TRINOMIAL TREES AND FINITE-DIFFERENCE SCHEMES

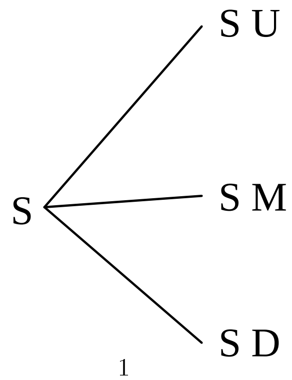
1. Trinomial model

This chapter discusses the implementation of trinomial probability trees for pricing derivative securities. These models have a lot more flexibility than binomial trees. As shown (early on) by Cox, Ross and Rubinstein, binomial models for stock and equity derivatives are dynamically complete. This may be one of the reasons for their popularity among practitioners: not only can they be used for pricing but the price actually represents the cost of a replicating strategy. However, a drawback of binomial trees is that they cannot be used to model processes with local means and variances which depend on the value of the underlying index. The reason is that there are not enough parameters to adjust or degrees of freedom. Therefore, binomial models are only useful to model stochastic processes with constant or, at best, with time-dependent parameters that do not vary with the underlying index or other factors. In certain situations, we would prefer to describe the underlying index by an Ito process with parameters that depend on the index itself or other factors. This is where trinomial trees become a useful numerical tool.

Trinomial trees permit us to specify different local volatilities and drifts at each node of a recombining tree. This means that we can model Ito processes of the form

$$\frac{dS}{S} \;=\; \sigma(S,\,t)\,dZ \;+\; \mu(S,\,t)\,dt\ , \tag{1}$$

in which the local parameters are functions of the asset price and time, with a structure that has $O(N^2)$ nodes if the number of periods is N. The elementary structure of the trinomial tree is shown below.



This structure is reproduced at each time to generate a "tree" in which each vertex has three offspring. A necessary and sufficient condition for the tree to recombine, i.e to have $3n$ nodes at time n, is that the parameters U, M and D satisfy

$$U \cdot D \ = \ M^2 \tag{2}$$

This constraint and the stochastic differential equation (1) lead us to parametrize the model as follows:

$$\begin{aligned} U \ &= \ e^{\overline{\sigma}\,\sqrt{dt} \ + \ \overline{\nu}\,dt} \\ M \ &= \ e^{\overline{\nu}\,dt} \\ D \ &= \ e^{-\overline{\sigma}\,\sqrt{dt} \ + \ \overline{\nu}\,dt} \ . \end{aligned} \tag{2}$$

Here $\overline{\sigma}$ and $\overline{\nu}$ are constants that will be determined and dt represents a (small) interval of time between successive shocks, measured in years.

We shall assign conditional probabilities to each of the three outcomes, P_U, P_M, and P_D. Let us denote by $\nu\,dt$ and $\sigma^2\,dt$ the mean and variance of the logarithm of the price shock over a period. We have

$$\nu\,dt \ = \ \overline{\sigma}\,\sqrt{dt} \cdot (\,P_U \ - \ P_D\,) \ + \ \overline{\nu}\,dt \tag{3}$$

and

$$\sigma^2\,dt \ = \ \overline{\sigma}^2\,dt \cdot (\,P_U \ + \ P_D\,) \ - \ (\,\nu\,dt\,)^2 \tag{4}$$

We will always assume that $dt \ \ll \ 1$ and that the parameters σ and ν are $O(1)$. Under these conditions, we are only interested in keeping terms of order dt in the calculation of the mean and variance.[1] In particular, we will replace equation (4) by the simpler equation

$$\sigma^2\,dt \ = \ \overline{\sigma}^2\,dt \cdot (\,P_U \ + \ P_D\,) \tag{5}$$

Equations (4) and (5) can be used to calculate the probabilities P_U, P_M and P_D in terms of the mean and the variance of the logarithm of the shock over a period. In fact, we can easily deduce from (3) and (5) that

[1] Lower-order terms of order $dt^{3/2}$ give a negligible contribution to the sum of the local variances.

$$P_U \;=\; \frac{1}{2}\left[\,\frac{\sigma^2}{\overline{\sigma}^2} \;+\; \left(\frac{\nu \,-\, \overline{\nu}}{\overline{\sigma}}\right)\sqrt{dt}\,\right] \;, \tag{6}$$

$$P_D \;=\; \frac{1}{2}\left[\,\frac{\sigma^2}{\overline{\sigma}^2} \;-\; \left(\frac{\nu \,-\, \overline{\nu}}{\overline{\sigma}}\right)\sqrt{dt}\,\right] \;, \tag{7}$$

and

$$P_M \;=\; 1 \;-\; \frac{\sigma^2}{\overline{\sigma}^2} \;. \tag{8}$$

Financial considerations typically require adjusting the drift $\mu(S,\,t)$ of the process in (1) to satisfy no-arbitrage conditions. For instance, in the world of currency derivatives, $\mu \;=\; r_d \;-\; r_f$ = the difference between the domestic and foreign interest rates. Thus, in practice, we would like to treat μ (i.e. the annualized expected return) as the input parameter rather than ν. Of course, we know from Ito calculus that

