Topics in Probability: Risk-Management in Finance
Homework # 1

1. Linear SDEs. (i) Show that the stochastic differential equation

$$dX(t) \;=\; (\, a(t) \;-\; b(t)\, X(t)\,)\; dt \;+\; \sigma(t)\, dZ(t) \tag{1}$$

admits the solution

$$X(t) \;=\; e^{-B(t,0)}\, X(0) \;+\; \int_0^t e^{-B(t,\,s)}\, a(s)\, ds \;+\; \int_0^t e^{-B(t,\,s)}\, \sigma(s)\, dZ(s)\ ,$$

where $B(t,\, s) \;=\; \int_s^t b(u)\, du$.

(ii) Show that if $b(t) \;\geq\; b_0 \;>\; 0$ and if the limits

$$\mu \;\equiv\; \lim_{t\,\to\,\infty} \frac{a(t)}{b(t)} \qquad \text{and} \qquad \Sigma^2 \;\equiv\; \lim_{t\,\to\,\infty} \frac{\sigma^2(t)}{2b(t)}$$

exist, then

$$\lim_{t \to \infty} \mathbf{P}\ \{\, X(t) \;<\; x\ \} \;=\; N\left(\frac{x \;-\; \mu}{\Sigma} \right)\ .$$

Thus, the *asymptotic distribution* of $X(t)$ is normal with these parameters. What happens if $b(t) \;\geq\; 0$?

(iii) You want to model a stochastic process of the form (1) with constant coefficients which has a long term mean $\mu \;=\; 0.2$ and which remains in the band $0.1 \;-\; 0.4$ with 90% probability under the equilibrium distribution. What choices can you make for a, b and σ ? Argue that the different solutions are related by a change of clock.

2. A Monte-Carlo calculation. Suppose that the price of a financial asset follows the stochastic differential equation

$$dS(t) \;=\; S(t)\,[\,\sigma\,dZ(t)\;+\;\mu\,dt\,]\;, \tag{2}$$

where $\sigma(t) \;=\; 0.16$ and $\mu \;=\; 0.20$ are the annualized volatility and expected return. Assume that the asset pays no dividends and that the annualized interest rate is 6%.

(i) Generate a large sequence of standard normal deviates $N_1,\, N_2,\, \ldots, N_k \ldots$ (for instance you can try using the Box-Muller technique). Then, simulate 1,000 different 180-day paths of the security, assuming that $S(0) \;=\; 100$, by means of the formula

$$S_k \;=\; S_{k-1}\cdot e^{\sigma\sqrt{\Delta t}\,N_k \;+\; \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t}\quad,\quad S_0 \;=\; 100.$$

(Take $\Delta t \;=\; 1/255$).

(ii) Simulate the profit/losses of 1,000 instances of Delta-hedging a short position in an at-the-money European call option with 180 days to maturity, assuming that the option is hedged (a) every 10 days, (b) every 5 days, (c) every day. Calculate the empirical variance of P/L in each case. In the three simulations, find the path that gave the biggest loss.

(iii) Repeat with $\mu = .70$. Do you see the effect of the stock return on the P/L in discrete trading?

3. More SDEs. (i) Show that the solution of the stochastic differential equation

$$\frac{dX}{X} \;=\; a\,X\,dt \;+\; \sigma\,dZ$$

(with a and σ constant) satisfies

$$X(t) \;=\; \frac{X(0)\cdot e^{\sigma Z(t)\;-\;\frac{1}{2}\sigma^2 t}}{1\;-\;a\,X(0)\int_0^t e^{\sigma Z(s)\;-\;\frac{1}{2}\sigma^2 s}\,ds}.$$

(ii) Conclude that trajectories can blow up if $a\cdot X(0) \;>\; 0$. (They may or may not blow-up according to the size of $\frac{\sigma^2}{a\,X(0)}$.) Can you say more about the blow-up time or blow-up probability?