ARBITRAGE PRICING THEORY : MULTI-PERIOD MODEL

(Notes for second lecture, to be completed later)

The analysis of inter-temporal investment/trading decisions is of crucial importance in financial modeling. This chapter develops a framework for discrete financial models with several trading periods. We define the notion of arbitrage in a multi-period setting and then derive a no-arbitrage theorem which can be viewed as an extension of the basic result of Chapter 1.

The basic multi-period model considers a sequence of **trading dates** in which agents can change their portfolios. On each date, the market will be in one of many possible "states". The states of the market are successively revealed as time passes. Prices of traded securities are determined by the present state of the market and the history of previous states (the "information vector"). The evolution of the market is thus fully characterized by a sequence of states, X_0, X_1, $...X_n$, ..., or "path" in state-space.

The main result that will be discussed is that the absence of arbitrage is equivalent to the existence of a a "risk-neutral" pricing probability defined on the *set of paths*, i.e. a probability measure that such that security prices are equal to the expected values of their discounted future cash-flows. This result is analogous to the one derived in Chapter 1 for one-period models, but implies additional constraints on the pricing measure that will turn out to be useful in practice. Namely, security prices, adjusted for dividends and measured in constant dollars, behave under the risk-neutral probability like **martingales** or conditionally unbiased time-series.

1. The multi-period model

We will assume a sequence of trading dates, t_0, t_1, $...t_n$, ... In each of these future dates, we assume that the market will be in one of many possible states, belonging to a finite set $\mathbf{X}^{(n)}$. The state of the market at date t_n is be denoted by the label X_n for $n = 0, 1, 2, ...$ Uncertainty is introduced in the model by assuming that the state of the market at date t_n is unknown before this date.

Traded securities are characterized by their prices at different dates and their dividend income (payoffs) at each date. More precisely, we assume that a traded security gives right to a sequence of dividends at different dates,

$$D_0,\, D_1,\, ..D_n,\, ...,$$

and that at each date, this security is available for trading *after* the dividend is paid at the market price P_n. Thus, we will have also a sequence of market prices[1]

$$P_0,\, P_1,\, ..\, P_n,\, ...$$

(Notice that with this convention the value of this security at time t_n *before* the dividend is paid out is $D_n \;+\; P_n$.) It is assumed that the prices and dividends of traded securities at any given date t_n are functions of the $n+1$ states of the market $X_0,\, X_1,\, ...X_n$, i.e.

$$D_n \;=\; D_n\left(X_0,\, X_1,\, ...\, X_n\right)\;, \qquad P_n \;=\; D_n\left(X_0,\, X_1,\, ...\, X_n\right)\;. \tag{1}$$

The vector $(X_0,\, X_1,\, ...X_n)$ represents the **information available at date t_n.**[2]

We assume that there are N securities available for trading at any given date, with associated price/dividend pairs $(P_{i\,n}\,,\; D_{i\,n}\;)\;, i = 1,\, 2,\, 3,\, ...N\;,\;\; n \;=\; 0,\, 1\,....$ In particular, there is **money-market** security (note, deposit, etc.) at each trading date t_n which has a market price of \$1 and a *known* dividend payable at the next trading date of

$$D_{n+1} \;=\; 1 \;+\; R_n \;=\; 1 \;+\; R_n\left(X_0,\, X_1,\, ...\, X_n\;\right)\;. \tag{2}$$

The quantity R_n represents the (gross) interest rate prevailing between the dates t_n and t_{n+1}. This rate is known in the market at the date that the loan is made, t_n. We will refer to it as the **riskless rate** prevailing over this period.[3]

A **trading strategy**, in the context of the multi-period model, is a *finite* sequence of N-vectors,

$$\theta_1,\, \theta_2,\, ...\, \theta_n,\, ...\, \theta_{n_0}\;,$$

in which

[1] Dividends and prices are expressed in terms of a fixed unit of account (dollars).

[2] Thus, it is implicitly assumed that all agents have access to this information and that prices are set according to it. We shall refer sometimes to $(X_0,\, X_1,\, ...\, X_n)$ as the **information vector** at date t_n.