$$\nu \;=\; \mu \;-\; \frac{1}{2}\sigma^2 \;. \tag{9}$$

Let us rewrite equations (6) and (7) in terms of μ. The result is:

$$P_U \;=\; \frac{1}{2}\frac{\sigma^2}{\overline{\sigma}^2}\left(1 \;-\; \frac{\overline{\sigma}\sqrt{dt}}{2}\right) \;+\; \left(\frac{\mu \,-\, \overline{\nu}}{2\overline{\sigma}}\right)\sqrt{dt} \tag{10}$$

and

$$P_D \;=\; \frac{1}{2}\frac{\sigma^2}{\overline{\sigma}^2}\left(1 \;+\; \frac{\overline{\sigma}\sqrt{dt}}{2}\right) \;-\; \left(\frac{\mu \,-\, \overline{\nu}}{2\overline{\sigma}}\right)\sqrt{dt} \tag{11}$$

2. Stability analysis

We would like to derive conditions that must be satisfied so that the three probabilities calculated above are non-negative. Introducing the parameters

$$p \;\equiv\; \frac{\sigma^2}{\overline{\sigma}^2} \;, \quad q \;\equiv\; \frac{\mu \,-\, \overline{\nu}}{\overline{\sigma}^2} \;, \tag{12}$$

the equations for the probabilities become

$$P_U \;=\; \frac{p}{2}\left(1 \;-\; \frac{\overline{\sigma}\,\sqrt{dt}}{2}\right) \;+\; q\,\frac{\overline{\sigma}\,\sqrt{dt}}{2}\;, \tag{13}$$

$$P_D \;=\; \frac{p}{2}\left(1 \;+\; \frac{\overline{\sigma}\,\sqrt{dt}}{2}\right) \;-\; q\,\frac{\overline{\sigma}\,\sqrt{dt}}{2}\;, \tag{14}$$

and

$$P_M \;=\; 1 \;-\; p\;. \tag{15}$$

These expressions are useful to determine whether P_U, P_M and P_D are are nonnegative and less than one. In fact, we conclude from (15) that we must have

$$0 \;\leq\; p \;\leq\; 1\;. \tag{16}$$

Moreover, (from (13) with $q \;=\; 0$),

$$\overline{\sigma} \;<\; \frac{2}{\sqrt{dt}}\;, \tag{17}$$

must hold as well. A further condition, which guarantees that P_U, P_M and P_D are positive, is that

$$p \;>\; \frac{2\,|q|\,\overline{\sigma}\,\sqrt{dt}}{2 \;-\; \overline{\sigma}\,\sqrt{dt}}\;. \tag{18}$$

3. Calibration

To calibrate our tree to given volatility and drift "surfaces" $\sigma(S,\,t)$ and $\mu(S,\,t)$, we consider the four numbers

$$\sigma_{min} \;\equiv\; \underset{S,t}{inf}\;\sigma(S,\,t) \quad , \quad \sigma_{max} \;\equiv\; \underset{S,t}{sup}\;\sigma(S,\,t)$$

$$\mu_{min} \equiv \inf_{S,t} \mu(S,t) \quad \text{and} \quad \mu_{max} \equiv \sup_{S,t} \mu(S,t) \; .$$

We shall assume that these numbers are uniformly bounded. Regarding the volatility parameter, we assume that $\sigma_{min} > 0$. This last requirement is not necessary — in fact, it is possible to model using the trinomial tree processes with a local volatility which vanishes at certain levels of spot prices.[2]

We begin with the calibration of volatility. Since equation (5) tells us that

$$\sigma^2 \;=\; p \cdot \overline{\sigma}^2 \; , \tag{19}$$

we must have $\sigma^2 \leq \overline{\sigma}^2$ in view of the restriction (16). We shall therefore make the choice

$$\overline{\sigma} \;=\; \sigma_{max} \; , \tag{20}$$

which corresponds to the minimal value of $\overline{\sigma}$ compatible with the volatility range that we are trying to model. Of course, any choice of $\overline{\sigma}$ greater than σ_{max} is also possible.[3] It follows from equation (19) that, by varying the parameter p in the interval

$$\frac{\sigma_{min}^2}{\sigma_{max}^2} \;\leq\; p \;\leq\; 1 \; , \tag{21}$$

we can achieve any value of σ in the desired range.

Let us turn next to the drift. Equation (12) suggests that $\overline{\nu}$ should be chosen in the range

$$\mu_{min} \;\leq\; \overline{\nu} \;\leq\; \mu_{max} \; .$$

This implies, in particular, that

$$|q| \;\leq\; max \left(\frac{\mu_{max} - \overline{\nu}}{\overline{\sigma}^2} , \; \frac{\overline{\nu} - \mu_{min}}{\overline{\sigma}^2} \right) \; . \tag{22}$$

Notice that the bound is tighter when we set

$$\overline{\nu} \;=\; \frac{\mu_{max} + \mu_{min}}{2} \; , \tag{23}$$

[2] This property will prove useful later in modeling the impact of transaction costs on hedging.

