VALUATION OF DERIVATIVE SECURITIES

This Chapter discusses derivative asset pricing from the point of view of Arbitrage Pricing Theory (APT). Under idealized market conditions – i.e., neglecting transaction costs, liquidity constraints or trading restrictions – the absence of arbitrage implies the existence of a probability measure such that the value of any derivative security is equal to the expectation of its discounted cash-flows. We discuss the Black-Scholes model from the APT point of view and revisit the important notion of dynamic hedging, which was introduced previously in the discrete framework (Ch. 3). In particular, we distinguish between dynamically complete and dynamically incomplete pricing models and discuss an example involving stochastic voaltility.

1. The general principle

If a market has no arbitrage opportunities, there exists a probability on forward market scenarios such that the prices of all traded securities are martingales, after adjusting for the cost of money and dividends. This is the basic proposal of APT. In mathematical terms, there exists a probability Q, defined on the set of forward price paths in the time-interval $[0,\,T]$ such that, for all $0\,<\,t\,<\,T$,

$$P(t) \;=\; \mathbf{E}_t^Q \left\{ e^{-\int_t^T (r(s)\,-\,d(s))\,ds}\; P(T) \right\} \;, \tag{1}$$

where $P(t)$ represents the price of any traded security at time t, $d(s)$ is the dividend yield and $r(s)$ is the riskless rate.[1] The symbol $\mathbf{E}_t^Q$ denotes the conditional expectation under the probability measure Q given the past up time t. A measure Q having this property is called a **martingale measure** or an **Arrow-Debreu measure** in Mathematical Finance. The reason is that (1) is equivalent to saying that the "adjusted" price processes

$$e^{-\int_0^t (r(s)\,-\,d(s))\,ds}\; P(t) \;, \tag{2}$$

are martingales. This last quantity represents the wealth accumulated by investing in one unit of a security at time $t=0$, with dividend reinvestment, measured in constant dollars.

[1] Strictly speaking, Q is a probability defined on the collection of possible paths that can be described by the state-variables defining the economy.

APT implies that a derivative security which entitles the holder to receive a series of cash-flows, say,

$$F(\tau_1),\ F(\tau_2),\ F(\tau_3),\ \ldots,\ F(\tau_N)\ ,$$

contingent on the value of other securities at a sequence of fixed or random times $\tau_1 < \tau_2, < \ldots, < \tau_N$ must have fair value

$$V(t) \;=\; \mathbf{E}_t^Q \left\{ \sum_{i\,:\ \tau_i \geq t} e^{-\int_t^{\tau_i} r(s)\,ds}\, F(\tau_i) \right\} . \tag{3}$$

A key feature of APT is (2) and (3) must hold with the *same* Q, independently of which security we consider. A martingale measure can be viewed as a *representation of the market's current opinion on the evolution of one or more indices and the prices of all derivatives contingent on them.* This includes not only traded derivatives but, more importantly, *any other derivative which may be traded in the future which matures before time* T. The knowledge of the martingale measure is all that is required, in principle, to value arbitrary derivative securities.

From a practical point of view, it is important to recognize that

(i) *the martingale measure is not unique.* The information available on a particular commodity and its derivatives consists of finitely many numbers (prices, interest rates, implied volatilities, volume traded, open interest, etc.). On the other hand, a probability measure defined on the space of Ito processes is determined by an infinite number of parameters. It is thus easy to understand non-uniqueness. Given a set of "underlying" indices or prices of assets which are correlated with the value of a derivative security, there is usually more than one martingale measure which is consistent with the data.[2]

(ii) *The martingale measure varies with time.* Derivatives markets are strongly affected by information shocks (elections, political announcements, takeover announcements, etc.), as well as by the price dynamics. Non-uniqueness of the martingale measure implies that the *measure itself* can change with time, as the market changes its expectations about the relative value of assets in the future.[3]

[2] This is known as **market incompleteness**: in general, there is not enough information available for the market to price every future state of the world. Recall that a **complete market** is an Arrow-Debreu market in which any state-contingent claim can be replicated, or synthesized, with traded securities. Market incompleteness is one of the forces that drives securities and commodities trading.