[3] Note that the time unit is not specified. This is the reason for using "gross" rather than annualized interest rates at this point.

$$\theta_n \;=\; \theta_n\,(X_0,\,X_1,\,...\,X_n\,)\;. \tag{3}$$

As in the one-period model, the entries of the vector θ represent the number of securities held long or short in the portfolio.[4] An agent following this strategy will "enter" the market at date t_0 and adjust the portfolio according to (3) after each trading period. At date t_{n_0} he or she will "divest out" of the market, closing out all positions.

A natural condition that must be satisfied by the sequence $\{\theta_n\}$ is that the strategy is **self-financing**. This means that changes in the holdings of different securities across time are financed exclusively in terms of profits/losses arising from previous trading (including cash borrowed or lent in the money-market). Mathematically, the self-financing condition can be expressed as

$$\sum_1^N \theta_{i\,n}\;(\,P_{i\,n+1}\;+\;D_{i\,n+1}\,)\;=\;\sum_1^N \theta_{i\,n+1}\,P_{i\,n+1}\;. \tag{4}$$

¿From a practical standpoint, self-financing is "taken care of" automatically if we make a distinction between the money-market account and the remaining ("risky") securities. Indeed, a self-financing trading strategy is simply one in which the cash balance resulting from market gains/losses and income and from changing the portfolio of risky assets (stocks, long-term bonds, etc.) is debited to the money market account, including interests. Accordingly, if the money-market security is identified by the index $i\;=\;1$, the self-financing condition can be written in the form

$$\theta_{1\,n+1}\;=\;\theta_{1\,n}\;(1\;+\;R_n\,)\;+\;\sum_2^N \theta_{i\,n}\,D_{i\,n+1}\;\;-\;\sum_2^N\,(\theta_{i\,n+1}\,-\,\theta_{i\,n}\,)\;P_{i\,n+1}\;. \tag{5}$$

or, simply,

$$\text{Current money-market balance} \;=\; \text{Past money-market balance} \;+\; \text{interest}$$

$$+\;\text{dividend income} \,-\; \text{cost of changing the portfolio of risky assets}\;.$$

[4] The assumption that the n^{th} investment decision depends on $X_0,\,X_1,\,...\,X_n$ reflects the fact that trading decisions are based on information available at that time.

2. Necessary condition for no-arbitrage: Existence of Arrow-Debreu measures

Definition: *The trading strategy*

$$\theta_1,\ \theta_2,\ ...\theta_n,\ ...\ \theta_{n_0}\ ,$$

is an arbitrage *if*

$$\sum_1^N \theta_{i\,0}\, P_{i\,0} \ = \ 0 \qquad (\textit{zero initial cost})$$

and

$$\sum_1^N \theta_{i\,n_0}\, P_{i\,n_0} \ \geq \ 0 \qquad (\textit{no final losses})$$

and

$$\sum_1^N \theta_{i\,n_0}\, P_{i\,n_0} \ > \ 0$$

$$(\textit{positive final value for at least one path } \ X_0,\ ...X_{n_0}\ )\ .$$

We shall derive necessary conditions for the absence of arbitrage strategies using the results of Chapter 1. Consider a trading strategy that will consist in investing at the (future) date t_n in a portfolio $\theta_n \ = \ \theta_n(X_0,\ X_1,\ ...X_n)$ and closing out the position in risky securities after one trading period, i.e.

$$\theta_{i\,n+1} \ = \ -\theta_{i\,n} \quad , \quad i\ \geq 2\ .$$

The i^{th} risky security ($i \geq \ 2$) is available for purchase at the price $P_{i\,n}$ and will produce a cash-flow of $P_{i\,n+1}\ +\ D_{i\,n+1}$ after closing the position. Therefore, by the one-period no-arbitrage theorem (Theorem 2 of Ch. 1), we conclude that, given the information vector at date t_n, there must exist at least one probability measure $\hat{\pi}_n$ defined over the set of states at the date t_{n+1}, $\mathbf{X}^{(n=1)}$, such that

$$P_{i\,n} \ = \ \frac{1}{1\ +\ R_n}\,\mathbf{E}^{\hat{\pi}_n}\left\{\ P_{i\,n+1}\ +\ D_{i\,n+1}\ \right\}\ . \tag{6}$$

