On Parabolic Equations with Gauge function term and Applications to the multidimensional Leland Equation

April 7, 2003

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(first version March 11, 2002)

Abstract

We obtain sufficient conditions for existence and a closed form probabilistic representation for solutions of nonlinear parabolic equations with gauge function term. In particular, our result applies to the generalised Leland equation

$$BS_n + 1/2 \sum_{i=1}^{n} A_i \sqrt{\sum_{j,k}^{n} \rho_{jk} D_{ij}^S V D_{ik}^S V} = 0,$$

where BS_n is the n-dimensional Black-Scholes operator, A_i are positive transaction cost numbers, ρ_{jk} are the correlations between returns of asset S_j and asset S_k and $D_{rk}^S V$ is an abbreviation of $\sigma_r \sigma_k S_r S_k \frac{\partial^2 V}{\partial S_r \partial S_k}$ along with the volatilities σ_r of the rth asset S_r. We show that the associated Cauchy problem has a solution for uniformily bounded continuous data if for all i, j, $i \neq j$ $0 \leq A_i < 1$ and

$$|\rho_{ij}| \leq -\frac{(A_i + A_j)}{2} + \sqrt{(1 - A_i)(1 - A_j)}.$$

We also comment on existence as $A_i \to 1$ for some i for small and large correlations between returns of assets.

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1 Introduction

Consider operators of the form

$$L_p u \equiv \frac{\partial u}{\partial t} + \sum_{i,j=1}^{n} a_{i,j}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} - c(x,t)u \quad (1)$$

with smooth bounded coefficient functions a_{ij}, b_i and $c \geq 0$ and where $A = (a_{i,j}) > 0$, i.e. A is a positive matrix. It is well known that for bounded continuous data the solution u of the final value problem

$$\begin{cases} L_p u = 0 & \mathcal{R}^n \times (0,T) \\ u(x,T) = f(x) & \text{in } \mathcal{R}^n \end{cases} \quad (2)$$

($\mathcal{R}$ the real numbers) has a probabilistic representation

$$u(x,t) = E(e^{-\int_t^T c(X_s,s)ds} f(X_T^{t,x})), \quad (3)$$

if $X_s^{t,x}$ is a stochastic Itô process starting at x at time t which solves

$$dX_t^i = b_i(X_t,t)dt + \sum_j^p \sigma_{ij}(X_t,t)dW_t^j, \quad (4)$$

and

$$a_{ij}(x,t) = \sum_l^p \sigma_{il}(x,t)\sigma_{jl}(x,t).$$

This expectation value formula and derivatives are used in finance to compute option values and sensitivities, especially in higher dimension. However, the Leland approach of pricing options in the presence of transaction costs leads to the final value problem

$$\begin{cases} \frac{\partial V}{\partial t} + 1/2\sum_{i,j=1}^{n} \rho_{ij} D_{ij}^S V + r(\sum_{i=1}^{n} S_i \frac{\partial V}{\partial S_i} - V) \\ +1/2\sum_{i=1}^{n} A_i \sqrt{\sum_{j,k=1}^{n} \rho_{jk} D_{ij}^S V D_{ik}^S V} = 0 \text{ in } \mathcal{R}_+^n \times (0,T) \\ V(S_1,...,S_n,T) = f(S_1,...,S_n) \text{ in } \mathcal{R}_+^n, \end{cases} \quad (5)$$

(where $\mathcal{R}_+$ are the positive real numbers and $\sigma_i, \rho_{ij}, r, A_i$ are constants and we used the abbreviation

$$D_{rk}^S V = \sigma_r \sigma_k S_r S_k \frac{\partial^2 V}{\partial S_r \partial S_k}, \; 1 \leq k, r \leq n.$$

Here, the question of existence (well-posedness) and probabilistic representations of solutions of nonlinear parabolic equations with gauge function term arises.

Equation (5) is a multi-dimensional form of the so-called Leland problem with convex payoffs (final value data) to model transaction costs for options written on one asset. Generalizations to payoffs with mixed convexity were proposed by Whalley and Wilmott (1993) and by Avellaneda and Paras (1994). Avellaneda and Paras introduced the Leland number A and noticed that the equation becomes ill-posed if $A > 1$. Pacelli, Recchioni and Zirilli (1999) modelled the multidimensional case and treated numerical aspects. In this paper we deal with the problem of existence and probabilistic solution representation. There are two contributions. The first one concerns existence and probabilistic representation of solutions for a class of nonlinear parabolic equations with gauge function term motivated by the multidimensional Leland equation. This type of equation can be connected to Hamilton-Jacobi-Bellmann equations for which probabilistic representations of solutions are studied by many authors, especially J.P. Lions (1983,1988). He studied probabilistic representations for equations of type

$$\frac{\partial u}{\partial t} + \sup_{\alpha \in A}(a^{ij}_\alpha \frac{\partial^2 u}{\partial x_i \partial x_j} + b^i_\alpha \frac{\partial u}{\partial x_i} + c_\alpha u + f_\alpha) = 0,$$

where A is a compact subset of a complete metric space and some regularity assumptions are imposed for the coefficients (we used Einstein summation). For example it is assumed that the entries of the square root of the matrix a^{ij}_α are differentiable with respect to spacial variables such that the derivatives are bounded and Lipshitz. In the case of the multi-dimensional Leland equation (5) the essential part of the differential operator

$$1/2 \sum_{i,j=1}^n \rho_{ij} D^S_{ij} V + (1/2) \sum_{i=1}^n A_i \sqrt{\sum_{j,k}^n \rho_{jk} D^S_{jk} V D^S_{ik} V},$$

