Pricing Interest Rate Contingent Claims in Markets with Uncertain Volatilities

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ABSTRACT

We consider a model of a financial market where the volatility of the interest-rate is not known exactly, but rather it is assumed to lie within two a-priori known bounds. These bounds may represent for instance the extreme values of the implied volatility of liquidly traded options. In this model, the interest-rate process consistent with no-arbitrage and with the initial term-structure of forward rates is not determined uniquely. More precisely, there exists one interest-rate process for for each volatility path within the "band" determined by the minimal and maximal volatilities.

Due to uncertainty in the volatility, the present values of an interest-rate sensitive security cannot be determined exactly, unless it is equivalent to a series of discount bonds. Nevertheless, it is possible to calculate extreme values , corresponding to to worst-case scenarios of future volatility for short positions ("ask price") and long positions ("bid price") in any security or portfolio of securities. These extreme values are functions of the time-to-maturity, the current spot rate and an additional variable: the "accumulated variance". We show that the extreme prices can be found by solving a simple, nonlinear partial differential equations. In these equations, the "instantaneous", or "local" volatility used for pricing a a particular claim is determined dynamically: it is either the minimal or the maximal volatility according to the claim's convexity with respect to the state-variables.

A new feature of the model is that the value of a portfolio of interest-rate or bond options is different than the sum of the prices of the options taken separately. Thus, the model shows how volatility risk is diversified by holding mixed- Gamma positions. In particular, the model suggests that the capital required to hedge an

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option portfolio will be less than the sum of the present values of the options taken individually if the entire portfolio is hedged as a single contingent claim.

Numerical evidence is provided by comparing the present model to the standard Ho-Lee model. In particular, the effectiveness of the new pricing scheme for diversifying volatility risk is illustrated in the case of a spread of two bond options.

1 Introduction

The increasing popularity of interest rate contingent claims establishes the valuation of these securities as a significant problem in financial research. The *No-Arbitrage Yield-Curve Approach* (NA) introduced by Ho & Lee [5] and followed by Black & Derman & Toy [1], Black & Karasinski [2], Hull & White [3], constitutes an important progress in this problem. No-Arbitrage Yield-Curve models have the following properties: i) calculated bond prices fit *exactly* the present yield curve ii) the yield curve evolves in such a way that there exist no arbitrage opportunities from buying/selling bonds of different maturities. Simple NA models fit the only the initial term structure of forward interest rates [5], while more complex ones also fit the initial volatility structure [1, 3] and even the cap curve [2]. The state-of-the-art achievement in the NA approach is the Heath, Jarrow & Morton model [6], a multifactor model based on the joint evolution of forward rates.

A one-factor NA model can be understood as a binomial tree describing the evolution of short-term interest rate as a function of time and state of the world. Hull & White [4] describe how a derivative valuation is carried out in practice for one-factor models: the change in interest rate[1] during a small time interval is assumed to consist of a random jump centered around a deterministic mean, called drift. No-arbitrage requirements imply that the drift is completely determined by the current yield-curve and the volatility [6]. The volatility of the change in the interest rate, which is simply the standard deviation of the random jump, is assumed to be known *a-priori*. The value of any interest-contingent claim is calculated by working back through the tree: the value at a node is the discounted expectation of the values taken at node's leaves.

We have emphasized in the previous paragraph the assumption of an *a-priori* known volatility on purpose. It seems to us that this parameter is known only to the extent that it can be estimated from historical data of by fitting the model's prices to market prices. There is not a unique procedure for determining the volatility(ies) to input in NA interest-rate models from market data. Volatility shocks may arise, for instance, from exogenous factors, such as central bank monetary policy, political events, etc. It would be therefore desirable incorporate into NA models the inherent uncertainty of future rate volatility. This is the purpose of this work.

We shall postulate here that the information available for pricing a interest rate contingent derivative security consists of *lower* and *upper* bounds σ_{min} and σ_{max} on the instantaneous volatility of the interest-rate [2]. In this case, there is not a *unique* stochastic process for the evolution of interest rates which is consistent with no-arbitrage and fits the initial term structure of forward rates. Instead, there exist infinitely many binomial trees for the evolution of the interest-rate, namely one for each volatility path contained within the band $\sigma_{min} \leq \sigma(t) \leq \sigma_{max}$. The situation

[1] In the sequel we call the short term interest rate simply the interest rate.

[2] More generally, these upper and lower bounds could be time-dependent.

is interpreted as follows: we assume that investors will have different views on the forward volatility and thus value securities accordingly. Each investor constructs his or her tree for the interest rate evolution. From a "node", representing a certain value of interest rate R at a certain trading date t, stem two nodes representing the two possible values of the rate at the next trading date. These two values are assigned probabilities by the investor. The corresponding variance of the interest rate change represents the investor's subjective view on the volatility over that trading period. The volatility at each node is restricted only by the a-priori known lower and upper bounds σ_{min} and σ_{max}.

Let us define the *risk-averse prices*, or extremal prices, of an interest-contingent claim as its value to investors under *worst case* volatility scenarios: for the short position, the risk-averse price will be the highest realizable price given by a NA model over all possible volatility scenarios. For the long position it will be the lowest realizable price under all possible volatility scenarios.

To fix ideas, we assume that investors construct interest-rate trees according to the well-known Ho & Lee model, allowing for local (deterministic) changes in the volatility of the spot rate. Heath, Jarrow & Morton [6] established that the No-Arbitrage condition forces a relation between the mean and the term structure of volatility for interest rate process. This relation has a fundamental importance for our pricing equation so we recall it briefly. Consider the trading period which begins at the date t. The absence of yield-curve arbitrage opportunities implies that the average change in the interest rate over this period is equal to the sum of the change in initial forward rate curve over this interval and the variance of the interest rate change from time zero to time t — the *accumulated variance up to time t*. Accordingly,

$$\begin{aligned}\mathrm{drift}(t) \;&=\; f(t+\Delta t) \,-\, f(\Delta t) \;+\; \sum_{j=0}^{t/\Delta t - 1} \sigma^2(j\Delta t) \\ &= f(t+\Delta t) \,-\, f(\Delta t) \;+\; V(t) \quad ,\end{aligned}$$

where $f(t)$ represents the forward rate curve observed at time zero. At time t, the investor is thus left with the choice of up- and down-variations and corresponding probabilities, which should be consistent with the above drift and the volatility over the next period, $\sigma(t)$ (unknown).

The risk-averse prices of an interest-contingent claim at time t are determined by the volatility bounds $\sigma_{min}\,\sigma_{max}$ and the two state-variables $R(t)$ and $V(t)$. We assume that the spot rate $R_(t)$ and the accumulated variance $V(t)$ are observable at time t by all investors. At this time, if $s > t$, then $R(s)$ is random and $V(s)$ will be viewed differently by different investors.