It is important to emphasize at this point that the "risk-neutral" probabilities associated with each of the states X_{n+1} will depend, in general, on the information available at time t_n. Therefore, we have

$$\hat{\pi}_n(X_{n+1}) \;=\; \hat{\pi}_n\,(X_{n+1} \mid X_0,\, X_1,\, \dots X_n\,)\;, \tag{7}$$

with

$$\sum_{X_{n+1}} \hat{\pi}_n\,(X_{n+1} \mid X_0,\, X_1,\, \dots X_n\,) \;=\; 1\;, \tag{8}$$

where the dependence on the information vector is made clear. These probabilities can be interpreted as *Arrow-Debreu probabilities for the states* X_{n+1} *at the date* t_{n+1} *conditional on the information at date* t_n. The pricing equation (6) can be written more explicitly in terms of the state variables and the probabilities in (6) as

$$P_{i\,n}\,(X_0,\, X_1,\, \dots X_n\,) \;=\; \frac{1}{1\;+\;R_n(X_0,\, X_1,\, \dots X_n\,)} \;\times$$

$$\sum_{X_{n+1}} \left[P_{i\,n+1}(X_0,\, X_1,\, \dots X_n\,) +\; D_{i\,n+1}(X_0,\, X_1,\, \dots X_n\,)\right] \hat{\pi}_n\,(X_{n+1} \mid X_0,\, X_1,\, \dots X_n\,)\;. \tag{9}$$

Suppose now that we make a selection, for each date and for each information vector $(\,X_0,\, X_1,\, \dots X_n\,), n\; >\; 0$, of a a set of Arrow-Debreu conditional probabilities $\left\{\hat{\pi}_n\,(X_{n+1} \mid X_0,\, X_1,\, \dots X_n\,)\;, X_{n+1} \in \mathbf{X}^{(n+1)}\,\right\}$. Using these probabilities, we can define the quantities

$$\begin{aligned} \hat{\Pi}_{0\,n}\,(X_1,\, \dots X_n\,) \;&=\; \hat{\Pi}_{0\,n}\,(X_1,\, \dots X_n \mid X_0\,) \\ &=\; \prod_{j=1}^{n} \hat{\pi}_j\,(X_j \mid X_0,\, X_1,\, \dots X_{j-1}\,)\;. \end{aligned} \tag{10}$$

It is straightforward to check, using (8), that the functions $\hat{\Pi}_{0\,n}\,(X_1,\, \dots X_n\,)$ are non-negative and that, for all n,

$$\sum_{X_1,\, \dots X_n} \hat{\Pi}_{0\,n}\,(X_1,\, \dots X_n \mid X_0\,) \;=\; 1\;,$$

where the sum is taken over all paths $X_1,\, \dots X_n$ that can be realized up to date t_n. Therefore, the quantities $\hat{\Pi}_{0\,n}\,(X_1,\, \dots X_n\,)$ can be interpreted as *probabilities on the set of forward market scenarios up to time* t_n *given the information at time* t_0.

In a similar fashion, we define probabilities on forward paths *conditional on the market information at date t_m for any m.* In fact, for $m < n$, let us define the quantities

$$\hat{\Pi}_{m\,n}\,(X_{m+1},\,...\,X_n\,) \;=\; \hat{\Pi}_{m\,n}\,(X_{m+1},\,...\,X_n \mid X_0\,X_1\,...\,X_m)$$

$$= \prod_{j=m+1}^{n} \hat{\pi}_j\,(X_j \mid X_0,\,X_1,\,...\,X_{j-1}\,)\;. \tag{11}$$

These quantities are non-negative and

$$\sum_{X_{m+1},\,...\,X_n} \hat{\Pi}_{m\,n}\,(X_{m+1},\,...\,X_n \mid X_0\,X_1\,...\,X_m) \;=\; 1\;, \tag{12}$$

a property that is easily verified using (8). The latter quantities can be interpreted as risk-neutral probabilities for the forward market trajectories conditional on the information up to time t_m.

