ITO PROCESSES, CONTINUOUS-TIME MARTINGALES AND GIRSANOV'S THEOREM

1. Martingales and Doob-Meyer decomposition

We have seen in the previous chapter that an Ito process can be represented by the stochastic differential

$$dX(t) \;=\; \sigma(t)\,dZ(t) \;+\; \mu(t)\,dt \;, \tag{1}$$

where σ and μ are (non-anticipative) local parameters. Equivalently, $X(t)$ can be written in integral form:

$$\begin{aligned} X(t) \;&=\; X(0) \;+\; \int_0^t \sigma(s)\,dZ(s) \;+\; \int_0^t \mu(s)\,ds \\ &\equiv\; X(0) \;+\; M(t) \;+\; B(t) \;. \end{aligned} \tag{2}$$

Let us analyze the processes $M(\cdot)$, and $B(\cdot)$ in this decomposition. For each t, $M(t)$ has mean zero since it is a stochastic integral. A more fundamental property of the process $M(\cdot)$ is that it is a **martingale**. This means that for all $T \;>\; t$, we have

$$\mathbf{E}_t\,\{M(T)\} \;=\; M(t)\;, \tag{3}$$

where $\mathbf{E}_t$ represents the **conditional expectation operator given the history up to time** t. A remark on this last point: in the last chapter, we assumed, for simplicity, that the non-anticipative functions $(\sigma,\,\mu)$ were completely determined at time t by the path $Z(s)\;,\;\; s \le t$. The "history up to time t " thus meant the oberved values of the Brownian path used to define the stochastic integral. It is useful to generalize this framework by assuming that the functions σ and μ may depend on other random processes in addition to the Brownian path $Z(\cdot)$ in (1). (For instance, we expect the volatility of an asset to be affected by changes in other economically correlated variables.) In these lectures, the expression "history up to time t " will be understood to mean the *the observed values up to time t of all processes which determine the local parameters.*

The martingale property of $M(t)$ follows from Proposition 2 in the previous chapter. In fact, note that

$$M(T) \;=\; M(t) \;+\; \int_t^T \sigma(s)\, dZ(s) \;.$$

Since $M(t)$ is non-anticipative, we have

$$\mathbf{E}_t \,\{\, M(t)\,\} \;=\; M(t) \;.$$

Moreover,

$$\begin{aligned} \mathbf{E}_t \,\{\, \sigma(s)\, dZ(s)\,\} \;&=\; \mathbf{E}_t \,\{\, \mathbf{E}_s \,\{\, \sigma(s)\, dZ(s)\,\}\,\} \\ &=\; \mathbf{E}_t \,\{\, \mathbf{E}_s \,\{\, \sigma(s)\,\} \cdot \mathbf{E}_s \,\{\, dZ(s)\,\}\,\} \\ &=\; 0 \end{aligned}$$

due to the independence of $dZ(s)$ from the past up to time s.

The concept of martingale in Probability Theory formalizes the intuitive notion of *fair game*. For example, a "coin-tossing" game, in which a gambler bets on the outcome of a coin toss several times in a row, has an accumulated wealth process which is a martingale. In contrast, the accumulated wealth for the game of roulette (betting, say, on the color red each time) is not a martingale because the house wins each time zero (green) occurs.[1]

Since the paths of $M(\cdot)$ are indistinguishable from Brownian paths after a change of time, the quadratic variation of $M(\cdot)$ is finite with probability 1. More precisely, the limit of the sums of squares of increments corresponding to a sequence of increasingly refined partitions satisfies

[1] Theoretical results on gambling in games of chance have been sought since the days of Laplace and surely earlier. See E. Thorp (1969), *Optimal gambling systems for favorable games*, Rev. Intl. Statistical Inst., 37, p.3, for interesting mathematical results on games of chance and speculative investing.

$$\lim_{\Delta t \to 0} \sum_j | \Delta M_j |^2 = \int_0^t \sigma^2(s)\, ds$$

in the mean-square sense.[2]

In contrast, $B(\cdot)$ has bounded first variation (and thus vanishing quadratic variation) since

$$\lim_{\Delta t \to 0} \sum_j | \Delta B_j | = \int_0^t | \mu(s) | \, ds$$

whenever the integral on the right-hand side exists.

The stochastic process $M(t)$ and $B(t)$ are known respectively as the **martingale component** and the **bounded variation component** of the Ito process $X(t)$. The last equation in (2) is often referred to as the **Doob-Meyer decomposition** of $X(\cdot)$.