[3] This variation of the martingale measure in time is not incompatible with the basic securities market model described in the previous section. The reader should distinguish between the probability describing the *likelihood* of different economic states and the martingale measure which describes the *relative value* of future economic states, i.e. a probability that makes adjusted prices (2) into martingales. These are two different concepts. Uniqueness of the martingale measure is consistent with market equilibrium with respect to some (aggregate) utility, in which case security prices are equal to the marginal utilities for investment. (See Darrel Duffie, *Dynamical Asset Pricing Theory*, Princeton Univ. Press, 1992.) In Finance, one is typically dealing with a relatively short time-scale. It is therefore important to keep in mind that expectations and valuations change in time due to the arrival of new information.

In view of these considerations, the art of pricing of derivative securities consists of:

- selection of a (simple) martingale measure which is consistent with (2) and (3) – i.e., including information embedded in the prices of traded derivatives;
- evaluation of the expectation value in (3) for the derivative of interest;
- estimation of the sensitivity of the resulting price with respect to the choice of probability.

This last point is very important for risk-management. For instance, bond options have different prices according to the model used for pricing them and there is no clear consensus on which model should be used.[4]

2. Black-Scholes Model

The Black and Scholes model for pricing stock options has been applied to many different commodities and payoff structures.[5] We shall assume that there is a single underlying asset, which is traded in the market, and that the price of this asset satisfies the stochastic differential equation

$$dS \;=\; S\,\sigma\,dZ \;+\; S\,\mu\,dt \tag{4}$$

where σ and μ are the local volatility and mean of short-term returns. We shall assume that the interest rate r, the dividend yield d and σ remain constant over the period of interest. However, all the mathematical arguments extend without modifications to more complicated models in which r, d and σ are functions of the spot price and of the current time (or time-to maturity.)

We will use Proposition 1 of the previous lecture. If there are no arbitrage opportunities, we know that there exists an Ito process $\lambda(t)$ such that the local expected return μ_P, the dividend yield d_P and the volatility σ_P of any derivative security satisfy the relation

$$\mu_P(t) \;+\; d_P(t) - \;r \;=\; \sigma_P(t)\,\lambda(t)\;. \tag{5}$$

This relation is true for *any* traded security which depends on $S(t)$. In particular, this includes the asset itself since it is a traded security. Thus,

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[4]This issue is discussed in M. A. Wong: *Trading and Investing in Bond Options*, Wiley, 1991.

[5]This includes not only financial options but also "real options", which are the options of building plants, drilling for oil, etc. (See Dixit and Pindyck: *Investment under Uncertainty*). Options pricing theory can be seen, in a broader sense, as a tool for decision-making.

$$\mu \ + \ d - \ r \ = \ \sigma \, \lambda(t) \ ,$$

or,

$$\lambda \ = \ \frac{\mu \ + \ d \ - \ r}{\sigma} \ . \tag{6}$$

Since there is a single Brownian motion and a there exists a traded security driven by it, the function $\lambda(t)$ can be computed explicitly. It is equal to the ratio of the interest-adjusted rate of return of the security and its volatility. λ is called the the **Sharpe ratio** of the underlying asset. From (5) and (6), we conclude that, for any derivative security contingent on S, we have

$$\frac{\mu_P \ + \ d_P \ - \ r}{\sigma_P} \ = \ \frac{\mu \ + \ d \ - \ r}{\sigma} \ . \tag{7}$$

Equation (7) states that the Sharpe ratio of any derivative is equal to the Sharpe ratio of the underlying asset.

Now, assume that the fair value of a derivative can be expressed as a deterministic function of the value of the underlying index and current time, i.e., that

$$P(t) \ = \ \tilde{P}(S(t), \, t) \ .$$

Let us show that this function can be characterized using (7). In fact, applying the Generalized Ito Lemma to $\tilde{P}(S(t), \, t)$, we find that

$$dP \ = \ P \ (\, \sigma_P \, dZ \ + \ \mu_p \, dt \ ) \tag{8}$$

where

$$\sigma_P \ = \ \frac{1}{\tilde{P}} \frac{\partial \tilde{P}}{\partial S} \, S \, \sigma \tag{9}$$

and

$$\mu_P \ = \ \frac{1}{\tilde{P}} \cdot \left\{ \frac{\partial \tilde{P}}{\partial t} \ + \ \mu \, S \, \frac{\partial \tilde{P}}{\partial S} \ + \ \frac{1}{2} \, \sigma^2 \, S^2 \, \frac{\partial^2 \tilde{P}}{\partial S^2} \right\} \ . \tag{10}$$