Our ultimate goal is to understand risk-averse pricing in a continuous-time economy with uncertain volatility with known a-priori bounds. We shall follow the Cox, Ross & Rubinstein [10] approach: we approximate a continuous economy by a discrete economy with a very large number of short trading periods. The volatility realized at nodes of investor's trees is scaled so that the *annualized* volatility remains bounded even when the number of trading periods within, say, ten years, is very large. We develop *recurrence relations* for the risk-averse (worst case scenario) prices. As the duration of the time period between trades decreases, the Taylor expansion of the recurrence relation yields limiting partial differential equations for the prices.

The notion of pricing contingent claims with respect to the lower and upper bounds on volatility was originally developed by Avellaneda, Levy & Paras [9], [8] in the context of equity derivatives. In that case, the pricing scheme is based on the hedge between the derivative and the stock. Here, hedging is done with cash (short-term funds) and longer-term fixed-income securities.

In our model the upper risk averse price ("ask price" for short) $\mathbf{B}$ of an interest-contingent claim satisfies the partial differential pricing equation in t, R, V:

$$\mathbf{B}_t + (f'(t) + V)\mathbf{B}_R + \sigma^2 \left[\mathbf{B}_V + \frac{1}{2}\mathbf{B}_{RR}\right] (\mathbf{B}_V + \frac{1}{2}\mathbf{B}_{RR}) - R\mathbf{B} = 0, \tag{1.1}$$

with the final condition defined at time $t = T$, the maturity of the claim, and determined by the payoff function. The *"ask" volatility coefficient* $\sigma[\cdot]$ is given by

$$\sigma[X] = \begin{cases} \sigma_{max} & \text{if} \quad X \geq 0, \\ \sigma_{min} & \text{if} \quad X < 0, \end{cases} \tag{1.2}$$

The lower bound on the price of the contingent claim ("bid price") is determined by equation (1.1) but with a *"bid" volatility coefficient*, obtained by interchanging (1.2) σ_{min} and σ_{max} in (1.2).

Equation (1.1) is used in practice as follows. Consider the case of a European-style call option, with exercise price K and maturity τ on a \$1 face-value zero-coupon bond with maturity $T > \tau$. To find the price of the option, we first find the (upper) price of the bond $\mathbf{B}(t, R, V; T)$. It satisfies the equation (1.1) for t between zero and T with the final condition $\mathbf{B}(T, R, V; T) = 1$. Then, we find the price of the option $\mathbf{O}(t, R, V; \tau)$ by solving equation (1.1) for t between zero and τ with the final condition at $t = \tau$

$$\max(\mathbf{B}(\tau, R, V; T) - K, 0). \tag{1.3}$$

As the accumulated variance is zero at time $t = 0$, the option's price is given by $\mathbf{O}(0, R, 0; \tau)$ where R is the initial interest rate.

The call option example illustrates a general property of our pricing scheme: at time $t = 0$, the risk-averse price of a contingent claim depends on the time to maturity τ, the minimal and maximal annualized volatilities σ_{min} and σ_{max}, the initial instantaneous forward rate curve f and the final payoff.

We offer the following financial interpretation of the pricing equation (1.1). In a short time interval Δt the interest rate changes by an amount ΔR and the accumulated variance changes by an amount ΔV. As a result, the value of the contingent claim will change by $\Delta\mathbf{B}$. $\Delta\mathbf{B}$ consists of a random, mean-zero component and a deterministic component which depends on the realized volatility. Because our analysis is carried out in a risk-neutral world, the value of the contingent claim is its *expected value discounted at the current interest rate*. Since the volatility of the interest rate over the next trading period is not known, a fully risk-averse investor will base the value of a short (long) position on the maximal (minimal) possible value of $\Delta\mathbf{B}$. Applying the Taylor expansions, we find that the volatility-dependent deterministic component in $\Delta\mathbf{B}$ is simply $\sigma^2 X$ where σ^2 is the interest rate volatility during time interval Δt and $X \equiv \mathbf{B}_V + \frac{1}{2}\mathbf{B}_{RR}$. Thus, in the case of a short position, the derivative is valued with $\sigma^2 = \sigma^2_{\text{max}}$ if $X \geq 0$ and $\sigma^2 = \sigma^2_{\text{min}}$ if $X < 0$. This is fully consistent with definition (1.2): the risk-averse price evolution equation is simply based on the curvature of the price as a function of the interest rate and volatility. The above intuitive argument can be made fully rigorous following Hull's derivation of the Black-Scholes equation [11]. In this paper a different approach, in which the pricing equation is obtained as the approximation to a backward-induction scheme for a discrete economy with many trading periods.

The interest of pricing contingent claims with equation (1.1) lies in the fact that the model is capable of accounting for diversification of volatility risk. In fact, equation (1.1) represents the premium that will be acceptable to an investor that will bear both the "price risk" and the "volatility risk" associated with the short position. It is then clear that the worst-case volatility scenario may be different according the particular contingent claim that is being considered. Thus, a portfolio of options, with some options held long and others held short, will have a lower "ask price" than if the options in the portfolio were priced separately using (1.1) and the prices added together. This result is easy to verify mathematically and follows almost immediately from the definition of extremal prices. The nonlinearity of the "ask" volatility coefficient causes the price of a portfolio *not* to be the sum of the prices of its components in general. This suggests that investors or market-makers with inventories of contingent claims will be willing to quote narrower bid/offer spreads for buying and selling derivative securities than those with smaller endowments, while assuming the same level of volatility risk.

We illustrate this risk-diversification effect by pricing a call spread in aggregate fashion and separately using (1.1) for a call spread. We also compare the prices obtained with (1.1) with those obtained using the Ho-Lee model with maximal and minimal volatilities.

The rest of this paper is organized as follows. In Section 2 we illustrate the effectiveness of our pricing scheme with numerical calculations of the calendar spread value. Section 3 and 4 describe the discrete economy: the No-Arbitrage-restricted interest rate process and the recurrence relation for the price of a derivative. In Section 5 we derive the continuous-time pricing equation from the discrete-time recurrence relation of Section 4. The exact form of the interest rate process is rigorously inferred from the No-Arbitrage condition in Appendix A. In Appendix

Figure 1: Pricing curves for a call spread on a bond with the face value of 100$ and 2 years to the maturity, with the maturity of the spread in 1 year and exercise prices 85$ and 95$. The continuous lines are the asking and bidding prices as given by our model. The dashed lines are the values obtained by pricing both calls separately. The dashed-dotted line is the value at the middle volatility. The initial instantaneous forward rate is flat. The minimal and maximal volatilities are $\sigma^2_{\min} = 0.05$ and $\sigma^2_{\max} = 0.15$.

B we present a hedge of the derivative against the zero-coupon bond.

2 Numerical Calculations

The asking and bidding prices of a derivative are found by solving the pricing equation (1.1) from the maturity of the derivative backwards to the present time. We note, however, that the bond prices in our model are known in a closed form. This fact has important consequences: solving the pricing equation for a bond-based derivative, we start at the maturity of the derivative instead of the maturity of the bond. The actual formula for the time t value of the bond with maturity T is simply[3]

$$\mathbf{P}(t,T) = \exp\Big(-R(T-t) - \frac{V}{2}(T-t)^2 - \int_t^T (f(s) - f(t))\, ds\Big). \qquad (2.1)$$

The effectiveness of our model is clearly visible in valuation of option portfolios. A simple portfolio consisting of a single call on a bond is valued at the highest

[3]Note that for $V = \sigma^2 t$ we recover the standard Ho-Lee value of the bond [11].