can be written pointwise in the form of an Hamilton-Jacobi-Bellman equation

$$(1/2) \sup_{Y \in \mathcal{S}(n)} \sum_{i,k=1}^n (\rho_{ik} + A_i \frac{\sum_{j=1}^n \rho_{jk} y_{ij}}{\sqrt{\sum_{j,k=1}^n \rho_{jk} y_{ij} y_{ik}}}) D^S_{ik} V$$

where we used convex analysis. Y are the entries of the Hessian $(y_{ij}) := (D^S_{ij} U)$ of some scalar twice differentiable function U and we look at the supremum pointwise, i.e. for any point $D^S_{ij} V(S,t)$ we search for supremum Y in the space of symmetric matrices $\mathcal{S}(n)$. From this representation we see that standard results for Hamilton-Jacobi-Bellman equation cannot be applied to obtain existence and probabilistic representation of solutions for two reasons: first, the search space for the sup is a function space and not compact. Second, the second order coefficients of the Hamilton-Jacobi-Bellmann equation of the generalised Leland equation are not even continuous. We solve the problem of the probabilistic solution of parabolic equations with gauge function term in general. Moreover, our proof emphasizes the PDE-side and is new in this respect.

The second contibution concerns the problem of well-posedness of the multidimensional Leland equation. We prove existence of solutions for bounded continuous data if the relations

$$|\rho_{ij}| \leq -\frac{(A_i + A_j)}{2} + \sqrt{(1 - A_i)(1 - A_j)} \tag{6}$$

for all $i \neq j$ and $A_i < 1$ for all i hold. Note that $|\rho_{ij}| \geq 0$ implies that

$$(1 - A_i)(1 - A_j) \geq \frac{1}{4}(A_i + A_j)^2,$$

which means that for all i, j A_i, A_j are inside some ellipse or hyperbola of the quadrant $A_i, A_j \geq 0$.

It turns out that these relations are also necessary for ellipticity in the sense that they cannot be generalised without further restrictions on correlations, Leland numbers or payoffs. We will comment on that at the end of the paper where we outline directions for future research. In the following we assume that f is bounded continuous. Standard techniques can be used then to extend our results to allow for solutions with linear growth condition (cf. [2]).

Concerning our first contribution we ask the following more general question. Let $(a, p, X) \in \mathcal{R} \times \mathcal{R}^n \times \mathcal{S}(n)$ where $\mathcal{S}(n)$ is the space of symmetric $n \times n$-matrices and let

$$N(x, t; a, p, X)$$

be a gauge function with respect to the Hessian, i.e. a positive continuous function such that for each point $(x, t; a, p)$ $H_{(x,t;a,p)}(X) := N(x, t; a, p, X)$ is convex and homogenous of degree one with respect to the Hessian X. Furthermore, let D^2u denote that Hessian of u and Du denote the gradient of u (with respect to spacial variables). Is there a probabilistic representation of the solution of the problem

$$\begin{cases} L_p u + N(x, t; u, Du, D^2u) = 0 & \text{in } \mathcal{R}^n \times (0, T) \\ u(x, T) = f(x) & \text{in } \mathcal{R}^n \text{ ?} \end{cases} \tag{7}$$

If some assumptions on N are satisfied then we answer this question affirmatively showing that

$$u(x,t) = \lim_{\epsilon \searrow 0} \sup_{v \in C^{1,2}} E(e^{-\int_t^T c(X_s,s)ds} f(X_t^{x,v,\epsilon})), \tag{8}$$

where $(X_s^{x,v,\epsilon})_{s\in[0,T]}$ is a process starting at x at time t (defined on a probability space $(\Omega, \mathcal{F}, P)$) and which solves

$$dX_t^{x,v,\epsilon,i} = b_i(X_t,t)dt + \sum_{j=1}^{p} \sigma_{ij}^{v,\epsilon}(X_t,t)dW_t^j, \tag{9}$$

where

$$\Sigma_{ij}^{v,\epsilon}(x,t) = \sum_l \sigma_{il}^{v,\epsilon}(x,t)\sigma_{jl}^{v,\epsilon}(x,t) \tag{10}$$

is from a smoothed linearisation of (7) about the Hessian. (For the definition of the matrix valued function $\Sigma_{ij}^{v,\epsilon}$ and $(\sigma_{il}^{v,\epsilon}(x,t))$ and $(\sigma_{il}^{v,\epsilon}(x,t))$ cf. p.10.) In section 2 of this paper we provide sufficient conditions for (7) to have a solution and derive a closed form probabilistic representation of the solution. In section 3 we show that the generalised Leland equation has a solution if for all $i \neq j$ $A_i < 1$ and

$$|\rho_{ij}| \leq -\frac{(A_i + A_j)}{2} + \sqrt{(1-A_i)(1-A_j)} \tag{11}$$

and provide a probabilistic representation if this condition holds. Note that the latter relation is always violated if $A_i \to 1$ for some i. We comment on that at the end of the paper.

2 Probabilistic solutions for nonlinear parabolic equations with gauge function

We ask for sufficent conditions for existence of solutions in the viscosity sense of the initial value problem (IVP)

$$\begin{cases} G(x,t;u,\frac{\partial u}{\partial t},Du,D^2u) = 0 & \text{in } \mathcal{R}^n \times (0,T) \\ u(x,0) = f(x) & \text{in } \mathcal{R}^n \end{cases} \tag{12}$$

where

$$G(x,t;u,\frac{\partial u}{\partial t},\nabla u, D^2u) := L_p^- u - N(x,t;u,\nabla u,D^2u),$$

and

$$L_p^- u = \frac{\partial u}{\partial t} - \sum_{i,j=1}^n a_{i,j}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=1}^n b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u,$$

and N is a gauge function. Applications to final value problems occurring in finance are immediate by time change $t \to T - t$ in the IVP.