Assume, for simplicity, that the derivative security pays no dividends to its holder, i.e., that $d_P \ = \ 0$. If we substitute the values from equations (9) and (10) for σ_P and μ_P into equation (7), we find that

$$\frac{1}{\tilde{P}} \cdot \left\{ \frac{\partial \tilde{P}}{\partial t} + \mu S \frac{\partial \tilde{P}}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \tilde{P}}{\partial S^2} \right\} - r = \frac{1}{\tilde{P}} \frac{\partial \tilde{P}}{\partial S} S (\mu + d - r) .$$

Rearranging terms, we conclude that $\tilde{P}(S,t)$ should satisfy

$$\frac{\partial \tilde{P}}{\partial t} + (r - d) S \frac{\partial \tilde{P}}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \tilde{P}}{\partial S^2} - r \tilde{P} = 0 . \tag{11}$$

This is the **Black-Scholes partial differential equation**. Notice that the parameters that enter the equation are the riskless rate of return, the dividend rate and the volatility. In particular, local rate of return of the underlying asset, μ, does not affect the price of a derivative security.

We can draw a connection between the Black-Scholes equation and the theory of martingale measures. Let us imagine a fictitious market in which the price of the underlying security satisfies the **risk-adjusted** SDE

$$dS = S (\sigma dZ + (r - d) dt) .$$

The solution of this stochastic differential equation is the well-known "geometric Brownian motion",[6]

$$S(t) = S(0) e^{\sigma Z(t) - \frac{\sigma^2}{2} t + (r-d) t} \tag{12}$$

Applying Ito's Lemma to $P(t) = \tilde{P}(S(t), t)$ we find that (8) holds with σ_P given by (9) and with a modified "return"

$$\mu_P = \frac{1}{\tilde{P}} \cdot \left\{ \frac{\partial \tilde{P}}{\partial t} + (r - d) S \frac{\partial \tilde{P}}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \tilde{P}}{\partial S^2} \right\} . \tag{13}$$

Using (13) and the Black-Scholes equation (11), we find that $P(t)$ satisfies

$$dP = P \sigma_P dZ + P (r - d) dt . \tag{14}$$

This implies, in turn, that

$$d \left(e^{t(r-d)t} P(t) \right) = \left(e^{t(r-d)t} P(t) \right) dZ ,$$

[6]See lecture on Ito Calculus.

i.e., that $e^{t(r-d)t}\, P(t)$ is a martingale. In particular, we have

$$\tilde{P}(S,\,t) \;=\; \mathbf{E}^{Q}\left\{\, e^{-\,r\,(T-t)}\;\tilde{P}(S(T),\,T)\;|S(t)\;=\;S\right\}\;, \tag{15}$$

where Q represents the probability associated with the risk-adjusted SDE in (14).

In conclusion, under the martingale measure, the underlying asset is a geometric Brownian motion with $\mu \;=\; r \,-\, d \,-\, \frac{\sigma^2}{2}$.

Example: Derivation of the Black-Scholes formula. The most important application of this theory is the Black-Scholes option pricing formula. Since the value of a European option is a known function of the price of the underlying asset at the expiration date, it can be found by solving a **final-value problem** for the Black-Scholes equation. Namely, the function $\tilde{P}(S,t)$ satisfies

$$\begin{cases} \frac{\partial\tilde{P}}{\partial t} \;+\; (r \;-\; d\,)\;S\,\frac{\partial\tilde{P}}{\partial S} \;+\; \frac{1}{2}\,\sigma^2\,S^2\,\frac{\partial^2\tilde{P}}{\partial S^2} \;-\; r\,\tilde{P} \;=\; 0 \\ \\ \tilde{P}(S,T) \;=\; F(S)\;, \end{cases} \tag{16}$$

where T is the maturity date and $F(S) \;=\; max(S - K,\,0)$ or $max(K - S,\,0)$ according to whether the option is a call or a put. This problem has a closed-form solution when the parameters σ, r and d are constant or dependent only on time. Otherwise, the equation must be solved numerically, as indicated in Section 4 below.