R (%)	$\mathrm{HL}_{\mathrm{min}}^{\mathrm{max}}$	Ask	$\mathrm{HL}_{\mathrm{mid}}$	Bid	$\mathrm{HL}_{\mathrm{max}}^{\mathrm{min}}$
4	9.767	8.431	6.905	5.391	4.632
8	8.227	6.732	5.344	3.943	2.812
12	6.266	5.073	3.800	2.528	1.124
16	4.409	3.656	2.453	1.244	−0.334
20	2.928	2.514	1.424	0.401	−0.467

Table 1: Numerical values corresponding to Figure 1 for different levels of the interest rate. The Ask, Bid are the prices given by our model; $\mathrm{HL}_{\mathrm{min}}^{\mathrm{max}}$, $\mathrm{HL}_{\mathrm{max}}^{\mathrm{min}}$ are the prices when the calls are valued separately at their maximal and minimal, minimal and maximal volatility according to the standard Ho-Lee model; $\mathrm{HL}_{\mathrm{mid}}$ is the price at the middle volatility.

volatility σ^2_{max} for selling and at the lowest volatility σ^2_{min} for buying. This fact is a result of the positive curvature $X = \frac{1}{2}\mathbf{B}_{RR}+\mathbf{B}_V$ of the final payoff function for the call and the *curvature preserving* character of the pricing equation. Consequently, the asking (bidding) price of the call in our model is given by the standard Ho-Lee formula [11] with volatility equal to σ^2_{max} (σ^2_{min}). However, a more complex portfolio that has a mixed curvature X is priced with a dynamically selected volatility. In effect, the asking price of such a portfolio is *lower* than the sum of the prices of its elements and the bidding price of the portfolio is *higher* than the sum of the prices of its elements.

We illustrate the above considerations by pricing a spread. In Figure 1 and Table 1 we present the equation (1.1) based asking and bidding prices for the spread together with the values obtained by decomposing the spread into two calls and pricing them separately. We show also the prices at middle volatility σ^2_{mid}. They are calculated independently by the finite-difference solver (with minimal and maximal volatilities set to σ^2_{mid}) and the standard Ho-Lee model formulas. The agreement between these numbers serves as a measure of quality of our finite-difference solver.

3 Interest Rate Process

We consider a discrete trading economy with the basic time unit, i.e. the minimal time between trades, equal Δt, and all trading dates being multiples of it. The economy consists of a large number of investors, each with its own view on the evolution of the interest rate. In this section we describe a simple model of the interest rate process as it is seen by a *particular investor*. This process is constrained by the No-Arbitrage condition. Heath, Jarrow & Morton [6] were the first to observe the relation between the drift, i.e. the mean value, of the interest rate and its volatility, i.e. its variance. While we derive this relation for a discrete economy with changing volatility in the Appendix, here we simply state the assumed form of the interest rate process.

We assume a particular investor sees the interest rate $R(t)$ as a binomial process. Given the value $R(t-\Delta t)$ a time $t-\Delta t$ there are *two* possible values of $R(t)$ time t : $R^U(t)$ and $R^D(t)$. These two values are taken with probabilities π_t^U and π_t^D, s.t. $\pi_t^U + \pi_t^D = 1$, i.e.

$$R(t) = \begin{cases} R^U(t) & \text{with probability } \pi_t^U, \\ R^D(t) & \text{with probability } \pi_t^D. \end{cases} \tag{3.1}$$

Denoting the *mean* value of $R(t)$ by $R^M(t)$ we write the upper and lower values of $R(t)$ as variations around the mean:

$$\begin{aligned} R^U(t) &= R^M(t) + U_t \Delta t^{1/2}, \\ R^D(t) &= R^M(t) - D_t \Delta t^{1/2}, \end{aligned} \tag{3.2}$$

where $(U_t, D_t, \pi_t^U, \pi_t^D)$[4] is constrained by the fact that $R^M(t)$ is a mean of $R(t)$:

$$U_t \pi_t^U - D_t \pi_t^D = 0. \tag{3.3}$$

Note that the variations of the interest rate $R(t)$ around it mean $R^M(t)$ are scaled according to $\Delta t^{1/2}$. This scaling achieves a finite annualized volatility of the interest rate for any Δt.

The No-Arbitrage condition demands: a *fourplet* $(U_t, D_t, \pi_t^U, \pi_t^D)$ is chosen by an investor according to its own view. The mean value $R^M(t)$, however, is *completely* determined by

- initial forward rate change $\Delta f(t) \equiv f(t) - f(t-\Delta t)$,
- the $t-\Delta t$ value of interest rate $R(t-\Delta t)$,
- *all previous* investor's choices of fourplets $(U_s, D_s, \pi_s^U, \pi_s^D)$ at trading dates between time zero and time $t-\Delta t$:

$$R^M(t) = R(t-\Delta t) + \Delta f(t) + \Delta t \sum_{j=1}^{t/\Delta t - 1} \Delta t\, \sigma^2(j\Delta t), \tag{3.4}$$

where $\sigma^2(s)$ is the annualized volatility of the interest rate over the trading period $[s, s+\Delta t]$:

$$\sigma^2(s) = \frac{1}{\Delta t} \mathrm{Var}_{s-\Delta t}\left\{R(s)\right\} = U_s^2 \pi_s^U + D_s^2 \pi_s^D. \tag{3.5}$$

Note that as $\Delta t\, \sigma^2(s)$ is the volatility realized at time s, the sum in formula (3.4) is the accumulated variance between time zero and time $t-\Delta t$. This sum has magnitude $O(1)$. Thus, the accumulated variance remains finite for any Δt.

[4] The simplest binomial process occurs when $U_t = D_t = r$ and $\pi_t^U = \pi_t^D = 1/2$, i.e. the rate moves equally likely up or down form its mean by r.

It is convenient to denote the accumulated variance at the time t as $V(t)$:

$$V(t) = \sum_{j=1}^{t/\Delta t} \Delta t \, \sigma^2(j\Delta t). \tag{3.6}$$

Concluding, a particular investor's view of the interest rate evolution process is fully described by a *binomial tree of fourplets* (U, D, π^U, π^D). A fourplet $(U_t, D_t, \pi_t^U, \pi_t^D)$ assigned to a particular time $t - \Delta t$ node of the tree determines the two possible values of the interest rate at time t according to the formula

$$R(t) = R(t-\Delta t) + \Delta f(t) + V(t-\Delta t)\Delta t + \begin{cases} +U_t \Delta t^{1/2} & \text{with probability} \quad \pi_t^U; \\ -D_t \Delta t^{1/2} & \text{with probability} \quad \pi_t^D. \end{cases} \tag{3.7}$$

The accumulated variance $V(t)$ at a node is determined as a sum of volatilities realized along the path of the binomial tree leading to this particular node. Its value at a time t node stemming from some time $t - \Delta t$ node with accumulated variance $V(t - \Delta t)$ is given by

$$V(t) = V(t-\Delta t) + \mathrm{Var}_{t-\Delta t}\{R(t)\} = V(t-\Delta t) + (U_t^2 \pi_t^U + D_t^2 \pi_t^D)\Delta t. \tag{3.8}$$

We stress here a following observation: although the evolution of interest rate $R(t)$ from time t onward depends on all choices of fourplets taken between time zero and time t, this dependence is through a simple quantity: the accumulated variance $V(t)$. Once this quantity is known, in order to build the binomial tree of interest rates for times $s > t$ starting from a particular node with the level of interest rate $R(t)$, it is enough to remember just $V(t)$: the accumulated variance necessary to construct $R(s)$ at s is a sum of the volatilities realized between time t and time s along the path leading to s and the accumulated variance $V(t)$. In other words, the formulas (3.7) and (3.8) constitute a *recurrence relation* for the interest rate R and the accumulated variance V that can be used to construct the tree of interest rates from the tree of fourplets.