Recall that u is a viscosity solution of (12) if u is a viscosity subsolution and viscosity supersolution. For the operator G of the Cauchy problem associated to (12) this means ($U = \mathcal{R}^n \times (0,T)$)

$$\forall (x,t) \in \mathcal{R}^n \times (0,T)\ \forall (a,p,Z) \in P_U^{2,+}u(x,t)$$
$$G(x,t;u,a,p,Z) \leq 0, \text{ and}$$
$$\forall (x,t) \in \mathcal{R}^n \times (0,T)\ \forall (a,p,Z) \in P_U^{2,-}u(x,t)$$
$$G(x,t;u,a,p,Z) \geq 0$$

where

$$(a,p,Z) \in P_U^{2,+}u(x,t) \text{ iff}$$
$$\begin{aligned} u(y,s) \leq \ & u(x,t) + a(s-t) + \langle p, y-x \rangle \\ + \ & \tfrac{1}{2}\langle Z(y-x),(y-x)\rangle + o(|y-x|^2 + |s-t|) \end{aligned}$$

as $(y,s) \to (x,t)$ and $P_U^{2,-}u(x,t) = -P_U^{2,+}(-u(x,t))$. Here, $\langle p, y-x \rangle$ means the standard scalar product. Proving existence via viscosity solution theory is possible if the operator is proper, i.e. for all $(x,t) \in D$ and $(a,p,Z) \in P_U^{2,+}u(x,t)$

$$\begin{aligned} u \leq v \quad &\Rightarrow\ G(x,t;u,a,p,Z) \leq G(x,t;v,a,p,Z), \\ Y \leq Z \quad &\Rightarrow\ G(x,t;u,a,p,Y) \geq G(x,t;u,a,p,Z). \end{aligned}$$

Hence, the first question to answer is properness of the essential part of G

$$-\sum_{i,j=1}^n a_{i,j}(x,t)z_{ij} - \sum_{i=1}^n b_j(x,t)p_i + c(x,t)u(x,t) - N(x,t,u,a,p,Z).$$

Clearly, we have to assume monotonicity of N with respect to u. For the second property of properness consider the Euler relation ($Z = (z_{ij})$)

$$N(x,t;a,p,Z) = \sum_{ij} \frac{\partial N}{\partial z_{ij}}(x,t;a,p,Z)z_{ij} \tag{13}$$

which holds everywhere where N, being a function homogenous of degree 1, is differentiable. We assume that for all $Z \in \mathcal{S}(n)$, $1 \leq i,j \leq n$ and $z_{ij} \neq 0$ the partial derivative $\frac{\partial N}{\partial z_{ij}}$ exists at each point in state space and is continuous. Note that for each ij $z_{ij} \rightarrow \frac{H_{(x,t;a,p)}}{\partial z_{ij}}$ is homogenous of degree zero and therefore bounded for $z_{ij} \neq 0$; defining $\frac{\partial N}{\partial z_{ij}} z_{ij} = 0$ if $z_{ij} = 0$ (13) holds for all (z_{ij}).

Since $H_{(x,t;a,p)}$ is a convex and homogenous of degree 1 for each $(x,t;u(x,t),a,p) \in \mathcal{R}^n \times (0,T) \times \mathcal{R} \times \mathcal{R}^n$ we can write

$$\begin{array}{l} G(x,t;u(x,t),a,p,Z) = \\ a - \sup_{(y_{ij})} \sum_{i,j=1}^{n} (a_{i,j}(x,t) + \frac{\partial N}{\partial z_{ij}}(x,t;a,p,Y)) z_{ij} \\ - \sum_{i=1}^{n} b_j(x,t) p_i + c(x,t) u(x,t). \end{array} \tag{14}$$

For monotonicity in the Hessian we observe that for $Z = (z_{ij}) \in \mathcal{S}(n)$ and positive definite $W = (w_{ij}) \in \mathcal{S}(n)$

$$\begin{array}{rl} & \sup_{(y_{ij})} \sum_{i,j=1}^{n} (a_{i,j}(x,t) + \frac{\partial N}{\partial z_{ij}}(..,Y))(z_{ij} + w_{ij}) \\ - & \sup_{(y_{ij})} \sum_{i,j=1}^{n} (a_{i,j}(x,t) + \frac{\partial N}{\partial z_{ij}}(..,Y)) z_{ij} \\ \geq & \sum_{i,j=1}^{n} (a_{i,j}(x,t) + \frac{\partial N}{\partial z_{ij}}(..,Z)(z_{ij} + w_{ij}) \\ - & \sum_{i,j=1}^{n} (a_{i,j}(x,t) + \frac{\partial N}{\partial z_{ij}}(..,Z)) z_{ij} \\ = & \sum_{i,j=1}^{n} (a_{i,j}(x,t) + \frac{\partial N}{\partial z_{ij}}(..,Z)) w_{ij}. \end{array}$$

Hence monotonicity in the Hessian holds if

$$(a_{i,j}(x,t) + \frac{\partial N}{\partial z_{ij}}(..,Z)) \geq 0.$$

We have

Theorem 1 *Assume that there exists $\gamma > 0$ such that*

$$\begin{array}{l} \gamma(q - r) \leq G(x,t;q,p,Z) - G(x,t;r,p,Z) \\ \text{for } (x,t) \in \mathcal{R}^n \times (0,T),\ q \geq r \text{ and } (a,p,Z) \in \mathcal{R} \times \mathcal{R}^n \times \mathcal{S}(n), \end{array} \tag{15}$$

and there is a function $\kappa : [0,\infty] \rightarrow [0,\infty]$ with $\kappa(0^+) = 0$ such that

$$\begin{array}{l} G(x,t;a,\alpha(x-y),X) - G(y,t;a,\alpha(x-y),Y) \\ \leq \kappa(\alpha|x-y|^2 + |(x,t) - (y,t)|) \\ \text{for } (x,t),(y,t) \in \mathcal{R}^n \times (0,T) \text{ and } a \in \mathcal{R} \text{ and } X \leq Y. \end{array} \tag{16}$$