In the case of constant coefficients, the problem reduces to the calculating the expected value of the random variable $F\left(\,e^{Y}\,\right)$, where Y is normal. For instance, the value of a call option with strike K expiring at time T is

$$\tilde{P}(S(t),t) \;=\; \mathbf{E}\left\{\,e^{-\,r\,(T-t)}\,\left(S(t)e^{Y(T)} \;-\; K\right)\;;\;S(t)\,e^{Y(T)} \;>\; K\,\right\}\;, \tag{17}$$

with

$$Y(T) \;=\; \sigma\,\sqrt{T-t}\,X \;-\; \frac{1}{2}\sigma^2\,(T\;-\;t) \;+\; (r\;-\;q\,)\;(T\;-\;t)\;,$$

where X is a standard normal. The right-hand side can be evaluated explicitly in terms of the cumulant of the normal distribution

$$N(x) \;=\; \frac{1}{2\pi}\int\limits_{-\infty}^{x} e^{-\frac{y^2}{2}}\,dy\;.$$

Accordingly, the value the European call is

$$C(S,\,t;\,K,\,T) \;=\; S\,e^{-d\,(T-t)}\,N(d_1) \;-\; K\,e^{-r\,(T-t)}\,N(d_2) \tag{18}$$

where

$$d_1 \;=\; \frac{1}{\sigma\,\sqrt{T-t}}\,ln\left(\frac{S\,e^{(r-q)\,(T-t)}}{K}\right) \;+\; \frac{\sigma\,\sqrt{T-t}}{2}$$

and

$$d_2 \;=\; d_1 \;-\; \sigma\,\sqrt{T-t}\;.$$

This is, of course, the celebrated Black-Scholes formula. The elegance and versatility of the Black-Scholes theory have resulted in its wide acceptance in securities markets as a pricing and hedging tool.

3. Dynamic hedging and dynamic completeness.

Let us pursue the subject of the previous section from a slightly different point of view. Suppose that a trader sells a European option (or, more generally, a European contingent claim) and wishes to implement a trading strategy using the underlying security to hedge the option exposure. Let $\Delta(t)$ and $B(t)$ represent, respectively, the number of units of the underlying asset held in the portfolio and the current balance in a money-market account. We assume that the strategy is self-financing.

The change in the value of the portfolio after a small period of time dt is

$$dV \;=\; \Delta\,dS \;+\; q\,\Delta\,S\,dt \;+\; r\,B(t)\,dt\;, \tag{19}$$

to leading order in dt. Let us assume momentarily that the value of the contingent claim is $P(t) \;=\; \tilde{P}(S(t),\,t)$, where $\tilde{P}$ is an unknown function. According to Ito's Lemma, the change in P from time t to time $t+dt$ is, to leading order,

$$dP \;=\; \frac{\partial\tilde{P}}{\partial S}\,dS \;+\; \left(\frac{\partial\tilde{P}}{\partial t} \;+\; \frac{1}{2}\,\sigma^2\,S^2\,\frac{\partial^2\tilde{P}}{\partial S^2}\right)\,dt\;. \tag{20}$$

Suppose that the strategy followed by the trader is such that,

- initially, $V(0) \; = \; \tilde{P}(S(0),\, 0)$ and $\Delta(t) \; = \; \frac{\partial \tilde{P}(S(0),0)}{\partial S}$;
- subsequently, $\Delta(t) \; = \; \frac{\partial \tilde{P}(S(t),\, t)}{\partial S}$
- dividends produced by the risky asset are reinvested in the asset.