The simplest model for the interest rate, as described above, involves just two fourplets: $(\sigma_{\min}, -\sigma_{\min}, \frac{1}{2}, \frac{1}{2})$ and $(\sigma_{\max}, -\sigma_{\max}, \frac{1}{2}, \frac{1}{2})$. In this process the interest rate increases and decreases with *equal probability* and the upward and downward variations from its mean value have *equal magnitude*. An investor chooses at a node only a lower or upper value of volatility applicable to the present trading period.

We assume that there exists only one restriction on the investor's fourplets. There exist two, a-priori known numbers $\sigma_{\min}^2$ and $\sigma_{\max}^2$, specific to the whole economy, that restrict all possible fourplets (U, D, π^U, π^D) according to

$$\sigma_{\min}^2 \leq U^2 \pi^U + D^2 \pi^D \leq \sigma_{\max}^2. \tag{3.9}$$

We finally note that a choice of a fixed fourplet (U, D, π^U, π^D) for all nodes of the tree makes the limiting (as $\Delta t \downarrow 0$) process for the interest rate R to be the standard Ho-Lee interest rate process with annualized volatility $\sigma^2 = U^2\pi^U + D^2\pi^D$. Thus, the multiple tree discrete economy contains approximations of all standard Ho-Lee interest rate process with annualized volatilities σ^2 between $\sigma^2_{\min}$ and $\sigma^2_{\max}$. However, it clearly contains also much more: interest rate processes where the volatility σ^2 is a function of the interest rate level itself $\sigma^2 = \sigma^2(R)$, processes where the volatility depends on the whole history of the interest rate until the current time (path-dependent models), etc. It is a truly "rich" economy.

4 Price Recurrence Relation

The risk-averse price of a derivative in an economy with many possible interest rate evolution process is a *worst case scenario* price. We derive here a recurrence relation for this price as a function of current time, interest rate and accumulated variance. To fix our attention we consider here the asking price, i.e. the price of a short derivative. The biding price, i.e. the price of a long derivative, is found analogously.

We first recall how a particular investor, say investor $\mathcal{I}$, prices a derivative on the basis of the binomial tree for the interest rate. We call this price the $\mathcal{I}$-price. Let $\mathrm{Tr}\{t\}$ denote a binomial tree of fourplets extending between the present time t and the maturity of the derivative T expressing investor $\mathcal{I}$ view on the interest rate evolution. Let the time t value of the interest rate and the accumulated variance by R and V. Construct a binomial tree of interest rates: at time t start from the interest rate R and the accumulated variance V and follow the interest rate evolution process given by formulas (3.7) and (3.8) in Section 3. The $\mathcal{I}$-price at time t of a derivative with a known payoff at maturity is found by working back through the interest rate tree from maturity T to the present time t according to the standard *expectations procedure:* Let at the node $\mathcal{N}$ the interest rate equal R. Let $\mathcal{N}^U$ and $\mathcal{N}^D$ be two nodes stemming out of the node $\mathcal{N}$ and describing the upper and lower values of the interest rate at the next trading date. The nodes $\mathcal{N}^U$ and $\mathcal{N}^D$ are assumed with probabilities π^U and π^D. If $\mathbf{B}^U$ and $\mathbf{B}^D$ are the $\mathcal{I}$-prices of the derivative at nodes $\mathcal{N}^U$ and $\mathcal{N}^D$ then the $\mathcal{I}$-price of the derivative at the node $\mathcal{N}$ is simply

$$e^{-R\Delta t}(\pi^U\mathbf{B}^U + \pi^D\mathbf{B}^D). \tag{4.1}$$

Applying rule (4.1) repeatedly, investor $\mathcal{I}$ finds its $\mathcal{I}$-price at time t. Let us denote this $\mathcal{I}$-price by $\mathbf{B}(t,R,V;\mathrm{Tr}\{t\})$. The *risk-averse asking price* of the derivative is simply the *maximal*[5] $\mathcal{I}$-price over all investors (or alternatively over all trees of fourplets):

$$\mathbf{B}(t,R,V) = \max_{\mathrm{Tr}=\mathrm{Tr}\{t\}} \mathbf{B}(t,R,V;\mathrm{Tr}). \tag{4.2}$$

In order to simplify the derivation of the recurrence relation, we assume that every investor chooses fourplets (U,D,π^U,π^D) from a finite, but perhaps very

[5] The biding price is the *minimal* $\mathcal{I}$-price over all investors.

large set $\{(U_i, D_i, \pi_i^U, \pi_i^D) : i = 1, M\}$. Denote the fourplet $(U_i, D_i, \pi_i^U, \pi_i^D)$ by $\mathcal{F}_i$, its variance by σ_i^2:

$$\sigma_i^2 = U_i^2 \pi_i^U + D_i^2 \pi_i^D. \tag{4.3}$$

Similarly, let $\mathcal{F}(t)$ denote a time t fourplet $(U_t, D_t, \pi_t^U, \pi_t^D)$. Finally, let $\mathrm{Tr}\{t|\mathcal{F}(t) = \mathcal{F}_i\}$ denote a binomial tree of fourplets starting at time t with the fourplet $\mathcal{F}_i$. Then, clearly

$$\mathbf{B}(t, R, V) = \max_i \left\{ \max_{\mathrm{Tr}=\mathrm{Tr}\{t|\mathcal{F}(t)=\mathcal{F}_i\}} \mathbf{B}(t, R, V; \mathrm{Tr}) \right\}, \tag{4.4}$$

as the time t fourplet $\mathcal{F}(t)$ in any tree Tr must be one of the fourplets $\mathcal{F}_i, i = 1, M$.