Furthermore, assume that N is a positive continuous function which is a gauge function with respect to the Hessian (as explained above) *and such that*

$$\begin{array}{l}(a_{i,j}(x,t) + \frac{\partial N}{\partial z_{ij}}(x,t;a,p,Z)) \geq 0 \\ \textit{for all } (x,t) \in \mathcal{R}^n \times (0,T) \textit{ and } (a,p,Z) \in \mathcal{R} \times \mathcal{R}^n \times \mathcal{S}(n),\end{array} \tag{17}$$

where for all $1 \leq i,j \leq n$, $x_{ij} \neq 0$ *and for all* $(x,t;a,p,X)$ *the partial derivatives* $\frac{\partial}{\partial z_{ij}} H_{(x,t;a,p)}(X) = \frac{\partial N}{\partial z_{ij}}(x,t;a,p,X)$ *exist and are continuous. Then problem (12) has a viscosity solution for continuous bounded data* f.

Proof. We have seen that the operator G is proper if

$$(a_{i,j}(x,t) + \frac{\partial N}{\partial z_{ij}}(x,t;r,a,p,Z)) \geq 0$$

for all $(x,t) \in \mathcal{R} \times (0,T)$, $(a,p,Z) \in R \times \mathcal{R}^n \times \mathcal{S}(n)$. In order to obtain the existence of viscosity solutions we prove a comparison theorem and apply Perrons method.

By comparison for $G = 0$ we mean : if u is a subsolution of $G = 0$ and v is a supersolution of $F = 0$ and $u(x,0) \leq v(x,0)$ for $x \in \mathcal{R}^n$ then $u \leq v$. We assume that u, v are bounded on $(0,T) \times \mathcal{R}^n$.

First note that if u is a subsolution of $G = 0$ then $u_\epsilon = u - \frac{\epsilon}{T-t}$, $(\epsilon > 0)$ is a subsolution of

$$G(x,t;u,\frac{\partial u}{\partial t},\nabla u, D^2 u) = -\frac{\epsilon}{(T-t)^2}. \tag{18}$$

Furthermore, $u \leq v$ follows from $u_\epsilon \leq v$ in the limit $\epsilon \searrow 0$. Hence we may assume that

$$\lim_{t \to T} u(t,x) = -\infty \tag{19}$$

for all $x \in \mathcal{R}^n$. Let $\overline{P_D}^{2,+}u(x,t)$ (rsp. $\overline{P_D}^{2,-}v(y,t)$) be the closure of the superjet $P_D^{2,+}u(x,t)$ (rsp. the closure of the subjet $P_D^{2,-}v(y,t)$). Starting with an upper semicontinuous (rsp. lower semicontinuous) subsolution u (rsp. supersolution v) we assume that for some $(x,t) \in (0,T) \times \mathcal{R}^n$

$$u(x,t) - v(x,t) \geq \delta > 0 \tag{20}$$

and then contradict this assumption. We consider the function

$$\phi(x,t,y,t) := u(x,t) - v(y,t) - \frac{\alpha}{2}|x-y|^2 - \epsilon(|x|^2 + |y|^2) \tag{21}$$

with a maximum at $(\hat{x},\hat{t}),(\hat{y},\hat{t})$. (Note that $(\hat{x},\hat{t}),(\hat{y},\hat{t})$ depend on ϵ in general; we drop this dependence in notation for sake of simplicity). If $\epsilon > 0$ is small enough then

$$\phi(\hat{x},\hat{t},\hat{y},\hat{t})) := u(\hat{x},\hat{t}) - v(\hat{y},\hat{t}) - \tfrac{\alpha}{2}|\hat{x}-\hat{y}|^2 - \epsilon(|\hat{x}|^2 + |\hat{y}|^2) \geq \tfrac{\delta}{2} > 0. \tag{22}$$

From the consideration above we may assume that $\hat{t} \neq T$. Similarly we may assume that $\alpha > 0$ is large enough such that $\hat{t} \neq 0$. Moreover, since u, v are bounded from above

$$\frac{\alpha}{2}|\hat{x}-\hat{y}|^2 + \epsilon(|\hat{x}|^2 + |\hat{y}|^2) \leq u(\hat{x},\hat{t}) - v(\hat{y},\hat{t}) \leq C. \tag{23}$$

for some $C > 0$ and all $t \in (0,T)$ (Recall that u, v are assumed to be bounded). By the main theorem of viscosity solution theory there exist $X, Y \in S(n)$ such that

$$(a, \alpha(\hat{x}-\hat{y}) + 2\epsilon\hat{x}, X + 2\epsilon I) \in \overline{P_U}^{2,+} u(\hat{x},\hat{t}), \tag{24}$$

$$(b, \alpha(\hat{x}-\hat{y}) - 2\epsilon\hat{y}, Y - 2\epsilon I) \in \overline{P_U}^{2,-} v(\hat{y},\hat{t}) \tag{25}$$

and $X \leq Y$ and $a - b = 0$. Since u is a subsolution and v is a supersolution of $G = 0$ we have

$$\begin{array}{rl} G(\hat{x},\hat{t},u(\hat{x},\hat{t}),a,\alpha(\hat{x}-\hat{y}) + 2\epsilon\hat{x}, X + 2\epsilon I) \leq & 0, \\ G(\hat{y},\hat{t},v(\hat{y},\hat{t}),a,\alpha(\hat{x}-\hat{y}) - 2\epsilon\hat{y}, Y - 2\epsilon I) \geq & 0. \end{array} \tag{26}$$