2. Exponential Martingales

Proposition 1. *Let $\sigma(t)$ and $\mu(t)$ be bounded non-anticipative processes with respect to some Brownian motion $Z(\cdot)$. Then, the stochastic process $S(t)$ satisfies the stochastic differential equation*

$$dS(t) = S(t) \; [\; \sigma(t)\, dZ(t) \; + \; \mu(t)\, dt \;] \tag{4}$$

with

$$S(0) = S_0$$

if and only if

$$S(t) = S_0 \; e^{\int_0^t \sigma(s)\, dZ(s) \; - \; \frac{1}{2} \int_0^t \sigma^2(s)\, ds \; + \; \int_0^t \mu(s)\, ds} \; . \tag{5}$$

[2] The difference between the successive sums and the integral have second moment tending to zero as $\Delta t \to 0$.

Setting $\mu = 0$ in (5), it follows from this Proposition that the solution of the stochastic differential equation

$$dM(t) = M(t)\,\sigma(t)\,dZ(t) \tag{6}$$

with $S(0) = 1$, is given by

$$M(t) = e^{\int_0^t \sigma(s)\,dZ(s) - \frac{1}{2}\int_0^t \sigma^2(s)\,ds} . \tag{7}$$

The latter processes are called **exponential martingales**. Notice here again the difference between standard Calculus and Ito calculus: in the "smooth world", the solution of the differential equation (6) is given by dropping the ds-integral from the exponent of (7).[3]

Proof of Proposition 1. We show that if $S(t)$ satisfies (4) then it must have the form (5). For this, let us apply the Generalized Ito Lemma to the function $\ln S(t)$. Accordingly, we find that

$$\begin{aligned} d\,(\ln S(t)\,) &= \frac{1}{S(t)}\,dS(t) + \frac{1}{2}\,\frac{-1}{S(t)^2}\,(dS(t))^2 \\ &= \frac{1}{S(t)}\,dS(t) + \frac{1}{2}\,\frac{-1}{S(t)^2}\,\sigma^2(t)\,S(t)^2\,dt \\ &= \sigma(t)\,dZ(t) + \mu(t)\,dt - \frac{1}{2}\,\sigma^2(t)\,dt\ , \end{aligned}$$

and thus

$$\ln S(t) = \int_0^t \sigma(s)\,dZ(s) - \frac{1}{2}\int_0^t \sigma^2(s)\,ds + \int_0^t \mu(s)\,ds + \text{const}\ .$$

This shows that (4) implies (5). The converse statement also follows from an application of the Generalized Ito Lemma; it is left to the reader as an exercise.

[3] Notice also that if $\sigma \equiv 0$, the differential equation (4) has the "classical" solution $S(t) = S_0\,e^{\int_0^t \mu(s)\,ds}$.

Remarks.

1. If $\sigma(t)$ and $\mu(t)$ are independent of the Brownian motion $Z(\cdot)$, the Proportion gives an explicit solution of the stochastic differential equation (SDE) with linear coefficients (4).. This is one of a few cases in which a closed-form solution for an SDE exists. If, on the other hand, $\sigma(t)$ of $\mu(t)$ depends on $S(t)$ (or $Z(t)$) then (5) should not be viewed as a "solution" of the differential equation, because the right-hand side of (5) may depend on $S(s),\ s \leq t$ (through σ and μ.) Nevertheless, this result will be very useful.

2. Stochastic differential equations such as (4) are often used to describe the accumulated wealth of investment strategies or the evolution of security prices: in fact, if we write equation (4) in the form

$$\frac{dS(t)}{S(t)} \;=\; \sigma(t)\, dZ(t) \;+\; \mu(t)\, dt \ ,$$

the parameters $\mu\, dt$ and $\sigma\,\sqrt{dt}$ can be interpreted as the infinitesimal mean and the infinitesimal volatility of returns.

3. Futures prices are often modeled as exponential martingales. The reason is that, in an ideal market, the expected return on an open futures position held for one day should be zero (after adjusting for the price of risk, as we shall see in the following chapter). We have already encountered this result in the context of the binomial pricing model.

3. Girsanov's Theorem

We present an important application of the concept of exponential martingale, which is related to the idea of *change of probability* in path-space[4].