Now, let i be fixed. We show that

$$\begin{aligned} \max_{\mathrm{Tr}=\mathrm{Tr}\{t|\mathcal{F}(t)=\mathcal{F}_i\}} \mathbf{B}(t, R, V; \mathrm{Tr}) &= e^{-R\Delta t} \times \\ \Big(\pi_i^U \max_{\mathrm{Tr}=\mathrm{Tr}\{t+\Delta t\}} &\mathbf{B}(t + \Delta t, R + U_i \Delta t^{1/2} + \Delta f + V\Delta t, V + \sigma_i^2 \Delta t; \mathrm{Tr}) \\ +\pi_i^D \max_{\mathrm{Tr}=\mathrm{Tr}\{t+\Delta t\}} &\mathbf{B}(t + \Delta t, R - D_i \Delta t^{1/2} + \Delta f + V\Delta t, V + \sigma_i^2 \Delta t; \mathrm{Tr})\Big) \end{aligned} \tag{4.5}$$

The relation (4.5) is a consequence of the following observation. The maximum in the L.H.S. of (4.5) is *achieved* at some tree $\mathrm{Tr}\,\{t|\mathcal{F}(t) = \mathcal{F}_i\}$, because there is only a finite number of possible trees. We denote the maximizing tree by Tr^*. We form two new trees $\mathrm{Tr}^D\{t + \Delta t\}$ and $\mathrm{Tr}^U\{t + \Delta t\}$ that start at time $t + \Delta t$ from the upper and lower nodes of Tr^*. In other words, the tree Tr^* consists of Tr^U and Tr^D bound together by time t node $\mathcal{F}_i$. By definition, the value $\mathbf{B}(t, R, V; \mathrm{Tr}^*)$ is discounted by $e^{-R\Delta t}$ expected value found on trees Tr^U and Tr^D:

$$\begin{aligned} \mathbf{B}(t, R, V; \mathrm{Tr}^*) = e^{-R\Delta t}(\pi_i^U &\mathbf{B}(t + \Delta t, R_i^U, V + \sigma_i^2 \Delta t; \mathrm{Tr}^U) \\ &+\pi_i^D \mathbf{B}(t + \Delta t, R_i^D, V + \sigma_i^2 \Delta t; \mathrm{Tr}^D)), \end{aligned} \tag{4.6}$$

where R_i^U, R_i^D are values of the interest rate $R(t + \Delta t)$ at time $t + \Delta t$ dictated by R, V and $\mathcal{F}_i$ according to the equation (3.7):

$$\begin{aligned} R_i^U &= R + U_i \Delta t^{1/2} + \Delta f(t + \Delta t) + V\Delta t, \\ R_i^D &= R - D_i \Delta t^{1/2} + \Delta f(t + \Delta t) + V\Delta t. \end{aligned} \tag{4.7}$$

Therefore, the R.H.S. of (4.5) is larger (or equal) than $\mathbf{B}(t, R, V; \mathrm{Tr}^*)$. Similarly, both maxima in the R.H.S. of (4.5) are achieved at some trees $\mathrm{Tr}^U\{t + \Delta t\}$ and $\mathrm{Tr}^D\{t + \Delta t\}$. By binding these two trees together with a time t node $\mathcal{F}_i$ we construct a new tree $\mathrm{Tr}^*\{t\}$ starting at time t. Again the value $\mathbf{B}(t, R, V; \mathrm{Tr}^*)$ is exactly equal to the R.H.S. of (4.5). Thus, the L.H.S. of (4.5) is larger (or equal) than $\mathbf{B}(t, R, V; \mathrm{Tr}^*)$. This proves the equality between both sides of (4.5).

The recurrence relation follows easily from (4.4) and (4.5):

$$\mathbf{B}(t, R, V) = \max_{\mathrm{Tr}=\mathrm{Tr}\{t\}} \mathbf{B}(t, R, V; \mathrm{Tr})$$

$$
\begin{aligned}
&= \max_i \Big\{ e^{-R\Delta t} \pi_i^U \max_{\mathrm{Tr}=\mathrm{Tr}\{t+\Delta t\}} \mathbf{B}(t+\Delta t, R_i^U, V+\sigma_i^2\Delta t; \mathrm{Tr}) \\
&\qquad + e^{-R\Delta t} \pi_i^D \max_{\mathrm{Tr}=\mathrm{Tr}\{t+\Delta t\}} \mathbf{B}(t+\Delta t, R_i^D, V+\sigma_i^2\Delta t; \mathrm{Tr}) \Big\} \\
&= e^{-R\Delta t} \max_i \Big\{ \pi_i^U \mathbf{B}(t+\Delta t, R_i^U, V+\sigma_i^2\Delta t) \\
&\qquad\qquad + \pi_i^D \mathbf{B}(t+\Delta t, R_i^D, V+\sigma_i^2\Delta t) \Big\}
\end{aligned} \tag{4.8}
$$

This recurrence relation is *nonlinear* and it relates the time t value of $\mathbf{B}$ at the interest rate R and the accumulated variance V to time $t+\Delta t$ values of $\mathbf{B}$ at the new up and down levels of interest rate *and* new accumulated variance.

5 Continuous Economy

Consider a discrete economy where the minimal time between trades Δt tends to zero: in the limit a binomial tree for the interest rate evolution over trading periods becomes a continuous time process, and any function of the trading period number becomes a function of the continuous time. The family of continuous time process which arises as limits of interest rate binomial trees of Section 3 constitutes the *continuous economy.*

We analyze her in detail the price recurrence relation derived in Section 4. We show that in the limit $\Delta t \downarrow 0$ the price satisfies a non-linear partial differential equation, which we call a *pricing equation in the continuous economy.* We also show that in a special case of *no uncertainty* in volatility, the equation reduces to the standard Ho-Lee model Black-Scholes equation.

The derivation is based on the Taylor expansion of the recurrence relation (4.8), which we repeat here for completeness:

$$
\begin{aligned}
e^{R\Delta t}\mathbf{B}(t,R,V) \quad = \quad & \max_i \Big\{ \pi_i^U \mathbf{B}(t+\Delta t, R+U_i\Delta t^{1/2}+\Delta f+V\Delta t, V+\sigma_i^2\Delta t) \\
& + \pi_i^D \mathbf{B}(t+\Delta t, R-D_i\Delta t^{1/2}+\Delta f+V\Delta t, V+\sigma_i^2\Delta t) \Big\}.
\end{aligned} \tag{5.1}
$$

We observe that the Δt change in the time coordinate t results in the $O(\Delta t^{1/2})$ change in the interest rate coordinate R and the $O(\Delta t)$ change in the accumulated variance coordinate V, as it is apparent from the R.H.S. of (5.1). Consequently, the following Taylor expansion of $\mathbf{B}$ around the point (t,R,V) is useful:

$$
\mathbf{B}(t+\Delta t, R, +\Delta R, V+\Delta V) \simeq \mathbf{B} + \Delta t\mathbf{B}_t + \Delta R\mathbf{B}_R + \Delta V\mathbf{B}_V + \frac{1}{2}(\Delta R)^2\mathbf{B}_{RR}, \tag{5.2}
$$

which is accurate up to the error of order $O(\Delta t^{3/2})$ *provided* that $\Delta V \sim O(\Delta t)$ and $\Delta R \sim O(\Delta t^{1/2})$. We apply the expansion (5.2) to a term under maximum in the R.H.S. of (5.7), obtaining, up to the error $O(\Delta t^{3/2})$,