Under these conditions, the value of the dynamic portfolio shall remain equal to $\tilde{P}(S(t),\, t)$ (to leading order in dt) if and only if the dt-contributions from (19) and (20) are equal. That is, if

$$\begin{aligned} \frac{\partial \tilde{P}}{\partial t} \; + \; \frac{1}{2}\,\sigma^2\, S^2\, \frac{\partial^2 \tilde{P}}{\partial S^2} \; &= \; d \cdot \, S\, \frac{\partial \tilde{P}}{\partial S} \; + \; r\, B \\ &= \; d \cdot \, S\, \frac{\partial \tilde{P}}{\partial S} \; + \; r\, \left(\tilde{P} \; - \; S\, \frac{\partial \tilde{P}}{\partial S} \right) \\ &= \; r\, \tilde{P} \; - (\, r \; - \; d \,)\; S\, \frac{\partial \tilde{P}}{\partial S} \; . \end{aligned} \tag{21}$$

This condition is equivalent to requiring that $\tilde{P}$ satisfy the Black-Scholes equation

$$\frac{\partial \tilde{P}}{\partial t} \; + \; (\, r \; - \; d \,)\; S\, \frac{\partial \tilde{P}}{\partial S} \; + \; \frac{1}{2}\,\sigma^2\, S^2\, \frac{\partial^2 \tilde{P}}{\partial S^2} \; - \; r\, \tilde{P} \; = \; 0 \; . \tag{22}$$

We have proved

Proposition 1: *Assume that an index or asset price follows the stochastic differential equation*

$$dS \; = \; S\; (\, \sigma\, dZ \; + \; \mu\, dt \,) \tag{23}$$

and that investing in this asset produces a dividend yield d, where σ,r and d are functions of time and the spot price. Suppose that $\tilde{P}$ satisfies

$$\begin{cases} \frac{\partial \tilde{P}}{\partial t} \; + \; (\, r \; - \; d \,)\; S\, \frac{\partial \tilde{P}}{\partial S} \; + \; \frac{1}{2}\,\sigma^2\, S^2\, \frac{\partial^2 \tilde{P}}{\partial S^2} \; - \; r\, \tilde{P} \; = \; 0 \;\; , \quad t \; < \; T \; , \\ \\ \tilde{P}(S,\, T) \; = \; F(S) \; . \end{cases} \tag{24}$$

Then, the trading strategy

$$B(t) \;=\; \tilde{P}(S(t),\, t) \;-\; S(t)\,\frac{\partial \tilde{P}(S(t),\, t)}{\partial S} \quad , \quad \Delta(t) \;=\; \frac{\partial \tilde{P}(S(t),\, t)}{\partial S} \qquad 0 \;\leq\; t \;\leq\; T, \tag{25}$$

is self-financed. Furthermore, the portfolio $(\,\Delta(T)\,,\; B(T)\,)$ *has value* $V(T) \;=\; F(S(T))$*. In particular, the no-arbitrage value of a European-style contingent claim with payoff* $F(S(T))$ *is* $\tilde{P}(S(t),\, t)$*.*

This proposition gives a concrete interpretation of APT in the context of the Black-Scholes model: the value of a derivative security is equal to its *replication cost.* If not, there is a clear arbitrage. For instance, if a derivative security trades below the value of its replicating portfolio, $(\,\Delta(t)\,,\; B(t)\,)$ at time t, a trader can purchase the derivative and implement a self-financing dynamic hedging strategy with $\Delta(t) \;=\; -\,\frac{\partial \tilde{P}(S(t).\,t)}{\partial S}$. The difference in prices implies that the trade produce an initial profit and is also riskless, since the position will have net value zero at time T (by Proposition 1). The reverse strategy can be implemented if the derivative is more expensive than the dynamic portfolio.

Is APT equivalent to pricing derivatives by dynamic replication? The answer is no, in general. The Black-Scholes model and more general parametrizations of the type

$$\sigma(t) \;=\; \tilde{\sigma}(S(t),\, t) \quad , \quad r \;=\; \tilde{r}(S(t), t) \quad , d(t) \;=\; \tilde{d}(S(t),\, t) \quad , \tag{26}$$

in which $\tilde{\sigma}$, $\tilde{r}$ and $\tilde{r}$ are deterministic functions of S and t, are examples of **dynamically complete models**. These are models in which every contingent claim can be replicated by dynamic trading with cash market instruments. A key property that distinguishes dynamically complete models from others is that the dimensionality of the space of "effective" factors is equal to the dimensionality of the space of cash instruments. We will formalize this statement below.