[3] We note, however, that the greater $\overline{\sigma}$, the smaller we have to choose dt in order to ensure that (17) holds.

when it becomes

$$|q| \;\le\; \frac{\mu_{max} \;-\; \mu_{min}}{2\,\overline{\sigma}^2} \;. \tag{24}$$

Combining equations (21) and (24), we conclude that the stability condition (18) is satisfied if we have

$$\frac{\sigma^2_{min}}{\sigma^2_{max}} \;>\; \frac{2\,\sigma_{max}\,\sqrt{dt}}{2 \;-\; \sigma_{max}\,\sqrt{dt}} \cdot \frac{\mu_{max} \;-\; \mu_{min}}{2\sigma^2_{max}}$$

or, more concisely, if

$$dt \;<\; \frac{4}{\sigma^2_{max}} \left(\frac{\sigma^2_{min}}{\sigma^2_{min} \;+\; \delta\mu} \right)^2 \;, \quad \delta\mu \;=\; \mu_{max} \;-\; \mu_{min} \;. \tag{25}$$

This condition is, in general, more restrictive than the one implied by (17), namely

$$dt \;<\; \frac{4}{\sigma^2_{max}} \;. \tag{26}$$

It reduces to the latter condition when the drift is constant ($\delta\mu \;=\; 0$). (In the latter case, it is possible to implement a trinomial tree with volatility that vanishes at certain nodes without generating negative probabilities.)

To construct a trinomial tree with transition probabilities corresponding to the diffusion process (1), we calculate the parameters σ_{min}, σ_{max}, μ_{min} and μ_{max} and define the parameters $\overline{\sigma}$ and $\overline{\nu}$ accordingly. We then select a time interval dt which satisfies the stability condition (25). Once this is done, we obtain a trinomial tree, or lattice, that describes the values of the index S in discrete increments. These values can be denoted by S^j_n, where n represents the time variable and j the "height" on the tree.

Next, we discretize the drift and volatility surfaces, setting

$$\sigma^j_n \;\equiv\; \sigma(S^j_n,\, t_n) \quad , \qquad \mu^j_n \;\equiv\; \mu(S^j_n,\, t_n) \;.$$

Finally, we set

$$p^j_n \;=\; \frac{\left(\sigma^j_n\right)^2}{\overline{\sigma}^2} \;,$$

$$q^j_n \;=\; \frac{\mu^j_n \;-\; \overline{\nu}}{\overline{\sigma}^2}$$

and define the probabilities at the node $(n,\, j)$ according to equations (13), (14) and (15), substituting p^j_n for p and q^j_n for q. In this way, we have specified a discrete approximation for the diffusion process (1) on a trinomial tree.

4. Finite-difference scheme for the Black-Scholes PDE with prescribed volatility and drift surfaces σ and μ

Pricing contingent claims on the trinomial tree by discounting expected cash-flows leads to a recursion relation for the value of the claim at the different nodes of the tree. This relation is analogous to the Cox-Ross-Rubinstein "backward induction" method for the binomial model, but we can incorporate volatility and drift "surfaces". The discrete pricing equation is

$$
\begin{aligned}
V_n^j \;&=\; F_n^j \;+ \\
e^{-\,r_n^j\,dt}\cdot\Big\{\;&\frac{1}{2}\,p_n^j\left(1-\frac{\overline{\sigma}\,\sqrt{dt}}{2}\right)V_{n+1}^{j+1}\;+\;\frac{1}{2}\,p_n^j\left(1+\frac{\overline{\sigma}\,\sqrt{dt}}{2}\right)V_{n+1}^{j-1} \\
&+\,(1\,-\,p_n^j\,)\,V_{n+1}^j\;+\;\frac{1}{2}\,q_n^j\,\overline{\sigma}\,\sqrt{dt}\,\left(V_{n+1}^{j+1}\,-\,V_{n+1}^{j-1}\right)\;\Big\}\,, \qquad (27)
\end{aligned}
$$

where F_n^j represents a cash-flow due at time t_n if the index value is S_n^j. This equation can be viewed as a finite-difference scheme for solving the Black-Scholes equation. The construction of the previous sections guarantees the stability and convergence of the scheme as $dt\,\to\,0$.

Of course, dynamic programming equations derived from (27) can be used for pricing American options and other contingent claims which involve stopping times. In particular, the trinomial tree provides a more accurate alternative than the binomial model for the pricing of barrier options because the barrier can be made to coincide with a particular level in the tree. This eliminates to some extent numerical roundoff error.

Other applications involve dynamical programming equations that are used in worst-case scenario analyis of portfolios (Avellaneda, Levy and Paras (1994) and Avellaneda and Parás (1995)), which will be discussed in other chapters.