$$\pi_i^U\left(\mathbf{B}+\Delta t\mathbf{B}_t+(+U_i\Delta t^{1/2}+f'\Delta t+V\Delta t)\mathbf{B}_R+\sigma_i^2\Delta t\mathbf{B}_V+\frac{1}{2}U_i^2\Delta t\mathbf{B}_{RR}\right)$$
$$+\pi_i^D\left(\mathbf{B}+\Delta t\mathbf{B}_t+(-D_i\Delta t^{1/2}+f'\Delta t+V\Delta t)\mathbf{B}_R+\sigma_i^2\Delta t\mathbf{B}_V+\frac{1}{2}D_i^2\Delta t\mathbf{B}_{RR}\right)$$
$$=\mathbf{B}+\Delta t\mathbf{B}_t+(f'+V)\Delta t\mathbf{B}_R+\sigma_i^2\Delta t\mathbf{B}_V+\frac{1}{2}\sigma_i^2\Delta t\mathbf{B}_{RR}, \tag{5.3}$$

where we used the fact that σ_i^2 is the volatility of the fourplet $(U_i, D_i, \pi_i^U\pi_i^D)$, see (4.3). Now, the approximation $e^{R\Delta t}=1+\Delta tR$ yields that the L.H.S. of (5.1) equals, up to the error $O(\Delta t^2)$,

$$\mathbf{B}+\Delta tR\mathbf{B}. \tag{5.4}$$

Applying expansions (5.3) and (5.4) to the recurrence relation (5.1) we obtain a relation that contains terms of order $O(\Delta t)$ and higher, only. Thus, dividing the resulting relation by Δt and letting Δt tend to zero, we derive the following equation

$$R\mathbf{B}=\max_i\left\{\mathbf{B}_t+(f'+V)\mathbf{B}_V+\sigma_i^2(\mathbf{B}_V+\frac{1}{2}\mathbf{B}_{RR})\right\}. \tag{5.5}$$

However, the maximum in the R.H.S. of (5.5) is achieved either for $\sigma_i^2=\sigma_{\max}^2$ or $\sigma_i^2=\sigma_{\min}^2$ depending on the sign of $X=\mathbf{B}_V+\frac{1}{2}\mathbf{B}_{RR}$:

$$\mathbf{B}_t+(f'+V)\mathbf{B}_V+\sigma^2[\mathbf{B}_V+\frac{1}{2}\mathbf{B}_{RR}](\mathbf{B}_V+\frac{1}{2}\mathbf{B}_{RR})-R\mathbf{B}=0, \tag{5.6}$$

Where the volatility coefficient $\sigma^2[\cdot]$ is given by

$$\sigma^2[X]=\begin{cases}\sigma_{\max}^2 & \text{if } X\geq 0\\ \sigma_{\min}^2 & \text{if } X<0.\end{cases} \tag{5.7}$$

Equation (5.6) constitutes the pricing equation for the continuous economy with uncertain volatility constrained by the lower and upper bounds $\sigma_{\min}^2$ and $\sigma_{\max}^2$.

The standard Ho-Lee model Black-Scholes equation is a special case of equation (5.6). This is seen as follows. Assume that the maximal and minimal values of volatility are equal: $\sigma_{\min}^2=\sigma_{\max}^2=\sigma^2$. The equation (5.6) becomes a *linear* differential equation as the volatility coefficient $\sigma^2[\cdot]$ becomes constant: $\sigma^2[\cdot]=\sigma^2$. Note that the dependence of the solution $\mathbf{B}$ of equation (5.6) on the interest rate variable R is significantly different from the dependence on the accumulated variance variable V [6]. It is not difficult to check that if $\mathbf{B}$ is a solution of equation (5.6) then a function $\mathbf{A}(t,R,V)$ defined by

$$\mathbf{A}(t,R,V)=e^{-Vt^2/2}\mathbf{B}(t,R+Vt,V+\sigma^2t), \tag{5.8}$$

[6] Mathematically, equation (5.6) is *parabolic* in R and *hyperbolic* in V.

satisfies the following partial differential equation:

$$\mathbf{A}_t + (f' + \sigma^2 t)\mathbf{A}_R + \frac{1}{2}\mathbf{A}_{RR} - R\mathbf{A} = 0. \tag{5.9}$$

However, as the equation (5.9) *does not* involve variable V, the accumulated variance V is only an external parameter in the solution $\mathbf{A}(t, R, V)$ of the equation (5.9). Because at time $t = 0$ the accumulated variance is zero, the *financially interesting* solution of equation (5.9) is the one with the parameter V set to zero, i.e. $\mathbf{A}(t, R, 0)$. According to the $\mathbf{A}$-to-$\mathbf{B}$ relation (5.8), $\mathbf{A}(t, R, 0) = \mathbf{B}(t, R, \sigma^2 t)$[7]. Thus, in the standard Ho-Lee model with volatility σ^2, the price of a derivative is given by the solution to equation (5.9), which is a well known fact [12]. This shows that our model of uncertain volatility economy is an extension of the standard Ho-Lee model.

6 Conclusions

We have developed a simple No-Arbitrage model for pricing interest rate contingent claims in a market where the volatility of the interest rate is not known exactly, and, consequently, the interest rate process is not uniquely determined. In our model, the value of a derivative satisfies a new, non-linear pricing partial differential equation, which generalizes the Black-Scholes equation for the Ho-Lee model. The derivation of the pricing equation is based on the worst case prediction for the possible value of the investor's position. We present a hedge against a zero-coupon bond which assures the pricing equation value of the position. The model generates the spread in the ask/bid prices in a natural way: it is caused by two different Δ-hedges, one for a short and the other for a long position, in the presence of non-uniqueness of the interest rate process.

The prices of zero-coupon bonds and simple puts and calls are known explicitly in our model. We illustrate the effectiveness of our scheme by pricing a calendar spread.

Our model does not posses the mean-reversion of the interest rate. However, an extension of our approach to the Vasicek mean-reverting model is straightforward and it will be carried out in a separate paper. Similarly, the generalization to a *multifactor* model is possible as well and it is in preparation.

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7 Appendix A: Interest Rate Process

The No-Arbitrage condition constrains the evolution of the interest rate in a significant way [6]. In this section we derive the No-Arbitrage form of the interest rate process as observed by one particular investor.

Let $f(t,T)$ denote the *forward rate* at time t for the time interval $[T, T+\Delta t]$. Let $\mathbf{P}(t,T)$ denote the price at time t of a *zero-coupon bond* with maturity T. The face value of the bond is 1\$, i.e. $\mathbf{P}(T,T)=1$. The relation between the price of the bond and the forward rate is given by

$$\mathbf{P}(t,T+\Delta t)=\mathbf{P}(t,T)\exp(-\Delta t f(t,T)). \tag{7.1}$$

Consequently

$$\mathbf{P}(t,T)=\exp\Big(-\Delta t\sum_{j=t/\Delta t}^{T/\Delta t-1} f(t,j\Delta t)\Big). \tag{7.2}$$

We assume that the forward rate $f(t,T)$, when observed as a function of time t, initiates as a fixed, *deterministic* rate $f(0,T)=f_0(T)$ and it evolves by *random* jumps $\xi(t,T)$:

$$f(t,T)=f(t-\Delta t,T)+\xi(t,T). \tag{7.3}$$

We consider here a simple *two-state* model: forward rates of all maturities T either all go up or all go down between time $t-\Delta t$ and the time t. If the *upstate* prevails, then $\xi(t,T)=a(t,T)$ for all T, if the *downstate* prevails then $\xi(t,T)=b(t,T)$, where $a(t,T)$ and $b(t,T)$ are *constants* such that $a(t,T)\geq b(t,T)$.