Hence,

$$\begin{array}{rl} 0 < & \gamma(u(\hat{x},\hat{t}) - v(\hat{y},\hat{t})) \\ \leq & G(\hat{x},\hat{t},u(\hat{x},\hat{t}),a,\alpha(\hat{x}-\hat{y}) + 2\epsilon\hat{x}, X + 2\epsilon I) \\ - & G(\hat{x},\hat{t},v(\hat{y},\hat{t}),a,\alpha(\hat{x}-\hat{y}) + 2\epsilon\hat{x}, X + 2\epsilon I) \\ = & G(\hat{x},\hat{t},u(\hat{x},\hat{t}),a,\alpha(\hat{x}-\hat{y}) + 2\epsilon\hat{x}, X + 2\epsilon I) \\ - & G(\hat{y},\hat{t},v(\hat{y},\hat{t}),a,\alpha(\hat{x}-\hat{y}) - 2\epsilon\hat{y}, Y - 2\epsilon I) \\ + & G(\hat{y},\hat{t},v(\hat{y},\hat{t}),a,\alpha(\hat{x}-\hat{y}) - 2\epsilon\hat{y}, Y - 2\epsilon I) \\ - & G(\hat{x},\hat{t},v(\hat{y},\hat{t}),a,\alpha(\hat{x}-\hat{y}) + 2\epsilon\hat{x}, X + 2\epsilon I) \\ \leq & G(\hat{y},\hat{t},v(\hat{y},\hat{t}),a,\alpha(\hat{x}-\hat{y}) - 2\epsilon\hat{y}, Y - 2\epsilon I) \\ - & G(\hat{x},\hat{t},v(\hat{y},\hat{t}),a,\alpha(\hat{x}-\hat{y}) + 2\epsilon\hat{x}, X + 2\epsilon I). \end{array} \tag{27}$$

By (24) we know that the term $\frac{\alpha}{2}|\hat{x}-\hat{y}|^2 + \epsilon(|\hat{x}|^2 + |\hat{y}|^2)$ is bounded independently $0 < \epsilon < 1$ for fixed α. Hence $\epsilon\hat{x}, \epsilon\hat{y} \to 0$ and $\alpha(\hat{x}-\hat{y})$ remains bounded as $\epsilon \to 0$. Hence, for $\epsilon \to 0$ the latter term in (28) converges to (for some $\hat{x}, \hat{y} \in \mathcal{R}^n$)

$$\begin{array}{rl} & G(\hat{y},\hat{t},v(\hat{y},\hat{t}),a,\alpha(\hat{x}-\hat{y}), Y) \\ - & G(\hat{x},\hat{t},v(\hat{y},\hat{t}),a,\alpha(\hat{x}-\hat{y}), X) \\ \leq & \kappa(\alpha|\hat{y}-\hat{x}|^2 + |(\hat{x},\hat{t}) - (\hat{y},\hat{t})|). \end{array} \tag{28}$$

By a standard argument of viscosity solution theory $\kappa(\alpha|\hat{y} - \hat{x}|^2 + |(\hat{x}, \hat{t}) - (\hat{y}, \hat{t})|) \to 0$ as $\alpha \to \infty$ and we have a contradiction.

To apply Perrons method we have to ensure that there exist supersolutions and subsolutions $\overline{u}$ and $\underline{u}$ such that $\overline{u}(x, 0) = f(x) = \underline{u}(x, 0)$. Clearly, the solution of

$$\begin{cases} L_p^- u = 0 & \text{in } \mathcal{R}^n \times (0, T) \\ u(x, 0) = f(x) & \text{in } \mathcal{R}^n \end{cases} \tag{29}$$

is subsolution of (18). For construction of supersolutions consider first the case where $f \in C^{2,b}$, i.e. f has second order bounded derivatives. Then for C large enough $f + Ct$ is a supersolution of (7). There are several ways to extend the result to bounded continuous data f. One is to derive a probabilistic representation of the solution for $f \in C^{2,b}$ as is done below and then use standard methods.

Next we derive the formula for a probabilistic solution. For given $v \in C^{1,2}$ consider the problem

$$(\epsilon_v) \begin{cases} \frac{\partial u}{\partial t} + \sum_{i,j=1}^n \Sigma_{ij}^{v,\epsilon}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} & \\ + \sum_{i=1}^n b_i(x, t) \frac{\partial u}{\partial x_i} - c(x, t)u = 0 & \text{in } \mathcal{R}^n \times (0, T) \\ u(x, T) = f(x), & \text{in } \mathcal{R}^n \end{cases} \tag{30}$$

where

$$\Sigma_{ij}^{v,\epsilon}(x, t) = (a_{ij}(x, t) + \frac{\partial N^\epsilon}{\partial z_{ij}}(x, t; v(x, t), Dv(x, t), D^2 v(x, t)))_{ij}$$

and $\frac{\partial N}{\partial z_{ij}}$ has been smoothed to $\frac{\partial N^\epsilon}{\partial z_{ij}}$ using a mollifier. We can assume that $\frac{\partial N^\epsilon}{\partial z_{ij}}$ is such that $(\Sigma_{ij}^{v,\epsilon}(x, t)) > 0$ for all (x, t). If $u_{v,\epsilon}$ is a solution of $(\epsilon)_v$ then $u_{v,\epsilon}$ has the representation

$$u_{v,\epsilon}(x, t) = E(e^{-\int_0^t c(X_s, s) ds} f(X_t^{x,v,\epsilon})) \text{ where} \tag{31}$$