Proposition 2 (Girsanov's Theorem). *Consider a probability measure P on the space of paths $Z(t),\, t \leq T$ such that $Z(\cdot)$ is a Brownian motion and assume that $b(\cdot)$ is a non-anticipative function. Set*

$$M(t) \;\equiv\; e^{\int_0^t b(s)\, dZ(s) \;-\; \frac{1}{2}\int_0^t (b(s))^2\, ds} \quad , \quad t \;\leq\; T \ ,$$

and define a new measure Q on the set of trajectories $\{Z(t),\, t \leq T\}$ by

$$Q\,\{\,\mathcal{S}\,\} \;\equiv\; \mathbf{E}^P\,\{\mathcal{S}\, M(T)\}\ , \tag{8}$$

[4] The Cameron-Martin theorem discussed in an earlier lecture can be viewed as a corollary of Girsanov's Theorem.

where $\mathcal{S}$ represents an arbitrary set of paths and $\mathbf{E}^P$ is the expectation operator with respect the probability P. Then, the random process

$$W(t) \;=\; Z(t) \;-\; \int_0^t b(s)\,ds \quad , t \,\leq\, T \,,$$

is a Brownian motion under the measure Q.

Proof. We shall verify that, for all real numbers λ, we have

$$\mathbf{E}_t^Q \left\{ e^{\lambda\,(W(T)\,-\,W(t))} \right\} \;=\; e^{\frac{\lambda^2}{2}(T\,-\,t)} \,. \tag{9}$$

Taking expectation values of both sides of this equation, we conclude that $W(T) - W(t)$ is Gaussian with mean zero and variance $T - t$.[5]

Equation (9) also implies that two successive increments, say, $W(t+a) - W(t)$ and $W(t+a+a') - W(t+a)$ are statistically independent (if $Z(\cdot)$ has probability distribution Q.) In fact, for any a, $a' > 0$ and all λ_1, λ_2, we have

$$\mathbf{E}^Q \left\{ e^{\lambda_1\,(W(t+a)\,-\,W(t))} \cdot e^{\lambda_2\left(W(t+a+a')\,-\,W(t+a)\right)} \right\}$$

$$= \; \mathbf{E}^Q \left\{ \mathbf{E}_{t+a}^Q \left\{ e^{\lambda_1\,(W(t+a)\,-\,W(t))}\; e^{\lambda_2\left(W(t+a+a')\,-\,W(t+a)\right)} \right\}\right\}$$

$$= \; \mathbf{E}^Q \left\{ e^{\frac{\lambda_1^2}{2}a}\; \mathbf{E}_{t+a}^Q \left\{ e^{\lambda_2\left(W(t+a+a')\,-\,W(t+a)\right)} \right\}\right\}$$

$$= \; e^{\frac{\lambda_1^2}{2}a} \cdot e^{\frac{\lambda_2^2}{2}a'}$$

$$= \; \mathbf{E}^Q \left\{ e^{\lambda_1\,(W(t+a)\,-\,W(t+a))} \right\} \cdot \mathbf{E}^Q \left\{ e^{\lambda_2\left(W(t+a+a')\,-\,W(t+a)\right)} \right\} \,.$$

[5] Recall that the moment-generating function of a Gaussian random variable with mean zero and variance σ^2 is $\mathbf{E}\,\{e^{\lambda X}\} \;=\; e^{\frac{\lambda^2\sigma^2}{2}}$.

These two properties – Gaussianity and independence of increments – characterize Brownian motion. We will therefore establish (9). Consider first the case $t = 0$. Then, from (8), we find that the moment-generating function of $W(\cdot)$ under Q is

$$\mathbf{E}^Q \left\{ e^{\lambda W(T)} \right\} = \mathbf{E}^P \left\{ e^{\lambda W(T)} M(T) \right\}$$

$$= \mathbf{E}^P \left\{ e^{\lambda \left(Z(T) - \int_0^T b(s)\, ds \right) + \int_0^T b(s)\, dZ(S) - \frac{1}{2} \int_0^T (b(s))^2\, ds} \right\}$$

$$= \mathbf{E}^P \left\{ \exp. \left[\int_0^T (\lambda + b(s))\, dZ(S) - \int_0^T \lambda\, b(s)\, ds - \frac{1}{2} \int_0^T (b(s))^2\, ds \right] \right\}$$

$$= \mathbf{E}^P \left\{ \exp. \left[\int_0^T (\lambda + b(s))\, dZ(S) - \frac{1}{2} \int_0^T (\lambda + b(s))^2\, ds + \frac{1}{2} \lambda^2 T \right] \right\}$$

$$= \mathbf{E}^P \left\{ \exp. \left[\int_0^T (\lambda + b(s))\, dZ(S) - \frac{1}{2} \int_0^T (\lambda + b(s))^2\, ds \right] \right\} \cdot e^{\frac{1}{2} \lambda^2 T}$$