The No-Arbitrage condition is easily understood in the context of the relation between bond prices at two consecutive trading dates. We derive from (7.2) and (7.3) that

$$\begin{aligned}
\mathbf{P}(t,T) &= \exp\Big(-\Delta t\sum_{j=t/\Delta t}^{T/\Delta t-1} f(t,j,\Delta t)\Big)\\
&= \exp\Big(-\Delta t\sum_{j=t/\Delta t-1}^{T/\Delta t-1} f(t-\Delta t,j\Delta t)+\Delta t f(t-\Delta t,t-\Delta t)\\
&\qquad\qquad -\Delta t\sum_{j=t/\Delta t}^{T/\Delta t-1}\xi(t,j\Delta t)\Big)\\
&= \exp\Big(-\Delta t\sum_{j=t/\Delta t}^{T/\Delta t-1}\xi(t,j\Delta t)\Big)\frac{\mathbf{P}(t-\Delta t,T)}{\mathbf{P}(t-\Delta t,t)}\\
&= h(t,T)\frac{\mathbf{P}(t-\Delta t,T)}{\mathbf{P}(t-\Delta t,t)}.
\end{aligned} \tag{7.4}$$

Note that according to our assumption of the two-state model, the function $h(t,T)$ takes only two values

$$h(t,T)=\begin{cases} h^U(t,T)\equiv\exp(-\Delta t\sum a(t,T)) \text{ if the upstate prevails,}\\ h^D(t,T)\equiv\exp(-\Delta t\sum b(t,T)) \text{ if the downstate prevails.}\end{cases} \tag{7.5}$$

The No-Arbitrage condition simply states that a portfolio of two bonds with different maturities that realizes a risk-free return[8] (i.e. a return independent of

[8] The risk-free hedge for a trading period $[t,t+\Delta t]$ contains one bond with maturity T_1 and $\mathbf{P}(t,T_1)(h^U(t+\Delta t,T_1)-h^D(t+\Delta t,T_1))/\mathbf{P}(t,T_2)(h^D(t+\Delta t,T_2)-h^U(t+\Delta t,T_2))$ of bonds with maturity T_2, see [5].

state of the economy on the next trading date) must make a return of a one-period bond. This condition constrains h^U and h^D in the following way: There exists a pair of constants π_t^U, π_t^D (independent of maturity T but not necessarily of present state of the economy and time) such that $\pi_t^U + \pi_t^D = 1$ and

$$\pi_t^U h^U(t,T) + \pi_t^D h^D(t,T) = 1. \tag{7.6}$$

Therefore, as h^U and h^D are the values taken by h in the upstate and the downstate, we think of π_t^U and π_t^D as *probabilities* of the upstate and the downstate. In other words, we may think of π_t^U and π_t^D as *given* probabilities of the events that the upstate or the downstate prevails on the next trading date. In that case, given the probabilities π_t^U and π_t^D, the condition (7.6) is just a restriction on the upstate and the downstate values of $\xi(t,T)$, i.e. a restriction on constants $a(t,T), b(t,T)$.

Condition (7.6) constitutes a set of equations on $a(t,T), b(t,T)$ that can be *solved*[9] in terms of probabilities π^U, π^D and volatilities of the forward rates

$$\mathrm{Var}_{t-\Delta t}\{f(t,T) - f(t-\Delta t, T)\} = \mathrm{Var}_{t-\Delta t}\{\xi(t,T)\}. \tag{7.7}$$

We assume that one period volatilities (7.7) are independent of maturity T and they are scaled so that the annualized volatility remains finite when the length Δt of the basic trading period decreases. This amounts to

$$\frac{1}{\Delta t}\mathrm{Var}_{t-\Delta t}\{\xi(t,T)\} = \sigma^2(t). \tag{7.8}$$

The conditional distribution of $\xi(t,T)$ is binomial, so we have

$$(a(t,T) - b(t,T))^2 \pi_t^U \pi_t^D = \sigma^2(t)\Delta t \tag{7.9}$$

Solving (7.9) for $a(t,T)$ in terms of $b(t,T), \sigma(t)$, and substituting the resulting formula into the restriction (7.6) we obtain

$$\sum_{j=t/\Delta t}^{T/\Delta t-1} \Delta t b\,(t, j\Delta t) = \log\left\{\pi_\pi^D + \pi_t^U \exp\Big(-\Delta t^{1/2}(T-t+\Delta t)\sigma(t)/(\pi_t^U\pi_t^D)^{1/2}\Big)\right\}. \tag{7.10}$$

By subtracting equations (7.10) with maturities T and $T+\Delta t$ we get

$$\begin{aligned} b(t,T) &= \frac{1}{\Delta t}\log\left\{\frac{\pi_t^D + \pi_t^U \exp\Big(-\Delta t^{1/2}(T-t+\Delta t)\sigma(t)/(\pi_t^U\pi_t^D)^{1/2}\Big)}{\pi_t^D + \pi_t^U \exp\Big(-\Delta t^{1/2}(T-t)\sigma(t)/(\pi_t^U\pi_t^D)^{1/2}\Big)}\right\} \\ &= -\sigma(t)\pi_t^U\Delta t^{1/2}/(\pi_t^U\pi_t^D)^{1/2} + \sigma^2(t)(T-t)\Delta t + O(\Delta t^{3/2}) \end{aligned} \tag{7.11}$$

where the last equation is a simple Taylor expansion result. Formula (7.11) together with equation (7.9) yield up to the error $O(\Delta t^{3/2})$,

$$\xi(t,T) = \sigma^2(t)(T-t)\Delta t + \begin{cases} +\sigma(t)(\pi_t^D/\pi_t^U)^{1/2}\Delta t^{1/2} & \text{with probability } \pi_t^U, \\ -\sigma(t)(\pi_t^U/\pi_t^D)^{1/2}\Delta t^{1/2} & \text{with probability } \pi_t^D. \end{cases} \tag{7.12}$$

[9] This observation is due to Heath, Jarrow & Morton [7].

Thus, $\xi(t,T)$ can be written, up to the error $O(\Delta t^{3/2})$, as

$$\xi(t,T) = \phi(t)\Delta t^{1/2} + \sigma^2(t)(T-t)\Delta t, \tag{7.13}$$

where $\phi(t)$ is a mean zero, variance $\sigma^2(t)$ random variable.