$(X_s^{x,v,\epsilon})_{s \in [t,T]}$ is a process starting at x at time t and solves

$$dX_{t,i}^{x,v,\epsilon} = b_i(X_t, t)dt + \sum_{j=1}^p \sigma_{ij}^{v,\epsilon}(X_t, t) dW_t^j \text{ and}$$

$$\Sigma_{ij}^{v,\epsilon}(x, t) = \sum_l \sigma_{il}^{v,\epsilon}(x, t) \sigma_{jl}^{v,\epsilon}(x, t).$$

Let u_ϵ solve

$$(\epsilon_u)\begin{cases} \frac{\partial u}{\partial t} - \sup_{v\in C^{1,2}} \sum_{i,j=1}^n \Sigma_{ij}^{v,\epsilon}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} \\ -\sum_{i=1}^n b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u = 0 \text{ in } \mathcal{R}^n \times (0,T) \\ u(x,T) = f(x), \text{ in } \mathcal{R}^n \end{cases} \tag{32}$$

We consider a subsequence $u_{\frac{1}{m}}(z,t)$ of the family u_ϵ) of solutions of problems (ϵ_u) at each (z,t) and define upper and lower semicontinuous functions $\overline{U}$ and $\underline{U}$ by

$$\overline{U}(z,t) = \lim_{m\to\infty} \sup{}^* u_{\frac{1}{m}}(z,t) = \\ \lim_{j\to\infty} \sup\{u_{\frac{1}{m}}(x,s) | m \geq j, x \in \mathcal{R}^n \times (0,T), \ \& \ |(z,t)-(x,s)| \leq \tfrac{1}{j}\}, \tag{33}$$

and

$$\underline{U}(z,t) = \lim_{m\to\infty} \inf{}_* u_{\frac{1}{m}}(z,t) = \\ \lim_{j\to\infty} \inf\{u_{\frac{1}{m}}(x) | m \geq j, x \in \mathcal{R}^n \times (0,T), \ \& \ |(z,t)-(x,s)| \leq \tfrac{1}{j}\}. \tag{34}$$

By standard arguments $\overline{U}(z,t)$ is a subsolution and $\underline{U}(z,t)$ is a supersolution of $G=0$. We get comparison for (ϵ) in an analogous way as in theorem 1 and have $\underline{U} \leq \overline{U}$ by definition and $\overline{U} \leq \underline{U}$ by comparison. By uniqueness of the viscosity solution we have $\lim_{\epsilon\searrow 0} u_\epsilon = u$. By comparison for the problems (ϵ_v) we also have

$$u_{v,\epsilon} \leq u_\epsilon \tag{35}$$

for all $\epsilon > 0$ and $v \in C^{1,2}$. Hence,

$$\sup_{v\in C^{1,2}} u_{v,\epsilon} = \sup_{v\in C^{1,2}} E(e^{-\int_0^t c(X_s,s)ds} f(X_t^{x,v,\epsilon})) \leq u_\epsilon \tag{36}$$

and

$$\lim_{\epsilon\searrow 0} \sup_{v\in C^{0,2}} E(e^{-\int_0^t c(X_s,s)ds} f(X_t^{x,v,\epsilon})) \leq \lim_{\epsilon\searrow 0} u_\epsilon = u. \tag{37}$$

In order to show that

$$\lim_{\epsilon\searrow 0} \sup_{v\in C^{1,2}} E(e^{-\int_0^t c(X_s,s)ds} f(X_t^{x,v,\epsilon})) \geq u(x,t) \tag{38}$$

we construct a sequence (v_m) where $v_m \in C^{0,2}$, converging to the solution u of (18). Starting with the continuous solution u let v_m

equal u on a grid $G_t^m \equiv \{(\frac{k_1}{m},...,\frac{k_n}{m},t)|k_i \in Z\}$ for all $t \in (0,T)$. Let $u_{m,\epsilon}$ solve

$$\left\{\begin{array}{ll} \frac{\partial u}{\partial t} - \sum_{i,j=1}^n \Sigma_{ij}^{v_m,\epsilon_m} \frac{\partial^2 u}{\partial x_i \partial x_j} & \\ -\sum_{i=1}^n b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u = 0 & \text{in } \mathcal{R}^n \times (0,T) \\ u(x,0) = f(x) & \text{in } \mathcal{R}^n \end{array}\right. \tag{39}$$

We denote the limit problem ($\epsilon \searrow 0$) with coefficient matrix $\Sigma_{ij}^{v_m,0}$ by (m). Let u_m be the solution of the limit problem (m). Then (since $C^{1,2}$ is dense in $C^{0,2}$) we have

$$\begin{array}{l} \lim_{\epsilon\searrow 0} \sup_{v\in C^{1,2}} E(e^{-\int_0^t c(X_s,s)ds} f(X_t^{x,v,\epsilon})) \geq \\ \lim_{\epsilon\searrow 0} \sup_m E(e^{-\int_0^t c(X_s,s)ds} f(X_t^{x,v_m,\epsilon})) = \\ \lim_{\epsilon\searrow 0} \sup_m u_{m,\epsilon} \geq \lim_{\epsilon\searrow 0} u_{m,\epsilon} = u_m \text{ for all } m. \end{array} \tag{40}$$

For all m we have $u_m \equiv u$ on G_t^m for all $t \in (0,T)$. Hence,

$$\lim_{\epsilon\searrow 0} \sup_{v\in C^{1,2}} E(e^{-\int_0^t c(X_s,s)ds} f(X_t^{x,v,\epsilon})) \geq u. \tag{41}$$

and we have

Theorem 2 *The probabilistic solution of (12) is*

$$u(x,t) = \lim_{\epsilon\searrow 0} \sup_{v\in C^{1,2}} E(e^{-\int_0^t c(X_s,s)ds} f(X_t^{x,v,\epsilon})). \tag{42}$$

