Stochastic Calculus Fall 2009
Homework 3

1. Multivariate models. Download end-of-day data for the components of the Dow Jones Industrials Average for the last year. Using this data, specify a model for the joint dynamics of the component stocks of the type

$$\frac{dS_i}{S_i} = \sum_{k=1}^{m} \sigma_{ik} dZ_k + \mu_i dt \tag{1}$$

2. Brownian reflection principle. Let $W(t)$ be a standard Brownian motion, let $a > 0$, and define τ_a as the first-passage time of the path through level a:

$$\tau_a = \inf \{t : W(t) = a\}.$$

We define the *reflected path around a* as

$$\begin{aligned} W_a(t) &= W(t) \qquad \text{if } t < \tau_a \\ &= 2a - W(t) \text{ if } t \geq \tau_a \end{aligned} \tag{2}$$

(a) Show that $W_a(t)$ is also a Brownian motion. (b) Show that the following equation holds

$$\begin{aligned} Prob\left\{\max_{0\leq t\leq T} W(t) > a\right\} &= Prob\{\tau_a < T, W(T) < a\} + Prob\{\tau_a < T, W(T) \geq a\} \\ &= Prob\{W_a(T) > a\} + Prob\{W(T) \geq a\} \\ &= 2\,Prob\{W(T) \geq a\} = 2\left(1 - N(\frac{a}{\sqrt{T}})\right) \end{aligned} \tag{3}$$

where $N(x)$ is the cumulative distribution function of a standard normal. (c) Using Monte Carlo simulation, compute $Prob\{\tau_x < 1\}$ for x=0.5, 1,1.5 and 2, and corroborate the result (b).

3. Ito's Lemma and polynomials. Let $W(t)$ be standard Brownian motion and let $P(x)$ be a polynomial of degree $n, n \geq 1$. Find values of the constants a, b, and a differential equation satisfied by $P(x)$, so that the process

$$X_t = t^a P\left(\frac{W(t)}{t^b}\right)$$

is a martingale. [Hint: apply Itô's Lemma to $t^a P(x/t^b)$.] (b) List polynomials that have this property for $n = 2, 3, 4$.

4. 2-D Brownian motion. (a) Let $W_1(t)$ and $W_2(t)$ be two independent Brownian motions and let f_t and g_t be two non-anticipative functions with respect to (W_1, W_2). Show that if $f_t^2 + g_t^2 = 1$ then

$$Y_t \ = \ \int_0^t (f_t \ dW_1(t) + g_t \ dW_2(t))$$

is a Brownian motion. What if $f_t^2 + g_t^2 \neq 1$? Is there a time-change under which this process is a Brownian motion? (b) With the notation of (a), define

$$\theta_t \ = \int_0^t \frac{W_1(s)dW_2(s) - W_2(s)dW_1(s)}{W_1^2(s) + W_2^2(s)},$$

where (W_1, W_2) is a BM and $W_1(0)+W_2(0) \neq 0$. Show that $\theta(t)$ is a martingale and that it can be viewed as a Brownian motion after a change of time. [Extra credit for geometrically inclined students: give a geometric interpretation of θ_t. Based on this interpretation, show an easy way to compute θ from the 2-d Brownian path. Hint: apply Itô's Lemma to $Arctan(\frac{W_2(t)}{W_1(t)})$].

5. Bessel processes. Let $(W_1(t), ..., W_n(t))$ be n independent Brownian motions. Define the new process

$$B_n(t) = \sqrt{W_1^2(t) + ... + W_n^2(t)}. \tag{4}$$

This process describes the distance to the origin of an n-dimensional Brownian motion. Show that B_n satisfies the equation

$$dB_n(t) = dZ(t) + \frac{n-1}{2B_n(t)}dt$$

where $Z(t)$ is a suitable Brownian motion. Assume that $B_n(0) \neq 0$. Argue that $B_n()$ can never hit the level zero.