The interest rate $R(t)$ is simply the forward rate $f(t,t)$. Thus, the evolution equation for forward rate (7.3) implies

$$R(t) = f_0(t) + \sum_{j=1}^{t/\Delta t} \xi(j\Delta t, t). \tag{7.14}$$

Using equations (7.13) and (7.14), we find the interest rate $R(t)$ to be, up to the error $O(\Delta t^{1/2})$,

$$R(t) = f_0(t) + \left(\sum_{j=1}^{t/\Delta t} \phi(j\Delta t)\right)\Delta t^{1/2} + \left(\sum_{j=1}^{t/\Delta t} \sigma^2(j\Delta t)(t-j\Delta t)\right)\Delta t \tag{7.15}$$

Thus, the interest rate as seen by an individual investor in the two-state Ho-Lee model may be modeled by the following evolution equation:

$$\begin{aligned} R(0) &= R_0, \\ R(t) &= R(t-\Delta t) + f_0(t) - f_0(t-\Delta t) + \phi(t)\Delta t^{1/2} \\ &\qquad + \Big(\sum_{j=1}^{t/\Delta t - 1} \sigma^2(j\Delta t)\Delta t\Big)\Delta t. \end{aligned} \tag{7.16}$$

where ϕ is a mean-zero, variance $\sigma^2(t)$, two-state random variable, $f_0(t)$ is the instantaneous forward rate for period $[t-\Delta t, t]$ as seen at the initial time $t=0$, and R_0 is the initial interest rate.

8 Appendix B: Hedging in Markets with Uncertain Volatility

The asking and bidding prices of an interest rate dependent derivative are found from the pricing equation (1.1). The hedging procedure to achieve these prices constitutes an important part of our model. Here we concentrate on a *bidding* price, i.e. the valuation of a long position in the derivative.

The hedge is based on a additional position in a zero-coupon bond. Let $\mathbf{P}(t,T)$ be the time t price of a bond with maturity T. This price changes during one trading period $[t-\Delta t, t]$ as follows:

$$\mathbf{P}(t,T) = \exp\Big(-\Delta t \sum_{j=t/\Delta t}^{T/\Delta t - 1} \xi(t, j\Delta t) + \Delta t R(t-\Delta t)\Big)\mathbf{P}(t-\Delta t, T), \tag{8.1}$$

where $R(t-\Delta t)$ is the interest rate at the time $t-\Delta t$ and $\xi(t,s)$ is the change in the forward rate for the period $[s, s+\Delta t]$ between times $t-\Delta t$ and t, cf. (7.3). The No-Arbitrage condition restricts $\xi(t,T)$ to be given by

$$\xi(t,T) = \phi(t)\Delta t^{1/2} + \sigma^2(t)(T-t)\Delta t, \tag{8.2}$$

where $\phi(t)$ is a mean zero, variance $\sigma^2(t)$ random variable. For simplicity we assume that $\phi(t)$ takes values $\pm\sigma(t)$ with equal probabilities[10]. Combining equations (8.1) and (8.2) we find that the price of the bond changes according to the following equation:

$$\Delta \mathbf{P}(t,T) \equiv \mathbf{P}(t,T) - \mathbf{P}(t-\Delta t, T) = \left[-\phi(t)(T-t)\Delta t^{1/2} + R(t-\Delta t)\Delta t\right] \mathbf{P}(t-\Delta t, T), \tag{8.3}$$

up to the error $O(\Delta t^{3/2})$.

Consider the portfolio consisting of one long derivative $\mathbf{B}$ and α units of the bond. Let $\mathbf{B}(t)$ denote the time t value of the derivative. During the trading period $[t-\Delta t, t]$, the interest rate R and the accumulated variance V change by

$$\begin{aligned} \Delta R &= R(t) - R(t-\Delta t) = [\Delta f + V\Delta t] + \phi(t)\Delta t^{1/2}, \\ \Delta V &= V(t) - V(t-\Delta t) = \sigma^2(t)\Delta t, \end{aligned} \tag{8.4}$$

where $\Delta f = f(t) - f(t-\Delta t)$ and $V = V(t-\Delta t)$, cf (3.7),(3.8). Consequently, the value of the derivative moves by

$$\begin{aligned} \Delta \mathbf{B} &\equiv \mathbf{B}(t) - \mathbf{B}(t-\Delta t) \simeq \mathbf{B}_t \Delta t + \mathbf{B}_R \Delta R + \mathbf{B}_V \Delta V + \frac{1}{2}\mathbf{B}_{RR}(\Delta R)^2 \\ &= [\mathbf{B}_t + \mathbf{B}_V \sigma^2(t) + \mathbf{B}_R(f' + V) + \frac{1}{2}\mathbf{B}_{RR}\sigma^2(t)]\Delta t + \mathbf{B}_R \phi(t)\Delta t^{1/2}, \end{aligned} \tag{8.5}$$

where all partial derivatives are evaluated at the time $t-\Delta t$ values of the interest rate and accumulated variance. Here we used the special form of the $\phi(t)$ random variable, utilizing the equality $\phi^2(t) = \sigma^2(t)$. Thus, combining equations (8.5) and (8.3), we see that a hedge constructed of one long derivative and of

$$\alpha \equiv \frac{\mathbf{B}_R}{(T-t)\mathbf{P}} \tag{8.6}$$

units of the bond is *risk-less* in the sense that its value does not depend on the random change in the interest rate. Note that both $\mathbf{B}_R$ and $\mathbf{P}$ in the hedge ratio (8.6) are evaluated at the time $t-\Delta t$ levels of the interest rate and the accumulated variance. When we recall that the totally risk-averse investor values its portfolio at a minimal price attainable for any possible volatility $\sigma^2(t)$ in the band $[\sigma^2_{\min}, \sigma^2_{\max}]$, we find that the value of the portfolio changes by

$$\begin{aligned} &\min_\sigma \{\Delta \mathbf{B} + \alpha \Delta \mathbf{P}\} \\ &\equiv \min_\sigma \left\{ \mathbf{B}_t + \mathbf{B}_R(f' + V) + (\mathbf{B}_V + \frac{1}{2}\mathbf{B}_{RR})\sigma^2(t) + R\mathbf{B}_R/(T-t) \right\} \Delta t. \end{aligned} \tag{8.7}$$

[10] This assumption, however, has no bearing on the generality of our hedging procedure.

Now, note that during the time interval Δt the risk-free portfolio makes $R\Delta t$ percent of its original worth, i.e. $R\Delta t(\mathbf{B} + \alpha\mathbf{P}) = R\Delta t(\mathbf{B} + \mathbf{B}_R/(T-t))$. This quantity equated with (8.7) yields again the pricing equation (1.1) for the derivative $\mathbf{B}$, closing the proof that our hedge recovers the equation (1.1) value of the derivative.

We conclude by observing the financial interpretation of the bidding volatility coefficient $\min\{(\mathbf{B}_V + \frac{1}{2}\mathbf{B}_{RR})\sigma^2\}$. It simply represents the risk exposure of the hedge to the uncertain volatility of the interest rate σ^2. In fact, the pricing equation (1.1) can be understood as a "fair" pricing of the difference between the value of the portfolio and the value attainable through the hedge. This difference is caused by fact that the hedge is a the linear approximation of the *curvature* in the price as a function of the interest rate *and* the accumulated variance.