3 Application to the Generalised Leland Equation

Now we can apply the results above to the generalised Leland equation describing the pricing problem for European options written on multiple assets in the presence of transaction costs. It seems that the equation was stated publicly in [8] for the first time (it also appeared later in text books). The pricing problem for an European options with payoff f and transaction cost numbers A_i reads

$$\left\{\begin{array}{l} \frac{\partial V}{\partial t} + 1/2\sum_{i,j=1}^n \rho_{ij} D_{ij}^S V + r(\sum_{i=1}^n S_i \frac{\partial V}{\partial S_i} - V) \\ +1/2\sum_{i=1}^n A_i \sqrt{\sum_{j,k=1}^n \rho_{jk} D_{ij}^S V D_{ik}^S V} = 0 \text{ in } R_+^n \times (0,T) \\ V(S_1,...,S_n,T) = f(S_1,...,S_n) \text{ in } R_+^n, \end{array}\right. \tag{43}$$

where V price of an option on n assets S_i driven by geometric Brownian motions with volatilities σ_i and correlations ρ_{jk}. As usual r denotes the constant interest rate. Recall that A_i are so called Leland numbers introduced by Avellaneda and Paras in [1]. They are proportional to round trip fees per standard deviation.

Transformation $x_i = log(S_i)$ and time reversal leads to a pure Cauchy problem on $\mathcal{R}^n$. We have

Theorem 3 *The Cauchy problem associated to (5) has a viscosity solution if f is uniformily bounded continuous and*

$$\inf_{Z\in\mathcal{S}(n)}(\rho_{ik} + A_i \frac{\sum_{j=1}^n \rho_{jk} z_{ij}}{\sqrt{\sum_{j=1}^n \rho_{jk} z_{ij} z_{ik}}}) \geq 0. \tag{44}$$

Especially, this holds if for all i, j, $i \neq j$ $A_i < 1$ and

$$|\rho_{ij}| \leq -\frac{(A_i + A_j)}{2} + \sqrt{(1 - A_i)(1 - A_j)} \tag{45}$$

holds. A probabilistic representation of the solution is

$$u(x,t) = \lim_{\epsilon \searrow 0} \sup_{v\in C^{1,2}} E(e^{-r(T-t)} f(X_t^{x,v,\epsilon})), \tag{46}$$

where $(X_t^{x,v,\epsilon})_{v,\epsilon}$ is a family of processes starting at x which solves

$$dX_{t,i}^{x,v,\epsilon} = (r - \Sigma_{ii}^{v,\epsilon})dt + \sum_{j=1}^{p} \sigma_{ij}^{v,\epsilon} dW_t^j$$

and $\Sigma_{jk}^{v,\epsilon} = \sum_{i=1}^p \sigma_{ji}^{v,\epsilon} \sigma_{ki}^{v,\epsilon}$ is from linearization as in (31).

Proof. The statements follow from theorem 1 and theorem 2 above if we can show that the Leland operator is proper under the assumption given. To see this introduce the notation

$$D_{ij}^x u := D_{ij} u - \delta_{ij} u$$

along with the Kronecker delta δ_{ij}. Then we may write the generalised Leland in coordinates $x_i = log(S_i)$ to be

$$\begin{array}{l} \frac{\partial u}{\partial t} - 1/2 \sum_{i,j=1}^n \sigma_i \sigma_j \rho_{ij} D_{ij}^x u - r \sum_{i=1}^n \frac{\partial u}{\partial x_i} + ru \\ -1/2 \sum_{i=1}^n A_i \sqrt{\rho_{jk} \sigma_i \sigma_j \sigma_i \sigma_k D_{ij}^x u D_{ik}^x u} = 0 \end{array} \tag{47}$$

In order to prove that the generalised Leland operator is proper if for all i, j $A_i \leq 1$ and

$$|\rho_{ij}| \leq -\frac{(A_i + A_j)}{2} + \sqrt{(1 - A_i)(1 - A_j)}. \tag{48}$$

we want to show that the latter condition is sufficient if for all $Z \in \mathcal{S}(n)$

$$M_n := (\rho_{ik} + A_i \frac{\sum_j \rho_{jk} z_{ij}}{\sqrt{\sum_{jk} \rho_{jk} z_{ij} z_{ik}}}) \geq 0. \tag{49}$$

First we observe that for all $X \in S(n)$

$$|\frac{\sum_{j=1}^n \rho_{jk} z_{ij}}{\sqrt{\sum_{j,k=1}^n \rho_{jk} z_{ij} z_{ik}}})| \leq 1. \tag{50}$$

This is an immediate consequence of

Lemma 1 *Let $Q = (\rho_{ij})$ be a correlation matrix (i.e. ρ_{ij} is symmetric, positive and $\rho_{ii} = 1$). Then*

$$\forall z \in \mathcal{R}^n \forall k \in \{1,..,n\} : \quad (Qz)_k \leq \sqrt{< z, Qz >}, \tag{51}$$

where $(Qz)_k$ denotes the kth component of the vector Rz.

Proof. Since Q is positive we have $Q = B^2$ for some matrix B. If $\|.\|_2$ denotes the euclidean norm and $y = Bz$ then we have for all $z \in \mathcal{R}^n$

$$(Qz)_k = (B^2 z)_k = (By)_k \leq \sqrt{\sum_{j=1}^n b_{kj}^2} \|y\|_2 = \|y\|_2 = \sqrt{< z, Qz >}, \tag{52}$$

since $\sum_{j=1}^n b_{kj}^2 = \sum_{j=1}^n b_{kj} b_{jk} = \rho_{kk} = 1$.