Notice that the case of the Black-Scholes model with constant parameters or its generalization (26), the spot price is the only source of risk. In the case of variable coefficients, it completely determines the parameters of the model. In paticular, for any other security contingent on $S(t)$, there exists a proportion, or hedge-ratio, of holdings among the two assets which results in a completely diversified portfolio (i.e. such that its local variance is zero). This property of the Black-Scholes model— the posibility of total diversification of local risk in the cash market — is the mechanism which allows for replication by dynamic hedging.

This discussion can be generalized to multi-factor models. We state the following proposition for future use:

Proposition 2. *Consider a securities market model with* M *factors,* $X_1(t)$, $X_2(t)$, $...,X_M(t)$*, satisfying the dynamic equations*

$$dX_i(t) \;=\; \sum_{0}^{\nu} \alpha_{i,k}\;(\mathbf{X}(t),\,t\,)\;dZ_k(t)\;+\;\beta_i\;(\mathbf{X}(t),t\,)\;dt\quad,\qquad \text{for}\;\; i\;=\;1\ldots M\;, \tag{27}$$

where $Z_k(\cdot)$, $k\ -\ 1\ldots\nu$ are independent Brownian motions. Suppose that the state-covariance matrix

$$A_{ij}(\mathbf{X},\,t)\;\equiv\;\sum_{k=1}^{\nu}\alpha_{i,k}(\mathbf{X},\,t)\,\alpha_{j,k}(\mathbf{X},\,t)\;,\qquad 0\;\le\;i,j\;\le\;M \tag{28}$$

has rank R ($R\;\le\;M$). A necessary and sufficient condition for this market to be dynamically complete is that there exist N traded securities with prices $P_l(t)\;=\;\tilde{P}_l(\mathbf{X}(t),\,t)$ such that the local price-covariance matrix

$$\Sigma_{lm}\;\equiv\;\sum_{k=1}^{\nu}\;\sum_{i=1\,,\,j=1}^{M}\;\alpha_{i,k}\,\alpha_{j,k}\,\frac{\partial\tilde{P}_l}{\partial X_i}\,\frac{\partial\tilde{P}_m}{\partial X_j} \tag{29}$$

has rank R. The condition can be expressed concisely in matrix notation as

$$\text{Rank}\,\left\{\alpha\cdot\alpha^t\right\}\;=\;\text{Rank}\,\left\{\,(\,\nabla\,\mathbf{P}\,)^t\cdot\,\alpha\,\cdot\,\alpha^t\cdot(\,\nabla\,\mathbf{P}\,)\,\right\}\;, \tag{30}$$

where the superscript $(\bullet)^t$ denotes matrix transposition.

The proof of this proposition follows essentially the same mathematical ideas as in the proof of Proposition 1 of Chapter 3. One way to visualize this theorem is to use the language of Differential Geometry. Accordingly, the rank of the matrix A can be viewed as the dimensionality of the tangent space to the "manifold" representing the economy, for with the state-variables are local coordinates. On the other hand, the rank of the price-covariance matrix corresponds to the dimensionality of the tangent subspace generated by portfolios of cash-market instruments. If the two dimensionalities were equal, a cmplete diversification of risk would be possible. In general, the dimensionality of the tangent space is higher and one must introduce parameters $\lambda_i(t)$, which may not be directly computable from the dynamic equations (27) and the security prices $\tilde{P}_l$. The mathematical proof consists in checking that these statements are correct, using Ito Calculus. The details are left to the interested reader.

Cox, Ingersoll and Ross call the parameters λ_i the **market prices of risk**. The number of unspecified market prices of risk corresponds therefore to the difference between the ranks or the matrices A and $(\,\nabla\,\mathbf{P}\,)^t\cdot\,\alpha\,\cdot\,\alpha^t\cdot(\,\nabla\,\mathbf{P}\,)$.[7]

[7] In the fancy language of Differential Geometry, this number is the *codimension* of the tangent subspace generated by security prices in the tangent space.