The following proposition expresses mathematically the fact that a selection of "local" Arrow-Debreu probabilities $\{\,\hat{\pi}_n\,(X_{n+1}|X_0,\,X_1,\,...\,X_n)\,\}$ actually defines a probability measure on the set of *infinite* paths "in the future" of any given date $t_m\,,\;\; m \geq 0$ given the information vector at time t_m.[5]

Proposition . *Let $\mathbf{E}_{m\,n}$ represent the expectation operator associated with the probabilities $\hat{\Pi}_{m\,n}$, i.e., for all functions $F \;=\; F(X_{m+1},\,...\,X_n)$,*

$$\mathbf{E}_{m\,n}\,\{F\} \;=\; \sum_{X_{m+1},\,...\,X_n} F(X_{m+1},\,...\,X_n)\,\hat{\Pi}_{m\,n}\,(X_{m+1},\,...\,X_n)\;\;, \tag{13}$$

(omitting the dependence on the information vector $X_0,\,...\,X_m$). Then, for all $n' > n$ we have

$$\mathbf{E}_{m\,n}\,\{F\} \;= \mathbf{E}_{m\,n'}\,\{F\}\;\;. \tag{14}$$

In particular, the collection of probabilities $\hat{\Pi}_{m\,n}$ with $n > m$ defines a unique probability measure on the set of forward paths

$$(X_{m+1},\,X_{m+2},\,...,X_n,\,...\,)\,, \qquad n \;>\; m\;.$$

[5]In practical applications, we shall always work with a finite "time-horizon", so that the probabilities $\hat{\Pi}_{m\,n}$ for some large n would be sufficient for our purposes. However, it is sometimes convenient (and less cumbersome) to think in terms of infinite paths.

Proof: Since the function F depends only on states $(X_{m+1}, \dots X_n)$, we have

$$\mathbf{E}_{m\,n'}\{F\} \;=\; \mathbf{E}_{m\,n}\{F\ \mathbf{E}_{n\,n'}\{1\ \}\ \}\ ,$$

where $\mathbf{E}_{n\,n'}\{1\ \} = 1$ follows from equation (12). Thus (14) holds.

The validity of (14) ensures that the probabilities $\hat{\Pi}_{m\,n}$ are *consistent* for different values of n: for any n, they produce the same expectation value and probabilities for any random variable which depends on a finite "portion"of the path, $(X_{m+1}, \dots X_{n_0})$, with $n_0 \;\leq\; n$. Thus, by the Kolmogorov Extension Theorem of Probability Theory, there exists a unique probability measure defined on semi-infinite paths, which we shall denote by $\hat{\Pi}_m$, such that

$$\mathbf{E}_m\{F\} \;\equiv\; \mathbf{E}^{\hat{\Pi}_m}\{\,F\,\} \;=\; \mathbf{E}_{m\,n}\{F\ \} \qquad \text{for all} \quad F \;=\; F(X_{m+1}, \dots X_n)\ , \quad n \geq\; m\ . \tag{15}$$

(For a rigorous exposition on the Kolmogorov Extension Theorem and a discussion of the properties of probabilities on sequence-spaces, see L. Breiman: *Probability*, Addison-Wesley 1968, Ch. 1, p. 23.)

It is left to the reader, as an exercise, to verify that for all $m \;<\; n$, and for any function $F \;=\; F(X_0, \dots X_{n_0})$, we have

$$\mathbf{E}_m\,\{\,\mathbf{E}_n\,\{\,F\ \}\,\} = \;\mathbf{E}_m\,\{\,F\,\}\ . \tag{16}$$

3. Martingale property of prices under the Arrow-Debreu measure

Arrow-Debreu measures on path-space allow us to represent prices as expected values of future cash-flows.

Proposition. *For any $m \;\geq\; 0$ and any $n \;>\; m$, define the quantities*

$$Z_{m\,n} \;=\; \prod_{j=m}^{n-1} \frac{1}{1\;+\;R_j}\ . \tag{17}$$

Then, the sequence of dividends and market prices $(D_j\ ,\ P_j\ )$ of any traded security satisfy the relations:

$$P_m \;=\; \mathbf{E}_m \left\{ \sum_{j=m+1}^{n} Z_{m\,j}\, D_j \;+\; Z_{m\,n}\, P_n \right\} . \tag{18}$$

Proof: The proof is by induction on n. We note that for $n = m + 1$, the result follows from the single-period pricing equation (6).