In order to check positiveness we consider the symmetrised matrix

$$M_n^s = (\rho_{ik} + (1/2)(A_i \frac{\sum_{j=1}^n \rho_{jk} z_{ij}}{\sqrt{\sum_{jk=1}^n \rho_{jk} z_{ij} z_{ik}}} + A_k \frac{\sum_{j=1}^n \rho_{ji} z_{kj}}{\sqrt{\sum_{jk=1}^n \rho_{jk} z_{ij} z_{ik}}}). \tag{53}$$

If $n = 2$ then we have positiveness for all $X \in \mathcal{S}(n)$ if $A_1 \leq 1$ and (using lemma 1)

$$det(M_2^s) \geq det(\rho_{ik} - (1/2)(A_i + A_k) + \delta_{ik}(A_i + A_k)) \geq 0 \tag{54}$$

where δ_{ik} denotes again the Kronecker delta. The latter inequality holds if for all i, j

$$|\rho_{ij}| \leq -\frac{(A_i + A_j)}{2} + \sqrt{(1 - A_i)(1 - A_j)}, \tag{55}$$

and we have finished in the two dimensional case. In the case of dimension $n > 2$ observe that

$$det(M_n^s) = \sum_{l=1}^{n} (1 - A_l) det(M_{n-1}^{A_l}), \tag{56}$$

where $M_{n-1}^{A_l}$ is the adjoint of M_n^s obtained from M_n^s by omitting the lth row and the lth column. However, $M_{n-1}^{A_l}$ is of the form

$$\rho_{ik} + (1/2)(A_i \frac{\sum_{j=1}^{n} \rho_{jk} z_{ij}}{\sqrt{\sum_{j.k=1}^{n} \rho_{jk} z_{ij} z_{ik}}} + A_k \frac{\sum_{j=1}^{n} \rho_{ji} z_{kj}}{\sqrt{\sum_{jk=1}^{n} \rho_{jk} z_{ij} z_{ik}}}), \tag{57}$$

where $i, k \neq l$, which is again "correlation matrix" (omitting the lth row and column of the original correlation matrix (ρ_{ik})) plus $(1/2)(A_i C_{ik} + A_k C_{ki})$ along with numbers $-1 \leq C_{ik}, C_{ki} \leq 1$. Hence we can proceed by induction and have done.

Finally we want to discuss the problem of ill-posedness. We have proved that that for all i, j, $i \neq j$ $A_i < 1$ and

$$|\rho_{ij}| \leq -\frac{(A_i + A_j)}{2} + \sqrt{(1 - A_i)(1 - A_j)}$$

is sufficient for the existence of solutions. Is it also necessary? First, we have remark that "ellipticity" and "solvability" are not interchangable terms since equations of mixed types can have solutions. However, a theory of mixed-type equations which covers the multi-dimensional Leland equation is not available nowadays. Second, even if a theory of mixed-type parabolic equations were available, the motivation of mathematical finance can lead to other types of equation motivated by transaction cost minimizing hedging strategy. Careful modelling is needed for studying pricing of options written on many assets with higher transaction costs. However, some people argue that low transaction costs are normal nowadays and for this case our analysis hints that solvability and ellipticity are close. And our sufficient condition for ellipticity becomes also a necessary condition in this case if correlations are strong.

This can be seen most easily in the case $n = 2$. The essential part of the function G

$$(z_{ij}) \to 1/2(z_{11} + 2\rho z_{12} + z_{22}) + 1/2 \sum_{i=1}^{2} A_i \sqrt{z_{i1}^2 + 2\rho z_{i1} z_{i2} + z_{i2}^2}.$$

can be written as

$$(z_{ij}) \to \begin{pmatrix} 1 + A_1 \frac{z_{11}+\rho z_{12}}{\|z\|_1} & \rho + A_1 \frac{\rho z_{11}+z_{12}}{\|z\|_1} \\ \rho + A_2 \frac{\rho z_{22}+z_{12}}{\|z\|_2} & 1 + A_2 \frac{z_{22}+\rho z_{12}}{\|z\|_2} \end{pmatrix} \begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix}$$

abbreviating $\|z\|_i = \sqrt{z_{i1}^2 + 2\rho z_{i1} z_{i2} + z_{i2}^2}$. For $\rho \to 1$ the coefficients of the A_i tend to $1, -1$. Choosing signs z_{ij} appropriately for ρ close to one (but not equal) the symmetrized coefficient matrix is close to

$$\begin{pmatrix} 1 - A_1 & \rho + 1/2(A_1 + A_2) \\ \rho + 1/2(A_1 + A_2) & 1 - A_2 \end{pmatrix}$$

and positivity of this is the proposed relation in the special case of dimension 2

$$|\rho| \leq -\frac{(A_1 + A_2)}{2} + \sqrt{(1 - A_1)(1 - A_2)}.$$

Our relations cannot be generalised without using further restrictive conditions. This also shows that for $\rho \to 1$ even for $A_2 \to 0$ (choosing signs z_{ij} appropriately) the determinant of the symmetrised coefficient matrix is close to

$$\det \begin{pmatrix} 1 - A_1 & \rho + (1/2)A_1 \\ \rho + (1/2)A_1 & 1 \end{pmatrix} \to (-1/2)A_1^2,$$

and ellipticity fails even for small transaction costs (small A_1) with respect to the asset S_1. However, if $\rho \to 0$ and $A_2 \to 0$ the coefficent matrix of the two dimensional generalised Leland operator is close to

$$\begin{pmatrix} 1 + A_1 \frac{z_{11}}{\|z\|_1} & A_1 \frac{z_{12}}{\|z\|_1} \\ 0 & 1 \end{pmatrix}$$

which stays positive for $0 \leq A_1 \leq 1$ as would be expected from the analysis of the original Leland equation.

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