Stochastic Calculus Fall 2009
Homework 4

1. First passage time of Brownian motion. Let $W(t)$ be a standard Brownian motion starting at zero. Derive the density of the first passage time τ_a of the level $W = a$. Verify that $E(\tau_a) = \infty$. [Hint: reflection principle.]

2. First exit time of Brownian motion from an interval. Let (a, b) be an interval, and let τ_{ab} represent the first exit time of $W(t)$ from this interval. Compute $E(\tau_{ab})$. [For extra credit only(much harder): can you compute the density of τ_{ab}?]

3. Let $X_i(t), i = 1, ..., N$, be independent Ornstein-Uhlembeck processes defined by

$$dX_i(t) = -\kappa X_i(t)dt + \sigma dW_i(t) \quad t > 0, \tag{1}$$

where $W_i(t)$ are independent standard Brownian motions. Define the process

$$Y(t) = \sum_{i=1}^{N} X_i^2(t). \tag{2}$$

(i) Show that $Y(t)$ satisfies

$$dY(t) = -\overline{\kappa}\left(\overline{\theta} - Y(t)\right)dt + \overline{\alpha}\sqrt{Y(t)}dZ(t) \quad t > 0, \tag{3}$$

where $Z(t)$ is a Brownian motion. (i) Calculate the coefficients of the above equations in terms of the coefficients of (1). (ii) Write a PDE to calculate the value of the expectation

$$E\left\{e^{-\lambda Y_T}|Y_t = y\right\} = \phi(y, t, \lambda, T),$$

and calculate this value. Verify the result using equation (2). (iii) Show that

$$\lim_{T-t\to\infty} \phi(y, t, \lambda, T)$$

is independent of y and corresponds to the Laplace transform of a χ^2 distribution.

4. I ♡ Monte Carlo. Calculate

$$E\left\{e^{-5\int_0^3 W_t^2 dt}\right\} \tag{4}$$

by Monte Carlo simulation. Then use a PDE to find a closed-form solution. [Hint: use the *ansatz* $\phi(x,t) = e^{a(t)x^2+b(t)}$.]. Check, please.

5. The Black Scholes formula & PDE. Derive *carefully* the PDE needed for calculating the expected value

$$C(S,T) = E\left\{e^{-rT}max(S_T - K, 0)|S_0 = S\right\}$$

where r is the interest rate and S_t is a diffusion defined by

$$\frac{dS_t}{S_t} = \sigma dW_t + \mu dt.$$

(Here μ represents the *cost of carry* of a stock.). Then solve the PDE using integration with respect to the Gaussian pdf and derive the classical Black-Scholes option-pricing formula for a European-style call option.