Assuming that (17) holds for some $n > m$, we prove that it also holds for $n + 1$. For this, we observe first that

$$\begin{aligned} P_n \;&=\; \frac{1}{1+\,R_n}\, \mathbf{E}_{n\;n+1} \{ D_{n+1} \;+\; P_{n+1} \,\} \\ &=\; \frac{1}{1+\,R_n}\; \mathbf{E}_n \{ D_{n+1} \;+\; P_{n+1} \,\} \;, \end{aligned} \tag{19}$$

an equation which follows again from (6)). Using this result and equation (17), we conclude that

$$\begin{aligned} Z_{m\,n}\, P_n \;&=\; Z_{m\;n+1}\, \mathbf{E}_n \{ D_{n+1} \;+\; P_{n+1} \,\} \;, \\ &=\; \mathbf{E}_n \{\, Z_{m\;n+1} \;(\, D_{n+1} \;+\; P_{n+1} \,)\} \;, \end{aligned} \tag{20}$$

where we used the fact that $Z_{m\;n+1}$ is a function of the information vector at date t_n. Finally, applying the expectation operator $\mathbf{E}_m$ to both sides of (19) and using (16), we conclude that

$$\begin{aligned} \mathbf{E}_m \{ Z_{m\,n}\, P_n \,\} \;&=\; \mathbf{E}_m \{ \mathbf{E}_n \{\, Z_{m\;n+1} \;(\, D_{n+1} \;+\; P_{n+1} \,)\} \;\} \\ &=\; \mathbf{E}_m \{ Z_{m\;n+1} \;(\, D_{n+1} \;+\; P_{n+1} \,)\} \;, \end{aligned} \tag{21}$$

as desired.

Let us apply this proposition to the special case of a security that pays no dividends over the time period from t_0 to t_{n_0}. According to equation (18), we must have

$$P_m \;=\; \mathbf{E}_m \{\, Z_{m\,n}\, P_n \,\} \;, \tag{22}$$

for all integers m, n such that $0 \leq m < n \leq n_0$. Multiplying this equation by $Z_{0\,m}$ on both sides, we obtain

$$Z_{0\,m}\,P_m \;=\; \mathbf{E}_m\,\{\,Z_{0\,n}\,P_n\,\}\;, \tag{23}$$

where we used the fact that $Z_{0\,m} \;=\; Z_{0\,m}\,(X_0,\,...\,X_{m-1})$.

Notice that that the quantity

$$\widetilde{P}_m \;\equiv\; Z_{0\,m}\,P_m$$

represents the market price (or value) of the security measured in *dollars at date* t_0. More precisely, an investor that would have deposited at date t_0 the amount $\widetilde{P_m}$ in the money-market and reinvested the interests would have a money-market account worth

$$\widetilde{P}_m \cdot \prod_{j=0}^{m-1}(1\,+\,R_j) \;=\; \widetilde{P}_m/Z_{0\,m} \;=\; P_m$$

at date t_m. With this notation, we can rewrite equation (22) as

$$\widetilde{P}_m \;=\; \mathbf{E}_m\left\{\,\widetilde{P}_n\,\right\}\;. \tag{24}$$

Thus, *if we measure values in constant dollars, the price at date* t_m *is equal to the expected value under the Arrow-Debreu probability of the price at date* t_n*, conditional on the information at date* t_m.

The sequence of "normalized" prices

$$\widetilde{P}_0,\,\widetilde{P}_1,\,...\,\widetilde{P}_n,\,...$$

can be viewed as a *random process*, or time-series, governed by the Arrow-Debreu measure $\hat{\Pi}_0$. Equation (24) tells us that $\left\{\widetilde{P}_n\,,n\,\geq\,0\right\}$ is a *martingale* sequence under the Arrow-Debreu measure. A martingale in Probability Theory is the mathematical formalization of the concept of a "fair game", in which a gambler's fortune today is the expected value of the fortune at any later date.[6]

We derive a more general result along these lines, assuming now that the security in question pays dividends. For a given path $(X_0,\,X_1,\,...\,X_n,\,...)$, the total dividend income form holding such security from dates t_1 to t_m expressed in dollars at date t_0 is