$$= e^{\frac{1}{2} \lambda^2 T} \; . \tag{10}$$

Notice that in this calculation we used the fact that

$$\exp. \left[\int_0^t (\lambda + b(s))\, dZ(S) - \frac{1}{2} \int_0^t (\lambda + b(s))^2\, ds \right]$$

is a martingale so, in particular, it has mean one. We have established equation (9) in the case $t = 0$. The calculation of the conditional moment-generating function of the increments $W(t+a) - W(t)$ is is analogous to (10). The key fact that needs to be used is a "conditional" version of (8) : the conditional probability Q_t is given explicitly by

$$Q_t\,\{\,\mathcal{S}\,\} \;\equiv\; \mathbf{E}_t^P\left\{\mathcal{S}\,\frac{M(T)}{M(t)}\right\}\;,$$

$$=\; \mathbf{E}_t^P\left\{\mathcal{S}\;;e^{\int_t^T b(s)\,ds \;-\; \frac{1}{2}\int_t^T (b(s))^2\,ds}\right\}\;, \tag{11}$$

where $\mathcal{S}$ is represents a set of paths. (Equation (11) is a consequence of (8) and the fact that $M(t)$ is a martingale. This point will be explained further in the Appendix.) Using (11), the moment-generating function (11) can be computed as in (10).

Girsanov's Theorem shows that the probability distributions of standard Brownian motion and of Brownian motion with "drift" $-b(s)$ are related in a simple way: namely, the following equation holds

$$Q \;=\; M(T)\,\cdot P\;.$$

In simple words, *one probability can be deduced from the other by multiplication by the exponential factor* $M(T)$. Therefore, the paths corresponding to Brownian motion with drift over a finite time-interval can be viewed as standard Brownian Paths after a change of measure. This theorem has an interesting consequence: events with probability zero for standard Brownian motion have probability zero for Brownian motion with drift and vice-versa. The two probability measure are said to be **mutually absolutely continuous** or **equivalent**.[6]

[6] This is true only true for events that depend on values taken by paths up to a *finite* time-horizon $T \;<\; \infty$. For instance, Brownian motion with postive drift converges with probability 1 to $+\infty$ as $t \to +\infty$ and standard Brownian motion does not.

Appendix: Proof of equation (11)

We shall use the elementary properties of conditional expectation operators with respect to a given sub-σ-algebra of events.[7] The conditional expectation operator with respect to the history of paths up to time t, $\mathbf{E}_t\{\cdot\}$, is characterized by the following property: let X be an arbitrary random variable. Then, $\mathbf{E}_t\{X\}$ is the unique random variable which is measurable with respect to the past up time t and satisfies

$$\mathbf{E}\{X\,Y\} \;=\; \mathbf{E}\{\mathbf{E}_t\{X\}\,Y\}. \tag{A.1}$$

for all random variables Y which are measurable with respect to the past up to time t.

Let us use this characterization to compute $\mathbf{E}_t^Q\{X\}$, where Q is defined in (8). From the definition of Q, we have

$$\begin{aligned}
\mathbf{E}^Q\{X\,Y\} &= \mathbf{E}^P\{X\,Y\,M(T)\} \\
&= \mathbf{E}^P\left\{X\cdot\left(\frac{M(T)}{M(t)}\right)\cdot M(t)\cdot Y\right\} \\
&= \mathbf{E}^P\left[\mathbf{E}_t^P\left\{X\left(\frac{M(T)}{M(t)}\right)\right]M(t)\,Y\right\} \\
&= \mathbf{E}^Q\left\{\mathbf{E}_t^P\left[X\left(\frac{M(T)}{M(t)}\right)\right]Y\right\}.
\end{aligned} \tag{A.2}$$

Here, we applied equation (A.1) with X replaced by $X\left(\frac{M(T)}{M(t)}\right)$ and Y replaced by $M(t)\,Y$ (notice that the latter variable is determined by the past up to time t.)

From the characterization of the conditional probability operator, we conclude that

$$\mathbf{E}_t^Q\{X\} \;=\; \mathbf{E}_t^P\left\{X\left(\frac{M(T)}{M(t)}\right)\right\}.$$

This is precisely what we wanted to show.

[7] The reader unfamiliar with these notions should consult, for instance, L. Breiman, *Probability*, or the first chapter of Bickel, *Introduction to Mathematical Statistics*.