Floor: A series of floorlets

Parity relations:

$$CAPLET \; + \; FLOORLET \; = \; FRA$$

$$CAP \; + \; FLOOR \; = \; SWAP$$

6. BINOMIAL PRICING MODELS

Goals:

- Model **arbitrage-free** evolution of r_t (or $\overline{r}_t$ for some Δt)
- Compute the value of instruments by backward induction ("roll-back")

Arbitrage-free pricing $\Longleftrightarrow$ adjusting the model so that the prices of liquid instruments (bonds, swaps, FRAs and even caps and floors) are reproduced

Yield-curve modeling: Constructing a model that matches a **zero-coupon yield curve**

- Black-Derman-Toy model: spot rates follow random walk
- Black-Karasinski: log of spot rates follow r.w.

7. Black-Derman-Toy Model

N periods (of duration dt each, say)

R_n = interest rate (simply compounded) for period $n\,dt\,,\,(N+1)dt$

D_n = discount factor

$$D_n = \frac{1}{1+R_n}\,, \qquad n = 0, 1, ..., N-1\,.$$

Price of a zero maturing in n periods:

$$B_n = \mathbf{E}\left\{\Pi_{j=1}^{n} D_j\right\}$$

- D_n varies on a binomial tree.
- Value at node (n, j) = D_n^j