[6] See L. Breiman: *Probability.*

$$\sum_{j=1}^{m} Z_{0\,m}\, D_j$$

Therefore, the value accumulated by holding this security from time t_0 to time t_m, measured in "t_0-dollars" would be

$$\widetilde{V}_m \;=\; \sum_{j=1}^{m} Z_{0\,m}\, D_j \;+ Z_{0\,m}\, P_m \;.$$

Using equation (18) we find that, for all $n \;>\; m$,

$$\begin{aligned}
\widetilde{V}_m \;&=\; \sum_{j=1}^{m} Z_{0\,m}\, D_j \;+ Z_{0\,m}\, P_m \\
&=\; \sum_{j=1}^{m} Z_{0\,m}\, D_j \;+ Z_{0\,m}\, \mathbf{E}_m \left\{ \sum_{j=m+1}^{n} Z_{m\,j}\, D_j \;+\; Z_{m\,n}\, P_n \right\} \\
&=\; \mathbf{E}_m \left\{ \sum_{j=1}^{m} Z_{0\,m}\, D_j \;+ Z_{0\,m} \left(\sum_{j=m+1}^{n} Z_{m\,j}\, D_j \;+\; Z_{m\,n}\, P_n \right) \right\} \\
&=\; \mathbf{E}_m \left\{ \, \widetilde{V}_n \, \right\} \;. \qquad (25)
\end{aligned}$$

Thus, the sequence of accumulated values, $\widetilde{V}_0$, $\widetilde{V}_1$, ... $\widetilde{V}_n$, behaves like a martingale sequence under the Arrow-Debreu probability.

We summarize the results of this section as

Theorem 1. *A necessary condition for the absence of arbitrage in this multi-period securities market is that there exists a probability measure $\hat{\Pi}_0$, defined on the set of forward state-paths $(X_0,\, X_1,\, \dots X_n,\, \dots)$ such that for all traded securities we have*

$$(i) \qquad P_0 \;=\; \mathbf{E}_0 \left\{ \sum_{j=1}^{n} \left(\prod_{i=0}^{j-1} \frac{1}{1+R_i} \right) D_j \;+\; \left(\prod_{i=0}^{n-1} \frac{1}{1+R_i} \right) P_n \right\} \qquad (26)$$

and, for all $0 \;<\; m \;<\; n$,

$$(ii) \qquad P_m = \mathbf{E}_m \left\{ \sum_{j=m+1}^{n} \left(\prod_{i=m}^{j-1} \frac{1}{1+R_i} \right) D_j + \left(\prod_{i=m}^{n-1} \frac{1}{1+R_i} \right) P_n \right\}, \tag{27}$$

where $\mathbf{E}_m$ represents the conditional expectation with respect to the probability $\hat{\Pi}_0$ conditional on the information at time t_m.

In particular the sequence of accumulated values between dates t_0 and t_n, measured in constant dollars,

$$\widetilde{V}_n = \sum_{j=1}^{n} \left(\prod_{i=0}^{j-1} \frac{1}{1+R_i} \right) D_j + \left(\prod_{i=0}^{n-1} \frac{1}{1+R_i} \right) P_n , \tag{28}$$

is a martingale sequence under this measure.

Notice how the concepts of martingale sequence and of risk-neutrality are strongly related: a risk-neutral individual will value a security as the expectation of its payoffs in the future measured in constant dollars, given the information available at the time.[7] In an inter-temporal setting, risk-neutral valuation under some statistical measure is equivalent to assuming that the accumulated value is statistically a martingale sequence.

Let us mention again the strong caveat regarding the distinction between "belief probabilities" of investors or historical probabilities (obtained from data analysis) and Arrow-Debreu probabilities. The latter are mathematical probabilities which must exist as a consequence of no-arbitrage. Their existence does not imply that accumulated gains from investments "are" martingales or that the markets present us with more or less permanent arbitrage opportunities – such contentions can easily dismissed by analyzing historical data.[8] The distinction comes from the fact that Arrow-Debreu measures provides a mathematical (objective) relation between today's prices and future values of investments, whereas historical and belief probabilities give rise to statistical (subjective) relations between prices and future cashflows. One can think of the Arrow-Debreu probabilities as being generated initially from historical probabilities by adjusting them according to the market's valuation of different states in the future.

(To be continued..)

[7] A *risk-averse* individual would not want to risk money in a fair bet, and would prefer to have the odds in his favor. In contrast, a *risk-prone* individual would think he can "beat the odds" and hence be willing to pay more than the fair value of the investment or game.

[8] It is a well-known fact that investment in the U.S. equity market (stocks) has a higher rate of return than the return on money-market securities (U.S. Treasury bills.)