Example: stochastic volatility model. A simple model for a traded security with stochastic volatility was introduced a few years ago by Hull and White[8]. These authors considered a model for the joint evolution of a stock price and its spot volatility, namely,

$$\begin{aligned} dS &= \sigma\, S\, dZ \; + \; \mu_S\, S\, dt \\ d\sigma &= \xi\, \sigma\, dW \; + \; \mu_\sigma\, \sigma\; dt\;, \end{aligned} \tag{31}$$

where Z and W are Brownian motions with correlation coefficient $\rho \;=\; \mathbf{E}\,\{\, dZ\, dW \,\}\;/\; dt \;\neq\; \pm 1$. This system can be regarded as a two-factor model in which the spot price and the volatility are the state-variables. If we assume that spot volatility is not traded (more on this in lectures to follow!), then we are in a situation in which the volatility risk cannot be fully diversified. Proposition 2 does not apply, since we have $\mathrm{Rank}(A) \;=\; 2$ and there is only one traded asset. No-arbitrage implies that the returns and "volatilities" of any traded asset must satisfy

$$\mu_P \; + \; d_P \; - \; r \; = \; \sigma_{P,1}\, \lambda_1 \; + \; \sigma_{P,2}\, \lambda_2 \;, \tag{32}$$

where the subscripts 1 and 2 refer, respectively to the stock price and the stock volatility. Since the stock is a traded security, we have

$$\mu_S \; + \; d_S \; - \; r \; = \; \sigma\, \lambda_1 \quad , \quad (\mathrm{since}\, \sigma_{S,2} \; = \; 0\,)\,. \tag{33}$$

We recover an old result: λ_1 is the Sharpe ratio of the stock. Substituting this into equation (32), we find that

$$\mu_P \; + \; d_P \; - \; r \; = \; \sigma_{P,1} \left(\frac{\mu_S \, + \, d_S \, - \, r}{\sigma} \right) \; + \; \sigma_{P,2}\, \lambda_2 \;. \tag{34}$$

If we assume, that the price of a derivative security is given by $P(t) \;=\; \tilde{P}(S(t), \sigma(t), t)$, this last equation becomes a relation between the partial derivatives of $\tilde{P}$¿ Specifically, using Ito's Lemma, (31) and (34) we find, after some calculation, that $\tilde{P}$ satisfies the equation

$$\begin{aligned} &\frac{\partial \tilde{P}}{\partial t} \; + \; (r \, - d_S\,)\, S\, \frac{\partial \tilde{P}}{\partial S} \; + \; (\mu_\sigma \; - \; \xi\, \lambda_2)\, \sigma\, \frac{\partial \tilde{P}}{\partial \sigma} \; + \\ &\frac{\sigma^2\, S^2}{2}\, \frac{\partial^2 \tilde{P}}{\partial S^2} \; + \; \xi\, \rho\, \sigma^2\, S\, \frac{\partial^2 \tilde{P}}{\partial S\, \partial \sigma} \; + \; \frac{\xi^2\, \sigma^2}{2}\, \frac{\partial^2 \tilde{P}}{\partial \sigma^2} \; = \; 0. \end{aligned} \tag{35}$$

[8] *J. of Finance*, Vol XLII, no.2, June 1987, p. 281

Notice that this equation involves the parameter λ_2 in the coefficient of the first derivative of $\tilde{P}$ with respect to σ. Therefore, to obtain a pricing measure for this stochastic volatility model, we must make additional assumptions on the value of λ_2. The model is not dynamically complete.

Equation (35) shows that the martingale measure corresponding to (31) is such that the asset-volatility pair satisfies the SDE

$$\begin{aligned} dS &= \sigma\, S\, dZ \;+\; (r \;-\; d)\; S\, dt \\ d\sigma &= \xi\, \sigma\, dW \;+\; (\,\mu_\sigma \;-\; \xi\, \lambda_2\,)\; \sigma\; dt\;. \end{aligned} \tag{36}$$

How can one specify the parameter λ_2 in order to generate a "closed" model? This is not an easy problem, for the following reason: to hedge volatility risk, one must obviously use options or some other volatility-sensitive instruments. But then we face the problem that in order to determine λ_2 we need to know the the option value $\tilde{P}$ as a function of σ and S. But this is precisely what we are trying to find! [9]

We will return to the problem of "calibration" of the stochastic volatility model in other lectures, as we discuss further the relation between the volatilities implied from option prices and "spot" volatility.

[9] It can be shown that the option price cannot be determined using a "self-consistent" argument along these lines. See L.O. Scott: *Option pricing when the variance changes randomly: theory and application*, 1986.